Rendiconti
del
Seminario Matematico
della
Università di Padova

Richard D. Carmichael

Distributional boundary values and the tempered ultra-distributions

Rendiconti del Seminario Matematico della Università di Padova, tome 56 (1976), p. 101-123

<http://www.numdam.org/item?id=RSMUP_1976__56__101_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 56 (1977)

Distributional Boundary Values and the Tempered Ultra-Distributions.

RICHARD D. CARMICHAEL (*)

1. Introduction.

Sebastião e Silva [7, 8] and Hasumi [3] have introduced and studied the tempered ultra-distributions in 1 and n dimensions. These objects are equivalence classes of analytic functions defined by a certain space of functions which are analytic in the 2^n octants in $\boldsymbol{C}^n$. These authors have shown that the tempered ultra-distributions are algebraically isomorphic to the distribution space $\mathcal{K}'$, which is the Fourier transform space of the distributions of exponential growth Λ_∞.

In [2] we have considered other problems concerning $\mathcal{K}'$ and Λ_∞: We have given necessary and sufficient conditions for functions analytic in tube domains corresponding to open convex cones to have $\mathcal{K}'$ boundary values in the distributional sense. In certain cases we have shown that the analytic functions which have $\mathcal{K}'$ boundary values can be recovered from the boundary value as the Fourier-Laplace transform of the inverse Fourier transform of the boundary value.

Yoshinaga [10] and Zieleźny [11] have also considered problems concerning the distributions Λ_∞ and $\mathcal{K}'$. Their results mainly concern the problems of convolution and multipliers in Λ_∞ and $\mathcal{K}'$.

A comparison of the results concerning tempered ultra-distributions in [7, 8] and [3] and the distributional boundary value results in [2] leads one to consider if there is a connection between the distributional boundary value process in $\mathcal{K}'$ and the tempered ultra-distri-

(*) Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina 27109, U.S.A.

butions. More explicitly we desire to consider in this paper if there is an isomorphic relationship between a quotient space of certain analytic functions in octants which we have studied previously and the set of all admissable distributional boundary values in $\mathcal{K}'$ with the isomorphism being defined by the boundary value process; and we desire to show the connection between this quotient space and the tempered ultra-distributions. We also desire in this paper to continue our investigation of distributional boundary values in $\mathcal{K}'$ for functions analytic in tubular cones, and some of the results which we obtain will have a relation to the tempered ultra-distributions.

In section 3 of this paper we obtain some new results concerning $\mathcal{K}'$ boundary values of functions analytic in tube domains. We then generalize our results here and in [2] to functions analytic in tubular cones corresponding to open disconnected cones and obtain related results. For these generalizations corresponding to certain open disconnected cones, the information we obtain will have an important connection with the tempered ultra-distributions. In section 4 we restrict our attention to the tubular cone defined by the union of the 2^n quadrants in $\boldsymbol{R}^n$ and show that a quotient space of functions analytic in this tubular cone is algebraically isomorphic to the elements of $\mathcal{K}'$ which are distributional boundary values. The isomorphism here is a mapping which is constructed from the $\mathcal{K}'$ boundary value mapping. We then relate this present quotient space with the tempered ultra-distributions. We show that there is a $1-1$ correspondence between the present quotient space and a subspace of the tempered ultra-distributions and that corresponding elements of these two quotient spaces generate exactly the same element of $\mathcal{K}'$ through the respective isomorphisms.

2. Notation and definitions.

All n-dimensional notation will be exactly as in Carmichael [2, p. 767]. Definitions and properties of the function spaces H and $\mathcal{K}$ and the distribution spaces Λ_∞ and $\mathcal{K}'$ are in [2, pp. 768-769]. The Fourier and inverse Fourier transforms of functions and of Λ_∞ and $\mathcal{K}'$ are defined in [2, pp. 767-769]. Our notation for these transforms for functions is $\mathcal{F}[\varphi(t); x]$ and $\mathcal{F}^{-1}[\varphi(t); x]$, respectively, [2, p. 767]; while $\mathcal{F}[V]$ and $\mathcal{F}^{-1}[U]$ will denote the Fourier transform of $V \in \Lambda_\infty$ and the inverse Fourier transform of $U \in \mathcal{K}'$, respectively, [2, pp. 768-769].

We refer to [1, p. 845] and [2, p. 767] for definitions and terminology concerning cones. We ask the reader to note the definitions of cone, compact subcone, and indicatrix of the cone C, $u_C(t)$. We assume that compact subcones of open cones are also open throughout this paper. $C^* = \{t: u_C(t) \leqslant 0\} = \{t: \langle t, y\rangle \geqslant 0, y \in C\}$ is the dual cone of the cone C. T^C will denote the set $T^C = \boldsymbol{R}^n + iC \subset \boldsymbol{C}^n$, where C is a cone. If C is open and convex T^C will be called a tube domain, while if C is only open, T^C will be called a tubular cone. $O(C)$ will denote the convex envelope of the cone C. The number

$$\varrho_C = \sup_{t \in C_*} \frac{u_{O(C)}(t)}{u_C(t)}, \quad C_* = \boldsymbol{R}^n \setminus C^*, \quad C^* = \{t: \langle y, t\rangle \geqslant 0,\ y \varepsilon C\},$$

characterizes the nonconvexity of the cone C. If C is an open disconnected cone of the form $C = \bigcup_{j=1}^{r} C_j$, $r < \infty$, where the C_j are open convex cones, we call C' a compact subcone of C if and only if C' is of the form $C' = \bigcup_{j=1}^{r} C'_j$, where the C'_j are compact subcones of C_j, $j = 1, \ldots, r$.

All distributional boundary value terminology and topological vector space terminology are exactly as discussed in [2, p. 767].

All notation with respect to the open cone $C = \bigcup_\sigma C_\sigma$, where the C_σ are the 2^n open convex cones that are the quadrants in $\boldsymbol{R}^n$, will be exactly like that of Hasumi [3, p. 94]. Thus σ denotes vectors $(\sigma_1, \ldots, \sigma_n)$ whose components are 0 or 1, and $(-1)^{|\sigma|} = (-1)^{\sigma_1 + \ldots + \sigma_n}$. The quadrant $C_\sigma = \{y \in \boldsymbol{R}^n: (-1)^{\sigma_j} y_j > 0,\ j = 1, \ldots, n\}$ for each of the 2^n vectors σ. Further, for section 4 of this paper we adopt the notation that $y_\sigma \in C_\sigma$ is the point $y_\sigma = \big((-1)^{\sigma_1}, \ldots, (-1)^{\sigma_n}\big)$ for each σ.

We now shall define the main space of analytic functions with which this paper is concerned. Let C be an open cone in $\boldsymbol{R}^n$, and let C' be an arbitrary compact subcone of C. Let $N(0, m)$ denote an open ball of the origin in $\boldsymbol{R}^n$ of radius $m > 0$. Denote $T(C', m) = \boldsymbol{R}^n + i\big(C' \setminus (C' \cap N(0, m))\big)$. We consider functions $f(z)$ which satisfy

$$(1) \qquad |f(z)| \leqslant K_m(C')(1 + |z|)^N, \quad z \in T(C', m),$$

where $m > 0$ is arbitrary, $K_m(C')$ is a constant depending on C' and m, and N is a nonnegative real number. We denote by $\mathcal{A}^*_{\omega, C}$ the set of all functions $f(z)$ which are analytic in $T^{C'} = \boldsymbol{R}^n + iC'$ and which sastify (1) where C' is an arbitrary compact subcone of C and $m > 0$ is arbitrary. We have used the notation $\mathcal{A}^*_{\omega,C}$ for our functions because for the case that $C = \bigcup_\sigma C_\sigma$, where the C_σ are the 2^n quadrants in $\boldsymbol{R}^n$, we shall see that $\mathcal{A}^*_{\omega,C}$ becomes a subspace of Sebastião e Silva's and Hasumi's space $\mathcal{A}_\omega$ [3, p. 100].

