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# On the Algebraic Compactness of Some Complete Modules.

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## Introduction.

Let $R$ be a commutative ring with unit.

An $R$-module $M$ is algebraically compact if every finitely soluble family of linear equations over $R$ in $M$ has a simultaneous solution.

If $R$ is a noetherian ring and $\Omega$ is the set of the maximal ideals of $R$, we can define over any $R$-module $M$ the $\Omega$-adic topology, by taking as a base of neighborhoods of 0 the submodels $IM$, where $I$ is a finite intersection of powers of the maximal ideals.

If $R$ is any ring and $M$ is any $R$-module, we can define on $M$ the $R$-topology, by taking as a base of neighborhoods of 0 the submodules $rM$ with $0 \neq r \in R$.

Warfield [W$_1$] has proved that any algebraically compact $R$-module is complete in the $\Omega$-adic topology, if $R$ is a noetherian ring, and in the $R$-topology if $R$ is any ring.

Moreover, Warfield has raised the problem to see if any complete Hausdorff module over a noetherian ring is necessarily algebraically compact.

In this work we answer in the affirmative to the question posed by Warfleld and we characterize the neotherian rings $R$ such that

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any $R$-module which is complete and Hausdorff in the $R$-topology is algebraically compact.

## 1. Complete modules in the $\Omega$-adic topology.

Let $R$ be a noetherian commutative ring with unit, $M$ a topological $R$-module equipped with the $\Omega$-adic topology.

We denote by $\hat{M}$ the $\Omega$-adic completion of $M$.

(« Complete module » means « Hausdorff complete module »).

The purpose of this section is to prove that, for any $R$-module $M$, $\hat{M}$ is an algebraic compact $R$-module.

First of all, we recall that $\hat{M}$ is topologically isomorphic to the product $\prod_{\mathfrak{m}\in\Omega} \hat{M}_{\mathfrak{m}}$ where $\hat{M}_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-adic completion of the localization of $M$ at $\mathfrak{m}$, so since the class of algebraically compact modules is closed under direct products, we shall have to settle the problem only with respect to the $\mathfrak{m}$-adic completion of a module.

By a suitable definition of pure submodule, Warfield has proved that the class of algebraically compact modules, coincides with the class of pure-injective modules. Therefore we now recall the principal definitions concerning the concept of purity and pure-injectivity.

DEFINITION 1. *Let $R$ be a ring, $\mathcal{S}$ a class of $R$-modules.*

*$N$ is an $\mathcal{S}$-pure submodule of an $R$-module $M$, if every element of $\mathcal{S}$ is projective for the exact sequence:*

$$0 \to N \to M \to M/N \to 0 .$$

An equivalent definition to definition 1 is the following: ([W$_2$])

DEFINITION 1'. *Let $\mathcal{S}$ be a class of $R$-modules.*

*$N$ is an $\mathcal{S}$-pure submodule of an $R$-module $M$ if it is a direct summand of any module $H$ such that: a) $N \leqslant H \leqslant M$, b) $H/N \in \mathcal{S}$.*

Walker ([W$_2$]) has introduced the notion of $\mathcal{S}$-copure submodule by dualizing the definition 1' in the following way:

DEFINITION 2. *A submodule $N$ of an $R$-module $M$ is $\mathcal{S}$-copure in $M$, if for every submodule $H$ of $N$ such that $N/H \in \mathcal{S}$, $N/H$ is a summand of $M/H$.*

We are interesting with a particular class of modules, namely we consider the class $\mathcal{F}$ of all finitely presented modules, so that we give the following definition:

DEFINITION 3. *A submodule of a module $M$ is pure (copure) in $M$ if it is $\mathcal{F}$-pure ($\mathcal{F}$-copure).*

*Moreover we say that a module is pure-injective if it is injective for any pure exact sequence.*

REMARK. If $R$ is a noetherian ring; the class $\mathcal{F}$ is the class of all finitely generated modules.

LEMMA 1. *Let $R$ be an artinian local ring.*
*If $N$ is a pure submodule of $M$, then it is also copure in $M$.*

PROOF. Let $H$ be a submodule of $N$ such that $N/H \in \mathcal{F}$.

We have to prove that $N/H$ is a summand of $M/H$.

Since $N$ is pure in $M$, $N/H$ is pure in $M/H$ ([$W_2$], Theor. 2.1); moreover, since the maximal ideal of $R$ is nilpotent, the $\Omega$-adic topology over any $R$-module is the discrete topology, so $N/H$ is a finitely generated and a complete module in the $\Omega$-adic topology.

