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A principle involving the variation of the metric tensor in a stationary space-time of general relativity

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A Principle Involving the Variation of the Metric Tensor in a Stationary Space-Time of General Relativity.

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Summary - Within general relativity we introduce a variational principle, involving the variation of the metric tensor in a stationary spacetime, and concerning the equilibrium of an elastic body capable of couple stresses but not of heat conduction.

1. Introduction.

In this work we consider an elastic body $\mathcal{C}$ capable of couple stresses but not of heat conduction and we assume absence of electromagnetic phenomena.

Basing ourselves on a certain variational theorem involving the variation of the space-time metric, firstly formulated by Taub in [4] and extended by Schöpf and Bressan to the non-polar and polar cases respectively — cf. [3] and [2]—, we introduce a variational principle concerning the rest of a body $\mathcal{C}$ of the type above.

More in detail we prove that if C_3 is a certain 3-dimensional region of a stationary spacetime S_4, the equilibrium of the body $\mathcal{C}$, in the stationary frame (x), is physically possible if and only if the functional

$$J = \int_{C_3} (R + 16\pi h c^{-4} \varrho) \sqrt{-g}\, dC_3 \,,$$

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is stationary with respect to certain variations of the metric, where R is the bicontracted Riemann tensor, h Cavedish's constant, c the velocity of light in vacuum and ϱ the proper actual density of matter energy.

This theorem is an analogue of the relativistic variational principle proved in [1].

2. Preliminaries.

We follow the theory of continuous media in general relativity constructed by Bressan, cf. [2] (cf. also [1] and the references therein).

Let $\mathcal{C}$ be a continuous body and $\mathcal{P}$ a process physically possible for $\mathcal{C}$ in a space-time S_4 of general relativity. We shall consider only regular motions for $\mathcal{C}$, e.g. without slidings and splittings; hence $\mathcal{C}$ can be regarded as a collection of material points.

By (x) we denote an admissible frame; by $g_{\alpha\beta} = g_{\alpha\beta}(x^\varrho)$ (1) the metric tensor corresponding to $\mathcal{P}$; by $u^0[A^0]$ the four velocity [acceleration] of $\mathcal{C}$ at the event point $\mathcal{E}$.

Then we consider a particular process $\mathcal{P}^*$ physically possible for the universe containing $\mathcal{C}$, the world-tube $W^*_{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{P}^*$ and an admissible frame (y). We call S^*_3 the intersection of $W^*_{\mathcal{C}}$ with the hypersurface $y^0 = 0$. We use the co-ordinate y^L of the intersection of S^*_3 with the world line of the point P^* of $\mathcal{C}$ as L-th material co-ordinate (2).

We represent the arbitrary (regular) motion of $\mathcal{C}$ in the system of co-ordinates (x) by means of the functions

$$x^\alpha = \hat{x}^\alpha(t, y^1, y^2, y^3)\,, \tag{1}$$

where t is an arbitrary time parameter.

If $T^{\cdots}_{\cdots}$ is a double tensor field associated to the event point x^ϱ and the material point y^L, we shall denote by $T^{\cdots}_{\cdots,\varrho}$ the ordinary partial derivative, by $T^{\cdots}_{\cdots/\varrho}$ the covariant derivative and by $T^{\cdots}_{\cdots|M}$ the lagrangian spatial derivative based on the map (1) and introduced by Bressan.

We are interested in an elastic body $\mathcal{C}$ capable of couple stress but not of heat conduction and we shall always assume thet electromagnetic phenomena are absent.

(1) Greek and Latin indices run over 0, 1, 2, 3 and 1, 2, 3 respectively.

(2) Capital and lower case latters represent material and space-time indices respectively.

We call ϱ the proper actual density of total internal energy, $X^{\alpha\beta}$ the stess tensor, $m^{\alpha\beta\gamma}$ the couple stress tensor and we shall assume as total energy tensor (3)

$$\mathfrak{U}^{\alpha\beta} = \varrho u^\alpha u^\beta + X^{(\alpha\beta)} + 2m^{(\alpha\lambda\beta)}/_\lambda + 2v^{(\alpha}u^{\beta)}, \tag{2}$$

where

$$v^\alpha = 2m^{(\alpha\varrho\sigma)}u_{\varrho/\sigma}.$$

This assumption is proved to be physically acceptable in the case considered here (cf. [2]).

3. A theorem concerning the variation of the metric tensor in S_4.

We always consider a body $\mathfrak{C}$ of the type specified above and we suppose assigned in S_4 the motion (1) of $\mathfrak{C}$ and the metric tensor $g_{\alpha\beta} = g_{\alpha\beta}(x)$ (4).