We shall also consider in this paper a space of functions which directly corresponds to the space $\mathcal{A}_\omega$ for tubular cones. Let C be an open cone and C' be an arbitrary compact subcone of C. We define the space $\mathcal{A}_{\omega,C}$ as the set of all functions $f(z)$ which are analytic in $T(C', m) = \boldsymbol{R}^n + i\big(C' \setminus (C' \cap N(0, m))\big)$, where $N(0, m)$ is now taken to be closed ball, and which satisfy (1) in $T(C', m)$ where $m > 0$ is a fixed real number which depends on $f(z)$ and C'. Thus $f(z) \in \mathcal{A}_{\omega,C}$ if for any compact subcone $C' \subset C$ there exists $m > 0$ depending on $f(z)$ and C' such that $f(z)$ is analytic in $T(C', m)$ and satisfies (1) there, where $N(0, m)$ is now taken to be the closed ball in $T(C', m)$. Again the notation $\mathcal{A}_{\omega,C}$ is chosen because of the direct relation to $\mathcal{A}_\omega$ for C being the union of the 2^n quadrants in $\boldsymbol{R}^n$. We have $\mathcal{A}^*_{\omega,C} \subset \mathcal{A}_{\omega,C}$ for any open cone C.

3. Functions analytic in tubular cones.

In this section we shall obtain information concerning the space of functions $\mathcal{A}^*_{\omega,C}$. We begin by considering this space for C being an open convex cone. The following result is like [2, Theorem 3] for the present more general space $\mathcal{A}^*_{\omega,C}$, and we shall use directly parts of the proof of [2, Theorem 3] where applicable.

THEOREM 1. *Let C be an open convex cone and let C' be an arbitrary compact subcone of C. Let $f(z) \in \mathcal{A}^*_{\omega,C}$. Then there exists a unique element $V \in \Lambda_\infty$ with $supp(V) \subseteq C^*$ such that*

$$f(z) = \langle V, \exp(2\pi i \langle z, t\rangle)\rangle\,, \qquad z \in T^{C'}\,, \tag{2}$$

$$f(z) = \mathcal{F}[\exp(-2\pi\langle y, t\rangle)\, V_t]\,, \qquad z \in T^{C'}\,, \tag{3}$$

where the equality (3) *is in* $\mathcal{K}'$,

(4) $\{f(z)\colon y = \mathrm{Im}(z) \in C' \subset C,\ |y| \leqslant Q\}$ *is a strongly bounded set in* $\mathcal{K}'$,

where $Q > 0$ is arbitrary but fixed,

(5) $f(z) \to \mathcal{F}[V] \in \mathcal{K}'$ *in the strong (and weak) topology of* $\mathcal{K}'$ *as* $y = \operatorname{Im}(z) \to 0, \; y \in C' \subset C$.

PROOF. Let $f(z) \in \mathcal{A}^{*}_{\omega,C}$. Because of (1), we may choose on n-tuple K of nonnegative integers such that

$$|z^{-K} f(z)| \leqslant K_m(C')(1 + |z|)^{-n-\varepsilon}, \quad z \in T(C', m), \tag{6}$$

where n is the dimension, $\varepsilon > 0$ is arbitrary, and $m > 0$ is arbitrary. Put

$$g(t) = \int_{R^n} z^{-K} f(z) \exp(-2\pi i \langle z, t\rangle)\, dx, \quad z = x + iy \in T(C', m). \tag{7}$$

By exactly the same methods of proof as in [1, Theorem 1] we have that $g(t)$ is continuous in $t \in \boldsymbol{R}^n$, is independent of $y = \operatorname{Im}(z)$, and has support in the dual cone $C^* = \{t: \langle y, t\rangle \geqslant 0, y \in C\}$. Also we have immediately from (6) that $g(t)$ defined in (7) satisfies

$$|g(t)| \leqslant M_m(C') \exp(2\pi\langle y, t\rangle), \tag{8}$$

and this inequality holds for all $y \in \big)C' \setminus (C' \cap N(0, m))\big)$ since $g(t)$ is independent of y. Further, it follows from (6) that $(z^{-K} f(z)) \in L^1 \cap L^2$ as a function of $x = \operatorname{Re}(z)$ for $y = \operatorname{Im}(z)$ fixed in $\big(C' \setminus (C' \cap N(0, m))\big)$. Thus from (7) and the Plancherel theorem we obtain

$$z^{-K} f(z) = \mathcal{F}[\exp(-2\pi\langle y, t\rangle) g(t); x], \quad z = x + iy \in T(C', m), \tag{9}$$

with this Fourier transform being in the L^2 sense.

We now define the differential operator Δ exactly as in [1, p. 847] and put $V = \Delta g(t)$ with the differentiation being in the distributional sense. Because of (8) and [3, Proposition 3] or [11, Theorem 1] we have $V \in \Lambda_\infty$; and $\operatorname{supp}(V) \subseteq C^*$ since $g(t)$ has support there. Because of our construction of $V = \Delta g(t)$ here and because of (8) we may proceed exactly as in the proof of [2, Theorem 3, equation (34)] to show that $(\exp(-2\pi\langle y, t\rangle) g(t)) \in L^p$, $1 \leqslant p < \infty$, for $y \in \big(C' \setminus (C' \cap N(0, m))\big)$ and to obtain

$$(10)\qquad \langle V, \exp(2\pi i\langle z, t\rangle)\rangle = z^{-K}\mathcal{F}[\exp(-2\pi\langle y, t\rangle)g(t); x]\,,$$
$$z = x + iy \in T(C', m)\,,$$

where K was chosen as in (6). (The details of these two facts are exactly the same as in obtaining [2, equations (39) and (40), p. 779].) The Fourier transforms in (9) and (10) can now be interpreted in both the L^1 and L^2 sense because as noted above $(\exp(-2\pi\langle y, t\rangle)g(t)) \in L^p$, $1 \leqslant p < \infty$. Thus combining (9) and (10) we have

$$f(z) = \langle V, \exp(2\pi i\langle z, t\rangle)\rangle\,, \qquad z \in T(C', m)\,.$$

Since $m > 0$ is arbitrary, this equality holds for each $z \in T^{C'}$ and (2) is obtained.

We now prove (3). Let $\psi \in \mathcal{H}$. There exists an element $\check{\varphi}(t) = \varphi(-t) \in H$ such that $\psi(x) = \mathcal{F}[\check{\varphi}(t); x]$ and $\check{\varphi}(t) = \mathcal{F}^{-1}[\psi(x); (-t)]$ where these transforms can be interpreted in both the L^1 and L^2 sense [2, p. 768]. Let y be an arbitrary but fixed point in $C' \subset C$. We have

$$(11)\quad \exp(-2\pi\langle y, t\rangle)\check{\varphi}(t) = \int_{R^n}\psi(x)\exp(2\pi i\langle z, t\rangle)\,dx\,, \qquad z = x + iy \in T^{C'}\,.$$

Recall that $V = \Delta g(t)$ where $\operatorname{supp}(g) \subseteq C^*$. Using (2), a change of order of integration, (19), and the definition of the Fourier transform mapping Λ_∞ onto $\mathcal{H}'$ [2, p. 768], we obtain for $z \in T^{C'}$

$$\begin{aligned}(12)\qquad \langle f(z), \psi(z)\rangle &= \langle\langle \Delta g(t), \exp[2\pi i\langle z, t\rangle]\rangle, \psi(x)\rangle = \\ &= (-1)^{|K|}\left(\frac{-1}{2\pi i}\right)^{|K|}(2\pi i)^{|K|}\int_{R^n}\psi(x)z^K\int_{C^*}g(t)\exp[2\pi i\langle z, t\rangle]\,dt\,dx \\ &= (-1)^{|K|}\langle g(t), \Delta\int_{R^n}\psi(x)\exp[2\pi i\langle z, t\rangle]\,dx\rangle \\ &= \langle V, \exp[-2\pi\langle y, t\rangle]\check{\varphi}(t)\rangle \\ &= \langle \mathcal{F}\,[\exp[-2\pi\langle y, t\rangle]V_t], \check{\varphi}(x)\rangle\,.\end{aligned}$$

Thus (12) proves (3). Using (3) we now proceed as in the proofs of [2, Theorem 3, equations (35) and (36)] and obtain (4) and (5) of this paper; the proofs are the same and will not be repeated. The proof of Theorem 1 is complete.