Then, by Theor. 3 of [$W_1$], $N/H$ is pure-injective and therefore it is a summand of $M/H$. //

LEMMA 2. *Let $R$ be a ring satisfying the hypotheses of the preceding lemma. If $N$ is a copure submodule of a module $M$, then it is a summand of $M$.*

PROOF. Let:

$$\mathcal{F}(N) = \{N_\alpha \leqslant N \colon N/N_\alpha \in \mathcal{F}\}$$

$$N^{\mathcal{F}} = \bigcap_{N_\alpha \in \mathcal{F}(N)} N_\alpha$$

For each $0 \neq x \in N$, let $H_x$ be a submodule of $N$ maximal with respect to the property of not containing $x$.

It is easy to verify that the submodule generated by $x + H_x$ is simple and essential in $N/H_x$.

Therefore the injective envelope $E(N/H_x)$ of $N/H_x$ is isomorphic to $E(R/\mathfrak{m}) = E$. Now, by [M] Theor. 3.4 and 3.11, $E = \bigcup_k E_k$ where $E_k$ is an increasing sequence of finitely generate submodules of $E$ with $E_k = \{x \in E \colon \mathfrak{m}^k x = 0\}$. Therefore, since $\mathfrak{m}^h = 0$, for a convenient integer $h$, we have $E = E_h$; then $N/H_x$ is finitely generated since it is a submodule of the noetherian module $E$ and then we have:

$$0 = \bigcap_{0 \neq x \in N} H_x \supseteq \bigcap_{\alpha} N_\alpha = N^{\mathcal{F}} .$$

Now, ([W$_2$], Corollary 2.9′) the group Copext $(L, N)$ of the copure extensions of $N$ by a generical module $L$, is the image of the homomorphism $f\colon \operatorname{Ext}(L, N^{\mathcal{F}}) \to \operatorname{Ext}(L, N)$ induced by the inclusion $N^{\mathcal{F}} \to N$. Then, since $N^{\mathcal{F}} = 0$, $N$ is a summand of every module in which it is a copure submodule. //

THEOREM 1. *Let $R$ be a noetherian ring, $\mathfrak{m}$ an element of $\Omega$ and $M$ an $R$-module. For every $k \in \boldsymbol{N}$, $M/\mathfrak{m}^k M$ is an algebraically compact $R$-module.*

PROOF. $M/\mathfrak{m}^k M$ is an $R/\mathfrak{m}^k$-module, then by lemmas 1 and 2, it is an algebraically compact $R/\mathfrak{m}^k$-module. Moreover we can easily deduce from the definition of algebraically compactness, that $M/\mathfrak{m}^k M$ is also an $R$-module algebraically compact //.

Let $M$ be an $R$-module, we denote by $B(M)$ the Bohr compactification of $M$, that is:

$$B(M) = \operatorname{Hom}_{\boldsymbol{Z}}(\operatorname{Hom}_{\boldsymbol{Z}}(M, K), K)$$

where $K$ is the circle group ([W$_1$], § 3).

Let $\omega_M$ be the natural homomorphism of $M$ in $B(M)$; then $\omega_M(M) = \tilde{M}$ is canonically isomorphic to $M$.

Warfield ([W$_1$], § 3), has proved that $B(M)$ is a topological compact $R$-module and that $M$ is a pure (and dense) submodule of $B(M)$.

Now we have the following:

THEOREM 2. *Let $R$ be a noetherian ring, $\mathfrak{m}$ a maximal ideal of $R$. The $\mathfrak{m}$-adic completion of any $R$-module $M$ is an algebraically compact $R$-module.*

PROOF. Let $M_k = M/\mathfrak{m}^k M$ and $\pi_h^k$ the natural homomorphisms $\pi_h^k\colon M_k \to M_h$ $(k > h,\ h, k \in \boldsymbol{N})$; then we have:

$$\hat{M} \approx \varprojlim_k \{M_k, \pi_h^k \quad k > h\}_{k \in N}$$

Let $B_k$ be the Bohr compactification of $M_k$ for every $k \in \boldsymbol{N}$, $\tilde{M}_k$ the copy of $M_k$ in $B_k$ and let $\tilde{\pi}_h^k$ be the homomorphisms induced by the $\pi_h^k$.

Then $M$ is isomorphic to $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ since $\omega_k$ are natural isomorphism for every $k \in \boldsymbol{N}$.

Therefore it will suffices to prove that $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ is algebraically compact. Let's consider the following diagram:

$$\begin{tikzcd}
\tilde{M}_k \arrow[r, "i_k"] \arrow[d, "\pi_h^k"] & B_k \arrow[ddl, dotted, "f_h^k"] \\
\tilde{M}_h \arrow[d, "i_h"] & \\
B_h &
\end{tikzcd}$$

The universal property of $B_k$, assures the existence of a unique continuous homormorphism $f_h^k$ such that the diagram commutes.