We take into account a bounded 4-dimensional domain C_4 of S_4, where the motion of $\mathfrak{C}$ is of class $C^{(2)}$ and we call $\mathcal{F}C_4$ its boundary oriented outwards. Furthermore let $\delta g_{\alpha\beta}$ be an arbitrary variation of $g_{\alpha\beta}$, of class $C^{(2)}$ in C_4 and such that

$$\delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta,\gamma} \quad \text{on } \mathcal{F}C_4. \tag{3}$$

Consider the functional

$$I = \int_{C_4} (R + 16\pi h c^{-4}\varrho)\sqrt{-g}\, dC_4. \tag{4}$$

In [2] it is proved that for every variation $\delta g_{\alpha\beta}$ of the aforementioned type we have

$$\delta\int_{C_4} (R + 16\pi h c^{-4}\varrho)\sqrt{-g}\, dC_4 = -\int_{C_4} (A^{\alpha\beta} + 8\pi h c^{-4}\mathfrak{U}^{\alpha\beta})\delta g_{\alpha\beta}\sqrt{-g}\, dC_4. \tag{5}$$

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(3) We use the notations $2T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$; $2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$.

(4) For the constitutive equations of an elastic body $\mathfrak{C}$ capable of couple stresses in the absence of heat conduction an electromagnetic phenomena see [1].

4. A variational principle concerning equilibrium in a stationary frame.

Let now S_4 be stationary and (x) a stationary frame. The metric tensor, that we always consider as assigned in S_4, satisfies

$$g_{\alpha\beta,0} = 0 \,. \tag{6}$$

We call C_3 the intersection of the world tube $W_{\mathcal{C}}$ of $\mathcal{C}$ with the hypersurface $x_0 = 0$.

We identify the arbitrary parameter in the equations (1) of the motion with x^0 and denote by $x^r = \chi^r(y^L)$ the configuration of $\mathcal{C}$ in C_3. We consider the following motion of $\mathcal{C}$:

$$\begin{cases} x^0 = t \\ x^r = x^r(t, y^L) = \chi^r(y^L)\,, \end{cases} \tag{7}$$

hence $\mathcal{C}$ is in equilibrium with respect to (x).

Consider an arbitrary variation $\underset{3}{\delta} g_{\alpha\beta} = \underset{3}{\delta} g_{\alpha\beta}(x^1, x^2, x^3)$ of $g_{\alpha\beta}$ on C_3, of class $C^{(2)}$ and such that

$$\underset{3}{\delta} g_{\alpha\beta} = 0 = \underset{3}{\delta} g_{\alpha\beta,\gamma} \quad \text{on } \mathcal{F}C_3 \qquad (\alpha, \beta = 0, 1, 2, 3)\,. \tag{8}$$

Consider the functional

$$J = \int_{C_3} (R + 16\pi h c^{-4} \varrho)\sqrt{-g}\, dC_3\,. \tag{9}$$

We shall prove that the rest (7) of the body $\mathcal{C}$, with respect to the stationary frame (x), is physically possible if and only if

$$\underset{3}{\delta} J = 0 \tag{10}$$

for every variation of $g_{\alpha\beta}$ of the aformentioned type.

Let a be a real positive number. We consider the following subsets of $W_{\mathcal{C}}$

$$C_4 = \{P \in W_{\mathcal{C}} \,|\, |x^0| \leqslant a + 1\}\,, \qquad C_4^a = \{P \in W_{\mathcal{C}} \,|\, |x^0| \leqslant a\}\,,$$

$$C_4^+ = \{P \in W_{\mathcal{C}} \,|\, a \leqslant x^0 \leqslant a + 1\}\,, \qquad C_4^- = \{P \in W_{\mathcal{C}} \,|\, -(a+1) \leqslant x^0 \leqslant -a\}\,,$$

where by x^ϱ we mean the co-ordinates of the point P of S_4.

Let then $\varphi(\xi)$ be an arbitrary function of the real variable ξ, of class $C^{(2)}$ in $[0, 1]$ and such that

$$\text{(11)} \qquad \begin{cases} \varphi(1) = \varphi'(1) = \varphi'(0) = \varphi''(0) = 0 \\ \varphi(0) = 1 \\ \int_0^1 \varphi(\xi)\, d\xi = 0 \,. \end{cases}$$

Consider the following variation $\delta g_{\alpha\beta}(x_0, x^1, x^2, x^3)$ in C_4

$$\text{(12)} \qquad \delta g_{\alpha\beta} = \begin{cases} \underset{3}{\delta} g_{\alpha\beta}(x^1, x^2, x\mathrm{f}) & \text{in } C_4^a \\ \varphi(x_0 - a)\, \underset{3}{\delta} g_{\alpha\beta} & \text{in } C_4^+ \\ \varphi(-x_0 - a)\, \underset{3}{\delta} g_{\alpha\beta} & \text{in } C_4^- \,. \end{cases}$$

On the basis of (11) this variation is of class $C^{(2)}$ in C_4 and satisfies the conditions

$$\text{(13)} \qquad \delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta,\gamma} \quad \text{on } \mathcal{F} C_4 \,.$$

Hence the variational theorem enounciated in the previous paragraph can be applied:

$$\text{(14)} \quad \delta \int_{C_4} (R + 16\pi h c^{-4} \varrho) \sqrt{-g}\, dC_4 = -\int_{C_4} (A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta}) \delta g_{\alpha\beta} \sqrt{-g}\, dC_4 \,.$$

The stationarity of spacetime and the equations (7) of equilibrium imply that $g_{\alpha\beta}$, R, $\mathfrak{U}_{\alpha\beta,\varrho}$ do not depend on x^0.