We now desire to extend Theorem 1 by considering functions in $\mathcal{A}^*_{\omega,C}$, where C is an open cone which is not necessarily connected but is the union of open convex cones. Let $C = \bigcup_{j=1}^{r} C_j$, $r < \infty$, where the C_j are the open convex cones. Let $f(z) \in \mathcal{A}^*_{\omega,C}$ for this open cone C. Then for z restricted to $T^{C_j} = \boldsymbol{R}^n + iC_j$, $j = 1, \dots, r$, we have $f(z) \in \mathcal{A}^*_{\omega,C_j}$. Applying Theorem 1 we thus obtain unique elements $V_j \in \Lambda_\infty$ with $\operatorname{supp}(V_j) \subseteq C_j^*$ and unique elements $U_j = \mathcal{F}[V_j] \in \mathcal{K}'$, $j = 1, \dots, r$ such that (2)-(5) hold for $z \in T^{C'_j} = \boldsymbol{R}^n + iC'_j$, $C'_j \subset C_j$, $j = 1, \dots, r$. With an additional assumption on the boundary values U_j, $j = 1, \dots, r$, of $f(z)$ from each connected component of T^C, which exist as above because of Theorem 1, we are able to obtain considerably more information concerning elements of $\mathcal{A}^*_{\omega,C}$ for $C = \bigcup_{j=1}^{r} C_j$ as seen in the following result.

THEOREM 2. *Let $C = \bigcup_{j=1}^{r} C_j$, $r < \infty$, where the C_j are open convex cones, and let C' be an arbitrary compact subcone of C. Let $f(z) \in \mathcal{A}^*_{\omega,C}$. Let the $\mathcal{K}'$ boundary values of $f(z)$, $U_j \in \mathcal{K}'$, which exist from each connected component T^{C_j}, $j = 1, \dots, r$, of T^C, be equal. Then there exists a unique element $V \in \Lambda_\infty$ with $\operatorname{supp}(V) \subseteq \{t: \langle y, t\rangle \geqslant 0,\ y \in 0(C)\}$ such that*

$$f(z) = \langle V, \exp(2\pi i\langle z, t\rangle)\rangle, \qquad z \in T^{C'}, \tag{13}$$

$$f(z) = \mathcal{F}[\exp(-2\pi\langle y, t\rangle) V_t], \qquad z \in T^{C'}, \tag{14}$$

where the equality (14) *is in* $\mathcal{K}'$,

(15) *$\{f(z): y = \operatorname{Im}(z) \in C' \subset C,\ |y| \leqslant Q\}$ is a strongly bounded set in $\mathcal{K}'$, where $Q > 0$ is arbitrary but fixed,*

(16) *$f(z) \to \mathcal{F}[V] \in \mathcal{K}'$ in the strong (and weak) topology of $\mathcal{K}'$ as $y = \operatorname{Im}(z) \to 0$, $y \in C'_j \subset C_j$, $j = 1, \dots, r$;*

and there exists a function $F(z) \in \mathcal{A}_{\omega,0(C)}$ which is the analytic extension of $f(z)$ to $T^{0(C)} = \boldsymbol{R}^n + i0(C)$, where $0(C)$ denotes the convex envelope of C.

PROOF. The $\mathcal{H}'$ boundary values $U_j = \mathcal{F}[V_j]$, $j = 1, \dots, r$, of $f(z)$ exist from each connected component of T^C as noted above, where the $V_j \in \Lambda_\infty$ with $\operatorname{supp}(V_j) \subseteq C_j^*$, $j = 1, \dots, r$. By hypothesis $U_1 = \dots = U_r$ in $\mathcal{H}'$, and we call this common value U. But $U_j = \mathcal{F}[V_j]$ implies $V_j = \mathcal{F}^{-1}[U_j]$, $j = 1, \dots, r$, [2, p. 769]. Since the inverse Fourier transform is a $1-1$ mapping of $\mathcal{H}'$ onto Λ_∞ [2, p. 769], we obtain $V_1 = \dots = V_r$ in Λ_∞. We call this common value V and thus have that $U = \mathcal{F}[V]$. Since $\operatorname{supp}(V_j) \subseteq C_j^* = \{t: \langle y, t\rangle \geq 0, y \in C_j\} = \{t: u_{C_j}(t) \leq 0\}$, $j = 1, \dots, r$, then V vanishes on $\bigcup_{j=1}^{r} \{t: u_{C_j}(t) > 0\}$. Now

$$u_C(t) = \max_{j=1,\dots,r} u_{C_j}(t); \tag{17}$$

and from the definition of ϱ_C (section 2), we have $u_{0(C)}(t) \leq \big(\varrho_C u_C(t)\big)$. Thus by (17) we have

$$u_{0(C)}(t) \leq (\varrho_C) \max_{j=1,\dots,r} u_{C_j}(t); \tag{18}$$

and by a lemma of Vladimirov [9, Lemma 3, p. 220], $1 \leq \varrho_C < +\infty$. Consider now the set $\{t: u_{0(C)}(t) > 0\}$. If $t \in \{t: u_{0(C)}(t) > 0\}$, then then by (18), $t \in \{t: \max_{j=1,\dots,r} u_{C_j}(t) > 0\}$. Thus $t \in \bigcup_{j=1}^{r} \{t: u_{C_j}(t) > 0\}$, and on this set V vanishes. Thus V vanishes if $t \in \{t: u_{0(C)}(t) > 0\}$ which implies that $\operatorname{supp}(V) \subseteq \{t: u_{0(C)}(t) \leq 0\} = \{t: \langle y, t\rangle \geq 0, y \in 0(C)\}$.

Now let C' be an arbitrary compact subcone of $C = \bigcup_{j=1}^{r} C_j$. Then $C' = \bigcup_{j=1}^{r} C_j'$ where each C_j' is a compact subcone of C_j, $j = 1, \dots, r$, (recall section 2). Applying Theorem 1 to $f(z)$, $z \in T^{C_j} = \boldsymbol{R}^n + iC_j$, $j = 1, \dots, r$, we obtain

$$f(z) = \langle V_j, \exp(2\pi i \langle z, t\rangle)\rangle, \quad z \in T^{C_j'}, \qquad C_j' \subset C_j, \quad j = 1, \dots, r,$$

$$f(z) = \mathcal{F}[\exp(-2\pi\langle y, t\rangle) V_j], \quad z = x + iy \in T^{C_j'},$$

$$C_j' \subset C_j, \quad j = 1, \dots, r,$$

with this last equality holding in $\mathcal{H}'$. Also by Theorem 1 $\{f(z): y = \operatorname{Im}(z) \in C_j' \subset C_j, |y| \leq Q\}$ is a strongly bounded set in $\mathcal{H}'$ where $Q > 0$ is arbitrary but fixed; and as we have already noted above, $f(z) \to \mathcal{F}[V_j]$ in the strong (and weak) topology of $\mathcal{H}'$ as $y = \operatorname{Im}(z) \to 0$, $y \in C_j' \subset C_j$, $j = 1, \dots, r$. Using these facts and recalling $V = V_1 =$

$= \ldots = V_r$, $U = U_1 = \ldots = U_r$, $U = \mathcal{F}[V]$, and $C' = \bigcup_{j=1}^{r} C'_j$, we have (13), (14), (15), and (16) as desired.

We now put

$$(19) \qquad F(z) = \langle V, \exp(2\pi i\langle z, t\rangle)\rangle , \qquad z \in T^{0(C)} = \boldsymbol{R}^n + i0(C) ,$$

and ask the reader to recall the definition of $\mathcal{A}_{\omega,C}$ for any open cone C contained in section 2. Since $V \in \Lambda_\infty$, $\operatorname{supp}(V) \subseteq \{t\colon u_{0(C)}(t) \leqslant 0\} = \{t\colon \langle y, t\rangle \geqslant 0, y \in 0(C)\}$, and $0(C)$ is an open convex cone, we have by exactly the proof of [2, Theorem 2] that $F(z) \in \mathcal{A}_{\omega,0(C)}$. ,Further we have by (13) that $F(z)$ exists and equals $f(z)$ for $z \in T^{C'}$ where C' is an arbitrary compact subcone of $C \subset 0(C)$. Thus $F(z)$ is the desired analytic extension of $f(z)$ to $T^{0(C)}$, and the proof is complete.