Now, by Theor. 1, $\tilde{M}_k$ is an algebraically compact $R$-module, so $B_k = \tilde{M}_k \oplus T_k$ for every $k \in \boldsymbol{N}$.

Let us consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{M}_k \arrow[r, "i_k"] \arrow[d, "\tilde{\pi}_h^k"] & B_k \arrow[r] \arrow[d, "f_h^k"] & T_k \arrow[r] \arrow[d, "g_h^k"] & 0 \\
0 \arrow[r] & \tilde{M}_h \arrow[r, "i_h"] & B_h \arrow[r] & T_h \arrow[r] & 0
\end{tikzcd} \qquad k > h$$

with $f_h^k \circ i_k = i_h \circ \tilde{\pi}_h^k$. Since the rows are exact, there is a unique homomorphisms $g_h^k$ such that the diagram commutes.

By the unicity of the $g_h^k$, the system $\{T_k;\ g_h^k \ \ k > h\}$ is an inverse system. Then we have:

$$\varprojlim_k \{\tilde{M}_k;\ \tilde{\pi}_h^k\} \oplus \varprojlim_k \{T_k;\ g_h^k\} \simeq \varprojlim_k \{B_k;\ \tilde{\pi}_h^k \oplus g_h^k\}\ .$$

Now, $\varprojlim B_k$ is a compact module in the topology induced by the product topology of the $B_k$, then by [W₁] Theor. 2, $\varprojlim M_k$ is algebraically compact. //

## 2. Complete modules in the $R$-topology.

Let $R$ be a noetherian commutative ring with unit.

The purpose of this section is to characterize the rings $R$ such that any $R$-module which is complete and $T_2$ in the $R$-topology is algebraically compact.

First of all we consider the case in which the $\Omega$-adic topology on $R$ is the discrete topology.

This hypothesis implies that every $R$-module is discrete in the $\Omega$-adic topology and so, any $R$-module is algebraically compact.

In the general case, that is, when the open ideals in the $\Omega$-adic topology on $R$ are always non zero, then the $R$-topology over any $R$-module $M$ is finer than the $\Omega$-adic topology.

Now, since $R$ is a noetherian ring, any ideal $rR$, with $r \neq 0$, contains a finite intersection of powers of prime non zero ideals of $R$.

Therefore, if $R$ has the following property:

$(P)$ every non zero prime ideal of $R$ is maximal

the $\Omega$-adic topology and the $R$-topology coincides over any $R$-module.

Then the results contained in Section 1, allow us to say that $(P)$ is a sufficient condition on $R$ to insure that any complete and $T_2$ module in the $R$-topology is algebraically ompact.

(The converse has been stated by Warfield, as we have just noted).

Now, we shall prove that $(P)$ is also a necessary condition.

Let us suppose that $R$ has a non zero and non maximal prime ideal $\mathfrak{I}$.

Let $\mathfrak{m}$ be a maximal ideal of $R$ containing $\mathfrak{I}$ and let $T$ be the localization of $R/\mathfrak{I}$ at $\mathfrak{m}/\mathfrak{I}$; we consider the $R$-module $A = T[x]$ where $x$ is a trascendental element over $T$.

Clearly the $R$-topology on $A$ is discrete, so $A$ is complete in such topology, but we shall prove that $A$ is not algebraically compact.

Infact, if $A$ were algebraically compact, Warfield's results would entail the completeness of $A$ in the $\Omega$-adic topology. But now, it is easy to verify that the $\Omega$-adic topology on $A$ is the same as the $\mathfrak{m}$-adic topology, so it suffices to find a nonconvergent Cauchy sequence of elements of $A$.

We denote by $\mathfrak{n}$ the maximal ideal of $T$.

The powers of $\mathfrak{n}$ give a strictly decreasing chain of ideals of $T$, since, by Krull Theorem, $\bigcap_i \mathfrak{n}^i = 0$ and by the hypotheses on $\mathfrak{I}$ we cannot have $\mathfrak{n}^i = 0$ for any $i \in \boldsymbol{N}$.

Let $a_i$ be an element of $\mathfrak{n}^i \setminus \mathfrak{n}^{i+1}$ for every $i \in \boldsymbol{N}$, and let us consider the following elements of $A$:

$$s_k = \sum_{i=0}^{k} a_i x_i \qquad k \in \boldsymbol{N} .$$

Now it is easy to prove that $\{s_k\}_k$ is a Cauchy sequence of element of $A$ which cannot converge to any element of $A$.

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