We have

$$\delta \int_{C_4^+} R \sqrt{-g}\, dC_4 = \int_{C_4^+} \frac{\partial R \sqrt{-g}}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta}\, dC_4 = \int_{C_4^+} \frac{\partial R \sqrt{-g}}{\partial g_{\alpha\beta}} \varphi(x^0 - a)\, \underset{3}{\delta} g_{\alpha\beta}\, dC_4 =$$

$$= \int_{C_3} \frac{\partial R \sqrt{-g}}{\partial g_{\alpha\beta}} \underset{3}{\delta} g_{\alpha\beta}\, dC_3 \int_a^{a+1} \varphi(x^0 - a)\, dx^0 = 0 \,,$$

and analogously

$$\delta\int_{C_4^+}\varrho\sqrt{-g}\,dC_4 = 0\,.$$

Furthermore in the same way we prove that

$$\delta\int_{C_4^-}R\sqrt{-g}\,dC_4 = 0 = \delta\int_{C_4^-}\varrho\sqrt{-g}\,dC_4\,.$$

Hence

$$\delta\int_{C_4^+}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_4 = 0 = \delta\int_{C_4^-}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_4\,. \tag{15}$$

Furthermore

$$\int_{C_4^a}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_4 = \tag{16}$$

$$= \int_{-a}^{a}dx^0\int_{C_3}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_3 = 2a\int_{C_3}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_3\,.$$

From (15) and (16) we have

$$\delta\int_{C_4}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_4 = 2a\,\underset{3}{\delta}\int_{C_3}(R + 16\pi hc^{-4}\varrho)\sqrt{-g}\,dC_3 \tag{17}$$

for the variation $\delta g_{\alpha\beta}$ and $\underset{3}{\delta} g_{\alpha\beta}$ above.

We also have

$$\int_{C_4^+}(A^{\alpha\beta} + 8\pi hc^{-4}\mathfrak{U}^{\alpha\beta})\,\delta g_{\alpha\beta}\sqrt{-g}\,dC_4 = \tag{18}$$

$$= \int_{C_4^+}(A^{\alpha\beta} + 8\pi hc^{-4}\mathfrak{U}^{\alpha\beta})\varphi(x^0 - a)\,\underset{3}{\delta} g_{\alpha\beta}\sqrt{-g}\,dC_4 =$$

$$= \int_{C_3}(A^{\alpha\beta} + 8\pi hc^{-4}\mathfrak{U}^{\alpha\beta})\,\underset{3}{\delta} g_{\alpha\beta}\sqrt{-g}\,dC_3\int_{a}^{a+1}\varphi(x^0 - a)\,dx^0 = 0\,.$$

and analogously

$$\int_{C_4^-}(A^{\alpha\beta} + 8\pi hc^{-4}\mathfrak{U}^{\alpha\beta})\,\delta g_{\alpha\beta}\sqrt{-g}\,\mathrm{d}C_4 = 0\,. \tag{19}$$

Furthermore

$$(20) \qquad \int_{C_4^a} (A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta})\, \delta g_{\alpha\beta} \sqrt{-g}\, dC_4 = $$
$$= \int_{C_3} (A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta})\, \underset{3}{\delta} g_{\alpha\beta} \sqrt{-g}\, dC_3 \int_{-a}^{a} dx^0 =$$
$$= 2a \int_{C_3} (A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta})\, \underset{3}{\delta} g_{\alpha\beta} \sqrt{-g}\, dC_3 \,.$$

From (14), (17), (18), (19), (20) we deduce

$$(21) \quad \underset{3}{\delta} \int_{C_3} (R + 16\pi h c^{-4} \varrho) \sqrt{-g}\, dC_3 = -\int_{C_3} (A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta})\, \underset{3}{\delta} g_{\alpha\beta} \sqrt{-g}\, dC_3$$

for the variations $\underset{3}{\delta} g_{\alpha\beta}$ specified above.

From (21) it follows that the variational condition $\underset{3}{\delta} J = 0$ is equivalent to the validity, in C_3, of the gravitational equations for $\mathcal{C}$: $A^{\alpha\beta} + 8\pi h c^{-4} \mathfrak{U}^{\alpha\beta} = 0$ $(\alpha, \beta = 0, 1, 2, 3)$. If we remember that ϱ, R, $A^{\alpha\beta}$, $\mathfrak{U}^{\alpha\beta}$ do not depend on x^0, we can conclude that the variational condition $\underset{3}{\delta} J = 0$ is equivalent to the validity of the gravitational equations for $\mathcal{C}$ in the whole world tube $W_{\mathcal{C}}$. This proves the theorem.

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