It is interesting to note in Theorem 2 that since $F(z) = f(z)$ in $T^{C'}$, $C' \subset C \subset 0(C)$, then $F(z)$ exists and is analytic in $T^{C'}$. However, $F(z)$ may not exist for all $z \in T^{C'}$, where C' is an arbitrary compact subcone of $0(C)$ because of the properties of being in $\mathcal{A}_{\omega,0(C)}$. Thus $F(z)$ in Theorem 2 actually has stronger properties than is indicated by saying $F(z) \in \mathcal{A}_{\omega,0(C)}$.

As we shall see in section 4, Theorem 2 has a connection with the tempered ultra-distributions. In order to begin seeing this connection we shall consider some special cases of Theorem 2. Our purpose is to show that for these special cases, even more information concerning $f(z) \in \mathcal{A}^*_{\omega,C}$, $C = \bigcup_{j=1}^{r} C_j$, can be obtained than in Theorem 2. The information obtained will then be used to motivate our work in section 4 where we connect the boundary value mapping in $\mathcal{K}'$ with the tempered ultra-distributions.

We consider the open cone $C = \bigcup_{\sigma} C_\sigma$, where the C_σ are the $r = 2^n$ quadrants in $\boldsymbol{R}^n$. (Recall the definition of the C_σ in section 2 and note that they are open convex cones.) We note that for this open cone C, the concept of compact subcone in the definition of $\mathcal{A}^*_{\omega,C}$ is not needed. Thus in the following result we make the convention that for $C = \bigcup_{\sigma} C_\sigma$, $\mathcal{A}^*_{\omega,C}$ is the set of all functions $f(z)$ that are analytic in $T^C = \boldsymbol{R}^n + iC$, $C = \bigcup_{\sigma} C_\sigma$, and which satisfy

$$(20) \qquad |f(z)| \leqslant K_m(1 + |z|)^N ,$$

$$z \in T(C, m) = \{z \in \boldsymbol{C}^n\colon |\operatorname{Im}(z_j)| \geqslant m > 0, j = 1, \ldots, n\} ,$$

where $m > 0$ is arbitrary. The conclusions in Theorem 2 corresponding to this open cone $C = \bigcup_\sigma C_\sigma$ then hold for $z \in T^C$.

COROLLARY 1. *Let* $C = \bigcup_\sigma C_\sigma$ *and let* $f(z) \in \mathcal{A}^*_{\omega,C}$. *Let the* $r = 2^n$ $\mathcal{H}'$ *boundary values of* $f(z)$, $U_\sigma \in \mathcal{H}'$, *which exists from each of the* $r = 2^n$ *octants* $T^{C_\sigma} = \mathbf{R}^n + iC_\sigma$, *be equal. Then* $f(z)$ *is a polynomial in* $z \in \mathbf{C}^n$ *and hence an entire analytic function in* $\mathbf{C}^n$. *Further, the conclusions* (13)-(16) *of* Theorem 2 *hold with respect to* $z \in T^C$, $C = \bigcup_\sigma C_\sigma$, *where* V *is a finite linear combination of distributional derivatives of the Dirac delta distribution.*

PROOF. If $C = \bigcup_\sigma C_\sigma$, then $0(C) = \mathbf{R}^n$, $T^{0(C)} = \mathbf{C}^n$, and $\{t: u_{0(C)}(t) \leqslant 0\} = \{\bar{0}\}$, where $\bar{0}$ denotes the origin in $\mathbf{R}^n$. Applying Theorem 2 we obtain an element $V \in \Lambda_\infty$ having support at the origin such that (13)-(16) hold for the present $f(z)$, $z \in T^C$. Since $\Lambda_\infty \subset \mathcal{D}'$, a direct application of Schwartz [5, Théorème XXXV, p. 100] yields that V must be a finite combination of distributional derivatives of the Dirac delta distribution. Thus $F(z)$ defined in (19) in Theorem 2 is immediately obtained to be a polynomial in $z \in \mathbf{C}^n$. Since $F(z)$ is the analytic extension of $f(z)$, then $f(z)$ is a polynomial. The proof is complete.

Of course an important special case of Corollary 1 is when $n = 1$ in which case $C = ((-\infty, 0) \cup (0, \infty))$, the open cone which defines the union of the upper and lower half planes in $\mathbf{C}^1$. Especially in this 1-dimensional setting, Corollary 1 has a direct connection with the tempered ultra-distributions as we shall see in section 4.

The proof of Corollary 1 indicates a way to strengthen the conclusions in Theorem 2 when the open cone $C = \bigcup_{j=1}^r C_j$, $r < \infty$, satisfies a certain property that is possessed by $C = \bigcup_\sigma C_\sigma$.

COROLLARY 2. *Let* $C = \bigcup_{j=1}^r C_j$, $r < \infty$, *where the* C_j *are open convex cones and such that* $\{t: u_{0(C)}(t) \leqslant 0\}$ *is a bounded set in* $\mathbf{R}^n$. *Let* C' *be an arbitrary compact subcone of* C, *and let* $f(z) \in \mathcal{A}^*_{\omega,C}$. *Let the* $\mathcal{H}'$ *boundary values of* $f(z)$, $U_j \in \mathcal{H}'$, *which exist from each connected component* T^{C_j}, $j = 1, \ldots, r$, *of* $T^C = \mathbf{R}^n + iC$, *be equal. Then there exists a unique element* $V \in \mathcal{E}'$ *with compact support in* $\{t: u_{0(C)}(t) \leqslant 0\}$ *such that* (13)-(16) *hold, and there exists an entire analytic function* $F(z)$ *which* *is the analytic extension of $f(z)$ to $\boldsymbol{C}^n$ and which satisfies*

$$|F(z)| \leqslant M(1+|z|)^N \exp(2\pi b|y|)\,, \qquad z = x+iy \in \boldsymbol{C}^n\,, \tag{21}$$

for some constant M and nonnegative real numbers N and b.

Proof. From Theorem 2 we obtain $V \in \Lambda_\infty$ with $\text{supp}(V) \subseteq \{t\colon u_{0(C)}(t) \leqslant 0\}$. By hypothesis $\{t\colon u_{0(C)}(t) \leqslant 0\}$ is a bounded set; hence $\text{supp}(V)$ is compact. This fact and the fact that $V \in \Lambda_\infty \subset \mathcal{D}'$ imply $V \in \mathcal{E}'$. Since $\text{supp}(V)$ is compact, we can choose $b \geqslant 0$ such that $\text{supp}(V) \subseteq \{t\colon |t| \leqslant b\}$. It follows by the Paley-Wiener-Schwartz theorem [4. p. 21] that $F(z)$ defined in (19) is an entire analytic function which satisfies (21). All other conclusions follow immediately from Theorem 2, and the proof is complete.

In addition to the open cones $C = ((-\infty, 0) \cup (0, \infty))$ in 1-dimension and $C = \bigcup_\sigma C_\sigma$ in n-dimensions, there is another important open cone to which Corollary 2 is applicable. Put

$$\Gamma^+ = \left\{y \in \boldsymbol{R}^n\colon y_1 > \left(\sum_{j=2}^n (y_j)^2\right)^{(\frac{1}{2})}\right\},$$

$$\Gamma^- = \left\{y \in \boldsymbol{R}^n\colon y_1 < \left(\sum_{j=2}^n (y_j)^2\right)^{(\frac{1}{2})}\right\}.$$

$\Gamma^+(\Gamma^-)$ is the future (past) light cone, and both are open convex cones in $\boldsymbol{R}^n$. Putting $C = \Gamma^+ \cup \Gamma^-$, we have $0(C) = \boldsymbol{R}^n$, $T^{0(C)} = \boldsymbol{C}^n$, and $(0(C))^* = \{t\colon u_{0(C)}(t) \leqslant 0\} = \{\bar{0}\}$. Thus Corollary 2 is applicable for C being $C = \Gamma^+ \cap \Gamma^-$, and more explicitly the conclusions as in Corollary 1 follow for $f(z) \in \mathcal{A}^*_{\omega,C}$, $C = \Gamma^+ \cup \Gamma^-$. (We note that as in the case for $C = \bigcup_\sigma C_\sigma$, the concept of compact subcone is not needed in the definition of $\mathcal{A}^*_{\omega,C}$ for $C = \Gamma^+ \cup \Gamma^-$.) The cones Γ^+ and Γ^- are of importance in quantum field theory where the notion of distributional boundary values plays an important role.

The results which we have obtained in this section have all been concerned with obtaining information concerning a given element $f(z) \in \mathcal{A}^*_{\omega,C}$. We desire now to prove a result in the converse directions for this will be useful in section 4. Recall that the constructed element $V \in \Lambda_\infty$ in Theorem 1 was the distributional derivative of a continuous function which had support in the dual cone of the given cone C and which had the growth (8) independent of y. We now prove that all elements $V \in \Lambda_\infty$ of this form define an element of $\mathcal{A}^*_{\omega,C}$ whose boundary value in $\mathcal{K}'$ is the Fourier transform of V. In the following D_t^α denotes the differential operator $(1/2\pi i)^{|\alpha|}(\partial^{|\alpha|}\cdot/\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n})$ with α being an n-tuple of nonnegative integers.

THEOREM 3. *Let C be an open convex cone and let C' be an arbitrary compact subcone of C. Let $V_t = D_t^\alpha(g(t))$, where $g(t)$ is a continuous function on $\mathbf{R}^n$ which has support in C^* and satisfies*

$$(22) \qquad |g(t)| \leqslant M_m(C') \exp(2\pi\langle \Omega, t\rangle)$$

for all $\Omega \in \big(C' \setminus (C' \cap N(0, m))\big)$, where $m > 0$ is arbitrary and $M_m(C')$ is a constant depending on C' and $m > 0$. Then $V \in \Lambda_\infty$ and there exists an element $f(z) \in \mathcal{A}^*_{\omega,C}$ such that (2)-(5) *hold.*

PROOF. Since $g(t)$ is continuous and satisfies (22), then $V \in \Lambda_\infty$ by [3, Proposition 3] or [11, Theorem 1]. Put

$$(23) \quad f(z) = \langle V, \exp(2\pi i\langle z, t\rangle)\rangle = (-1)^{|\alpha|} z^\alpha \int_{C^*} g(t) \exp(2\pi i\langle z, t\rangle)\, dt .$$

(With $V_t = D_t^\alpha(g(t))$ we have used the definition of distributional derivative in (23). This operation is valid by what we prove below.) Now put

$$(24) \qquad p(z) = \int_{C^*} g(t) \exp(2\pi i\langle z,t\rangle)\, dt .$$

Let C' be an arbitrary compact subcone of C. Let z_0 be an arbitrary but fixed point of $T^{C'} = \mathbf{R}^n + iC'$. Now let $R(z_0, r) \subset T^{C'}$ be a neighborhood of z_0 with radius $r > 0$ whose closure is in $T^{C'}$, and then choose $m' > 0$ such that the closure of $R(z_0, r)$ is also contained in $T(C', m')$. The choice of $R(z_0, r)$ and subsequently the $m' > 0$ can obviously be made as indicated since C' is open (recall section 2). Now let $z \in R(z_0, r)$ be arbitrary but fixed and let γ be an arbitrary n-tuple of nonnegative integers. From Vladimirov [9, Lemma p. 223] we obtain the existence of a real number $d > 0$ depending on C' such that

$$(25) \qquad \langle t,y\rangle \geqslant d|t| , \qquad z = x + iy \in R(z_0, r) ,$$

for all $t \in C^* = \{t\colon \langle y, t\rangle \geqslant 0,\ y \in C\}$. Since (22) holds for all $\Omega \in \in \big(C' \setminus (C' \cap N(0, m))\big)$, $m > 0$ arbitrary, we now choose $\Omega = (y/2)$,

$z = x + iy \in R(z_0, r)$. (Since $z = x + iy \in R(z_0, r)$ and the closure of $R(z_0, r)$ is contained in $T(C', m')$, we have $|y| > m'$. Thus $|\Omega| = (|y|/2) > (m'/2)$. Further, since $z = x + iy \in R(z_0, r) \subset T(C', m')$, then $y \in C'$ and hence $\Omega = (y/2) \in C'$ because C' is a cone. Thus letting $m = (m'/2)$ we have $\Omega = (y/2) \in (C' \setminus (C' \cap N(0, m)))$; hence (22) holds by assumption for this choice of Ω and for $m = (m'/2)$.) With the above choice of Ω in (22) and using (25) we have

$$(26) \qquad \left| \int_{C^*} g(t) t^\gamma \exp(2\pi i \langle z, t\rangle)\, dt \right| \leqslant M_m(C') \int_{C^*} |t^\gamma| \exp(-\pi\langle y, t\rangle)\, dt$$

$$\leqslant M_m(C') \int_{C^*} |t^\gamma| \exp(-\pi d|t|)\, dt \leqslant M_m(C') S_n \int_0^\infty u^{|\gamma|+n-1} \exp(-\pi d u) du,$$

where this last inequality is obtained by applying Schwartz [6, Theorem 32, p. 39] and S_n is the area of the unit sphere in $\mathbf{R}^n$. The last integral in (26) is finite and the last bound in (26) is independent of $z = x + iy \in R(z_0, r)$. We thus conclude from (26) that the integral defining $p(z)$ in (24) and any derivative $D_z^\gamma(p(z))$ of it converge uniformly for $z \in R(z_0, r)$. Since z was an arbitrary point of the neighborhood $R(z_0, r)$, this proves $p(z)$ is analytic at z_0. But z_0 was an arbitrary point of $T^{C'}$. Thus $p(z)$ in (24) is analytic in $T^{C'}$. Hence $f(z)$ in (23) is analytic in $T^{C'}$, where C' is an arbitrary compact subcone of C.

We now prove that $f(z)$ satisfies the desired growth condition. Again using Vladimirov [9, Lemma 2, p. 223] we obtain the existence of a real number $d > 0$ depending on $C' \subset C$ such that

$$(27) \qquad \langle t, y\rangle \geqslant d|y||t|, \quad y \in C', \; t \in C^*.$$

Let $m > 0$ be arbitrary and $y = \mathrm{Im}(z) \in (C' \setminus (C' \cap N(0, m)))$. Arguing as in the preceeding paragraph we have $\Omega = (y/2) \in (C' \setminus (C' \cap N(0, (m/2))))$; hence (22) holds for this choice of Ω. Thus by (27) and (22) with this choice of Ω, we argue as in (26) and obtain that $p(z)$ in (24) satisfies

$$(28) \qquad |p(z)| \leqslant M_m(C') \int_{C^*} \exp(-\pi d m |t|)\, dt$$

$$\leqslant M_m(C') S_n \int_0^\infty u^{n-1} \exp(-\pi d m u)\, du,$$

for $z = x + iy \in T(C', m)$, $m > 0$ being arbitrary. In the last inequality of (28) we have again used Schwartz [6, Theorem 32, p. 39] as in (26) where S_n denotes the area of the unit sphere in $\boldsymbol{R}^n$, and this last integral in (28) is finite for each $m > 0$. (28) shows that $p(z)$, $z \in T(C', m)$, is bounded by a constant depending on C' and $m > 0$. Thus from (23) the desired growth (1) for $f(z)$ now follows immediately for $z \in T(C', m)$, $C' \subset C$, $m > 0$ being arbitrary. This completes the proof that $f(z)$ in (23) is in $\mathcal{A}^*_{\omega,C}$.

Now that we have (2) in the conclusion of this theorem (recall (23)), we proceed exactly as in obtaining (3) in the proof of Theorem 1 (recall (12)) and obtain (3) in the conclusion of the present theorem. Then using (3), we obtain (4) and (5) in the conclusion of this theorem exactly as (4) and (5) were obtained in Theorem 1. This completes the proof of Theorem 3.

By combining Theorems 1 and 3 we note the following facts which will be important to us in section 4. Let C be an open convex cone. Let Λ^C_∞ denote the subset of Λ_∞ coinsisting of distributional derivatives of functions $g(t)$ which are continuous on $\boldsymbol{R}^n$, have support in C^*, and satisfy (22). Let $\mathcal{K}'_C$ denote the subset of $\mathcal{K}'$ defined by $\mathcal{K}'_C = \{U \in \mathcal{K}' \colon U = \mathcal{F}[V], V \in \Lambda^C_\infty\}$. Then Theorems 1 and 3 yield that the construction $f(z) \rightarrow V = \Delta g(t)$ given through (7) in Theorem 1 maps $\mathcal{A}^*_{\omega,C}$ onto Λ^C_∞, and the boundary value mapping given in (5) maps $_{\omega,C}$ on o $_{C}$. so e images un er ese mappings are unique. Further, it is easily seen that both of these mappings are $1 - 1$. This follows immediately because of the recovery of the elements $f(z) \in \mathcal{A}^*_{\omega,C}$ as the Fourier-Laplace transform of $V \in \Lambda^C_\infty$ in (2) and the fact that the Fourier and inverse Fourier transforms from Λ^C_∞ to $\mathcal{K}'_C$ and $\mathcal{K}'_C$ to Λ^C_∞, respectively, are isomprhisms. Thus we conclude from Theorems 1 and 3 that $\mathcal{A}^*_{\omega,C}$ is algebraically isomorphic to both Λ^C_∞ and $\mathcal{K}'_C$ with the vector space isomorphisms being the construction $f(z) \rightarrow V = \Delta g(t)$ given through (7) in Theorem 1 and the $\mathcal{K}'$ distributional boundary value mapping given in (5), respectively. (These mappings obviously satisfy the desired linearity properties.) We thus have the following corollary of Theorems 1 and 3.

COROLLARY 3. *Let C be an open convex cone. Then $\mathcal{A}^*_{\omega,C}$ is algebraically isomorphic to both Λ^C_∞ and $\mathcal{K}'_C$.*

4. The boundary value mapping and tempered ultra-distributions.

Throughout this section $C = \bigcup_{\sigma} C_\sigma$ is the open cone in $\boldsymbol{R}^n$ defined by the 2^n quadrants $C_\sigma = \{y \in \boldsymbol{R}^n: (-1)^{\sigma_j} y_j > 0,\ j = 1, \dots, n\}$. We consider the functions $\mathcal{A}^*_{\omega,C}$ corresponding to $C = \bigcup_{\sigma} C_\sigma$. As in the paragraph preceeding Corollary 1, we note that $\mathcal{A}^*_{\omega,C}$ is defined without use of compact subcones for this $C = \bigcup_{\sigma} C_\sigma$. Thus throughout this section $f(z) \in \mathcal{A}^*_{\omega,C}$, $C = \bigcup_{\sigma} C_\sigma$, if and only if $f(z)$ is analytic in $T^C = \boldsymbol{R}^n + iC$ and satisfies (20) for $m > 0$ being arbitrary. $\mathcal{A}^*_{\omega,C_\sigma}$ is similarly defined for each of the quadrants C_σ; and we note that the conclusions of Theorems 1 and 3, corresponding to the open convex cone C being any of the C_σ, will hold for $z \in T^{C_\sigma}$.

Corresponding to the open cone $C = \bigcup_{\sigma} C_\sigma$ we define subspaces of Λ_∞ and $\mathcal{K}'$ like those defined in the last paragraph of section 3 prior to Corollary 3. We use the same notation Λ^C_∞ and $\mathcal{K}'_C$ for the subspaces of Λ_∞ and $\mathcal{K}'$ corresponding to $C = \bigcup_{\sigma} C_\sigma$ that we are about to define as the notation in the last paragraph of section 3. The reader should note now that throughout this section Λ^C_∞ and $\mathcal{K}'_C$, $C = \bigcup_{\sigma} C_\sigma$, will denote the subspaces of Λ_∞ and $\mathcal{K}'$ corresponding to the open disconnected cone C that we are about to define. On the other hand $\Lambda^{C_\sigma}_\infty$ and $\mathcal{K}'_{C_\sigma}$ will denote the subspaces of Λ_∞ and $\mathcal{K}'$ as defined in the last paragraph of section 3 because each of the C_σ are open convex cones. As the reader will see the only difference in Λ^C_∞ and $\Lambda^{C_\sigma}_\infty$ ($\mathcal{K}'_C$ and $\mathcal{K}'_{C_\sigma}$) is that a support property is held by the function $g(t)$ corresponding to the open convex cones C_σ and not for the open disconnected cone $C = \bigcup_{\sigma} C_\sigma$.

We now define Λ^C_∞ and $\mathcal{K}'_C$, $C = \bigcup_{\sigma} C_\sigma$. Throughout this section Λ^C_∞, $C = \bigcup_{\sigma} C_\sigma$, will denote the set of all elements $V \in \Lambda_\infty$ which are distributional derivatives of continuous functions $g(t)$ satisfying

$$(29) \qquad |g(t)| \leqslant M_m \exp(2\pi\langle \Omega, t\rangle)$$

for all $\Omega \in C = \bigcup_{\sigma} C_\sigma$ such that $|\Omega_j| \geqslant m$, $j = 1, \dots, n$, where $m > 0$

is arbitrary. We then put

$$(30)\qquad \mathcal{K}'_C = \{U \in \mathcal{K}' : U = \mathcal{F}[V],\ V \in \Lambda^C_\infty\}\ , \qquad C = \bigcup_\sigma C_\sigma\ ,$$

with this Fourier transform being that defined from Λ_∞ $1-1$ and onto $\mathcal{K}'$ [2, p. 768].

In this section we show that for the open cone $C = \bigcup_\sigma C_\sigma$, the space $\mathcal{K}'_C$ is isomorphic to a quotient space of $\mathcal{A}^*_{\omega,C}$ by a certain subspace that will be determined below. The isomorphism will be generated by a boundary value mapping. We then relate our isomorphic representation to the tempered ultra-distributions.

We now define the mapping that will generate the isomorphism. Let y_σ denote $y_\sigma = ((-1)^{\sigma_1}, \ldots, (-1)^{\sigma_n}) \in C_\sigma$ for each of the 2^n n-tuples σ. Let $f(z) \in \mathcal{A}^*_{\omega,C}$, $C = \bigcup_\sigma C_\sigma$. Then the $\mathcal{K}'$ boundary value of $f(z)$, denoted $BV(f(z))$, is defined to be the element $U \in \mathcal{K}'_C$ such that

$$(31)\qquad BV(f(z)) = \lim_{\varepsilon \to 0+} \sum_\sigma (-1)^{|\sigma|} \langle f(x + i\varepsilon y_\sigma), \psi(x)\rangle = \langle U, \psi\rangle\ , \quad \psi \in \mathcal{K}\ ,$$

with the convergence being in the str n n
of $\mathcal{K}'$. Obviously the 2^n limits in (31) exist because of Theorems 1 and 3 corresponding to each of the 2^n open convex cones C_σ. Further it is obvious that the element $U \in \mathcal{K}'$ so obtained is in $\mathcal{K}'_C$; because for each σ,

$$(32)\qquad \lim_{\varepsilon \to 0+} \langle f(x + i\varepsilon y_\sigma), \psi(x)\rangle = \langle \mathcal{F}[V_\sigma], \psi(x)\rangle$$

by Theorem 1; and by the construction of Theorem 1, V_σ is the distributional drivative of a continuous function having support in $(C_\sigma)^*$ and satisfying (29) for all $\Omega \in C_\sigma$ such that $|\Omega_j| \geq m$, $j = 1, \ldots, n$ $m > 0$ being arbitrary.

We now wish to identify the kernel of the mapping BV. Before doing so we define pseudo-polynomial. A pseudo-polynomial $p(z)$ is any function of the variable $z \in T^C = \mathbf{R}^n + iC$, $C = \bigcup_\sigma C_\sigma$, which is a finite linear combination of functions of the form

$$(z_j)^s g_{s,j}(z_{[j]})\ , \qquad j = 1, \ldots, n\ ,$$

where s is a nonnegative integer and $g_{s,j}(z_{[j]}) \in \mathcal{A}^*_{\omega, C_{[j]}}$, $j = 1, \dots, n$. Here $z_{[j]} = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in C_{[j]}$ and $C_{[j]}$ denotes $C_{[j]} = \bigcup_{\sigma_{[j]}} C_{\sigma_{[j]}}$ where the now 2^{n-1} quadrants $C_{\sigma_{[j]}}$ are defined

$$C_{\sigma_{[j]}} = \{(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \\ \in \boldsymbol{R}^{n-1}: (-1)^{\sigma_i} y_i > 0, i = 1, \dots, j-1, j+1, \dots, n\} .$$

Here also $\mathcal{A}^*_{\omega, C_{[j]}}$ denotes the functions defined as before but restricted to the tubular cone defined by the now 2^{n-1} quadrants $C_{\sigma_{[j]}}$ in $\boldsymbol{R}^{n-1}$. Obviously $\big((z_j)^s g_{s,j}(z_{[j]})\big) \in \mathcal{A}^*_{\omega, C}$ if $g_{s,j}(z_{[j]}) \in \mathcal{A}^*_{\omega, C_{[j]}}$. Thus a pseudo-polynomial $p(z)$ is any element of $\mathcal{A}^*_{\omega, C}$, $C = \bigcup_{\sigma} C_\sigma$, of the form

$$(33) \quad p(z) = \sum_{j=1}^{n} \sum_{s=0}^{r_j} (z_j)^s g_{s,j}(z_{[j]}) , \qquad z \in T^C = \boldsymbol{R}^n + iC , \; C = \bigcup_{\sigma} C_\sigma ,$$

where $g_{s,j}(z_{[j]}) \in \mathcal{A}^*_{\omega, C_{[j]}}$, $s = 0, 1, \dots, r_j$, $j = 1, \dots, n$. We denote the set of all pseudo-polynomials in $\mathcal{A}^*_{\omega, C}$ by Π_C and prove the following theorem.

THEOREM 4. *Let $C = \bigcup_{\sigma} C_\sigma$. The kernel of the mapping BV defined in* (31) *is the set Π_C of all pseudo-polynomials in $\mathcal{A}^*_{\omega, C}$.*

PROOF. We first show that any pseudo-polynomial $p(z)$ is in the kernel of BV. Because of the obvious linearity of BV, it suffices to consider $p(z)$ of the form

$$p(z) = (z_j)^s g_{s,j}(z_{[j]})$$

for a fixed $j = 1, \dots, n$ and a fixed nonnegative integer s. Obviously, the strong $\mathcal{K}'$ boundary value of $(z_j)^s$ from the upper (lower) half plane $\operatorname{Im}(z_j) > 0$ $\big(\operatorname{Im}(z_j) < 0\big)$ is $(x_j)^s$, $x_j = \operatorname{Re}(z_j)$. Denoting the strong $\mathcal{K}'$ boundary value of $g_{s,j}(z_{[j]})$ from each of the 2^{n-1} octants in $\boldsymbol{C}^{n-1}$ defined by the $C_{\sigma_{[j]}}$ as $U_{\sigma_{[j]}}$, it follows that the strong $\mathcal{K}'$ boundary value from each of the 2^n octants T^{C_σ} is $\big((x_j)^s U_{\sigma_{[j]}}\big) \in \mathcal{K}'$. By the construction of BV it follows immediately that $p(z)$ is in the kernel of BV. Thus the set of all pseudo-polynomials Π_C is in the kernel of BV.

To show that the reverse containment also holds we first note that because of the construction of $B V$ in (31), the image $U \in \mathcal{K}'_C$ of $f(z) \in \mathcal{A}^*_{\omega, C}$ must be of the form $U = \sum_{\sigma} (-1)^{|\sigma|} \mathcal{F}[V_\sigma] = \mathcal{F}\Big(\sum_{\sigma} (-1)^{|\sigma|} V_\sigma\Big)$ where each $V_\sigma \in \Lambda^{C_\sigma}_\infty$. (Recall the terminology Λ^C_∞ for an open convex cone C

in the last paragraph of section 3.) But the Fourier transform and inverse Fourier transform of Λ_∞ onto $\mathcal{K}'$ and $\mathcal{K}'$ onto Λ_∞, respectively are both vector space isomorphisms. Thus the kernel of the mapping BV is exactly the kernel of the mapping S from $\mathcal{A}^*_{\omega,C}$ to Λ_∞ defined by

$$S\colon \mathcal{A}^*_{\omega,C} \to \left\{V \in \Lambda_\infty \colon V = \sum_\sigma (-1)^{|\sigma|} V_\sigma\right\}, \tag{34}$$

where each $V_\sigma \in \Lambda_\infty^{C_\sigma}$ is constructed from $f(z) \in \mathcal{A}^*_{\omega,C}$ for $z \in T^{C_\sigma}$ as in Theorem 1. The determination of the kernel of S in (34) can be achieved by exactly the argument of Hasumi [3, p. 101] in determining his kernel Π; for the mapping $\mathcal{F}$ of Hasumi [p. 101, line 7] restricted to our space $\mathcal{A}^*_{\omega,C}$ is exactly the mapping S in (34). Since we now desire to show that if $f(z)$ is in the kernel of S in (34), then $f(z) \in \Pi_C$ and since Hasumi has omitted the details in this direction [3, p. 101] in his argument, we now give an argument which proves this implication for completeness of our proof. The argument is completely determined by the 2-dimensional case. If $n = 2$, then for $f(z)$ in the kernel of S in (34) we have $(V_{(0,0)} - V_{(0,1)} - V_{(1,0)} + V_{(1,1)}) = 0$ in Λ_∞. Thus

$$V_{(0,0)} - V_{(0,1)} = V_{(1,0)} - V_{(1,1)} \tag{35}$$

an

$$V_{(0,0)} - V_{(1,0)} = V_{(0,1)} - V_{(1,1)}\,. \tag{36}$$

For any quadrant C_σ, its dual cone $(C_\sigma)^*$ is exactly the closure of C_σ. Thus from (35) and (36) we obtain

$$\begin{aligned}
\operatorname{supp}(V_{(0,0)} - V_{(0,1)}) &\subseteq \{t \in \mathbf{R}^2\colon t_1 = 0\}\\
\operatorname{supp}(V_{(1,0)} - V_{(1,1)}) &\subseteq \{t \in \mathbf{R}^2\colon t_1 = 0\}\\
\operatorname{supp}(V_{(0,0)} - V_{(1,0)}) &\subseteq \{t \in \mathbf{R}^2\colon t_2 = 0\}\\
\operatorname{supp}(V_{(0,1)} - V_{(1,1)}) &\subseteq \{t \in \mathbf{R}^2\colon t_2 = 0\}\,.
\end{aligned}$$

A direct application of Schwartz [5, Théorème XXXVI, p. 101] yields

$$\left\{\begin{aligned}
V_{(0,0)} - V_{(0,1)} &= V_{(1,0)} - V_{(1,1)} = \sum_{s=0}^{r_1} Y_s(t_2)\,\delta^{(s)}(t_1)\\
V_{(0,0)} - V_{(1,0)} &= V_{(0,1)} - V_{(1,1)} = \sum_{s=0}^{r_2} W_s(t_1)\,\delta^{(s)}(t_2)\,.
\end{aligned}\right. \tag{37}$$

Here r_1 and r_2 are nonnegative integers, δ is the Dirac delta distribution, $Y_s(t_2)$ and $W_s(t_1)$ are elements of Λ_∞ in 1-dimension with respect to t_2 independent of t_1 and with respect to t_1 independent of t_2, respectively, for each $s = 0, \ldots, r_1$ and each $s = 0, \ldots, r_2$, respectively. From (37) it follows that $(V_{(0,0)} - V_{(0,1)} - V_{(1,0)} + V_{(1,1)}$ is of the form

$$\sum_{s=0}^{r_1} Y_s(t_2)\delta^{(s)}(t_1) + \sum_{s=0}^{r_2} W_s(t_1)\delta^{(s)}(t_2),$$

for some $Y_s(t_2)$ and $W_s(t_1)$ as described above. From this it follows in this 2-dimensional case that

$$f(z) = \sum_{s=0}^{r_1} (z_1)^s g_{s,1}(z_{[1]}) + \sum_{s=0}^{r_2} (z_2)^s g_{s,2}(z_{[2]})$$

for $f(z)$ in the kernel of S in (34). This technique extends immediately to n-dimensions. We thus conclude that if $f(z)$ is in the kernel of the mapping$S$ in (34), then $f(z)$ must be a pseudo-polynomial. Since the kernel of BV in (31) is exactly the kernel of S in (34), then if $f(z)$ is in the kernel of BV, it must be a pseudo-polynomial. We conclude from this and the previously proved converse direction that the kernel of BV is exactly Π_C, the set of all pseudo-polynomials defined in (33). The proof of Theorem 4 is complete.

We now define the space $\mathcal{U}_C$ to be the quotient space of $\mathcal{A}^*_{\omega,C}$ by the set of all pseudo-polynomials Π_C, $C = \bigcup_\sigma C_\sigma$. That is

$$\mathcal{U}_C = \mathcal{A}^*_{\omega,C}/\Pi_C, \tag{38}$$

with this being a quotient space of vector spaces. We prove that $\mathcal{H}'_C = \{U \in \mathcal{H}' : U = \mathcal{F}[V], V \in \Lambda^C_\infty\}$, $C = \bigcup_\sigma C_\sigma$, as defined in (30) is isomorphic to $\mathcal{U}_C$ under an isomorphism that is generated by the mapping BV defined in (31).

THEOREM 5. *Let $C = \bigcup_\sigma C_\sigma$. Then $\mathcal{U}_C$ is isomorphic to the space $\mathcal{H}'_C$ with the isomorphism being defined by the mapping BV. If $u(z) \in \mathcal{U}_C$ and $U \in \mathcal{H}'_C$ is the corresponding element under the isomorphism, then*

$$BV(f(z)) = \lim_{\varepsilon\to 0+} \sum_\sigma (-1)^{|\sigma|}\langle f(x + i\varepsilon y_\sigma), \psi(x)\rangle = \langle U, \psi\rangle, \quad \psi \in \mathcal{H} \tag{39}$$

where $f(z)$ is any representative of $u(z)$.

PROOF. We denote elements of $\mathcal{U}_C$ by $u(z)$. Define the mapping $\overline{BV}$ from $\mathcal{U}_C$ to $\mathcal{H}'_C$ as follows. If $u(z) \in \mathcal{U}_C$, put $\overline{BV}(u(z)) = BV(f(z))$ where the mapping BV is defined in (31) and $f(z)$ is any representative of $u(z)$. We show that $\overline{BV}$ is the desired isomorphism. Since BV is linear, then $\overline{BV}$ obviously satisfies the desired linearity properties. We also have immediately that $\overline{BV}$ is $1-1$ because of the kernel Π_C of the mapping BV. Thus $\overline{BV}$ is a vector space isomorphism from $\mathcal{U}_C$ to $\mathcal{H}'_C$. It remains to be proved that $\overline{BV}$ is an onto mapping. Let $U \in \mathcal{H}'_C$; Then $U = \mathcal{F}[V]$ for some element $V \in \Lambda^C_\infty$, and hence $V = \mathcal{F}^{-1}[U]$. Recalling the definition of Λ^C_∞ in the second paragraph of this section 4, we have that $V = D^\alpha_t(g(t))$, where $g(t)$ is a continuous function satisfying (29). We now put

$$V_\sigma = (-1)^{|\sigma|} D^\alpha_t(I_\sigma(t)g(t))$$

where $I_\sigma(t)$ denotes the characteristic function of the dual cone $(C_\sigma)^*$ of each of the 2^n open convex cones C_σ. Corresponding to each V_σ we put

$$f_\sigma(z) = \langle V_\sigma, \exp(2\pi i\langle z, t\rangle)\rangle , \quad z \in T^{C_\sigma} . \tag{40}$$

define a function $f(z)$, $z \in T^C$, $C = \bigcup_\sigma C_\sigma$, from the $2^n f_\sigma(z)$ as follows: for each of the 2^n values of the n-typle σ, we put

$$f(z) = f_\sigma(z) , \quad z \in T^{C_\sigma} . \tag{41}$$

We thus have

$$V = \sum_\sigma ((-1)^{|\sigma|} V_\sigma) \quad \text{so that} \quad U = \mathcal{F}[V] = \sum_\sigma ((-1)^{|\sigma|}\mathcal{F}[V_\sigma]) .$$

Further, by Theorem 3, the function $f_\sigma(z)$ defined in (40) is an element of $\mathcal{A}^*_{\omega,C_\sigma}$ and satisfies $f_\sigma(x + i\varepsilon y_\sigma) \to \mathcal{F}[V_\sigma]$ in the strong topology of $\mathcal{H}'$ as $\varepsilon \to 0^+$. Combining these facts we immediately see that the function $f(z)$ defined in (41) is an element of $\mathcal{A}^*_{\omega,C}$, $C = \bigcup_\sigma C_\sigma$, which maps to the given element $U \in \mathcal{H}'_C$ under the mapping BV defined in (31). Hence $f(z)$ is a representative of an element $u(z) \in \mathcal{U}_C$ such that $u(z)$ maps to the given $U \in \mathcal{H}'_C$ under $\overline{BV}$. This proves that $\overline{BV}$ is an onto mapping from U_C to $\mathcal{H}'_C$; hence U_C and $\mathcal{H}'_C$ are isomorphic.

Further, (39) follows immediately from our construction of $\overline{BV}$. The proof of Theorem 5 is complete.

We now shall relate the present space $\mathcal{U}_C$ of equivalence classes of analytic functions with the tempered ultra-distributions $\mathcal{U}$ of Sebastião e Silva [7, p. 70] and Hasumi [3, p. 101]. First we restrict our attention to 1-dimension. In this case $C = ((-\infty, 0) \cup (0, \infty))$; and as a direct consequence of Corollary 1 and our definition of BV in (31), any element of $\mathcal{A}^*_{\omega,C}$ which is in the kernel of BV must be a polynomial in $z \in C^1$. This of course forces Π_C to be exactly the set of all polynomials in $z \in C^1$ for $C = ((-\infty, 0) \bigcup (0, \infty))$. But we know from [7, p. 70] and [3, p. 101] that for 1-dimension the kernel Π in the definition of $\mathcal{U}$ is also the set of polynomials in $z \in C^1$. Thus since $\mathcal{A}^*_{\omega,C} \subset \mathcal{A}_\omega$, $\mathcal{A}_\omega$ being defined in [3, p. 100], then elements of $\mathcal{U}_C$ defined in (38) are tempered ultra-distributions for $n = 1$ and $C = ((-\infty, 0) \cup (0, \infty))$.

But if the dimension n is larger than 1, then we have $\mathcal{A}^*_{\omega,C} \subset \mathcal{A}_\omega$ and $\Pi_C \subset \Pi$, $C = \bigcup_\sigma C_\sigma$, with both containments being proper. Thus if $n \geqslant 2$, elements of $\mathcal{U}_C$ are not tempered ultra-distributions. But since we obviously have $(\Pi \cap \mathcal{A}^*_{\omega,C}) = \Pi_C$, then we can define a $1-1$ correspondence between elements of $\mathcal{U}_C$ and the subspace of $\mathcal{U}$ generated by $\mathcal{A}^*_{\omega,C}$, $C = \bigcup_\sigma C_\sigma$, as in the following. If $u(z) \in \mathcal{U}_C$ then $u(z) = \{f(z) + p(z)\}$ for some fixed $f(z) \in \mathcal{A}^*_{\omega,C}$ where $p(z)$ ranges over Π_C. For this same $f(z)$, $u'(z) = \{f(z) + q(z)\}$, where $q(z)$ ranges over Π, is an element of $\mathcal{U}$; that is $u'(z)$ is a tempered ultra-distribution. Conversely, given an element $u'(z) \in \mathcal{U}$ defined by an element $f(z) \in \mathcal{A}^*_{\omega,C}$, then an element $u(z) \in \mathcal{U}_C$ can be constructed from this same $f(z)$ by adding to it $p(z)$ ranging over Π_C. We thus correspond

$$u(z) \leftrightarrow u'(z)\,, \tag{42}$$

where $u(z) \in \mathcal{U}_C$ and $u'(z) \in \mathcal{U}$ both have the same $f(z) \in \mathcal{A}^*_{\omega,C}$ as a representative. Since $(\Pi \cap \mathcal{A}^*_{\omega,C}) = \Pi_C$, this is a $1-1$ correspondence between the elements of $\mathcal{U}_C$ and the subspace of the tempered ultra-distributions generated by $\mathcal{A}^*_{\omega,C}$.

We now let the dimension n be arbitrary. We extend the correspondence (42) to $n = 1$ by considering the symbol $\leftrightarrow$ to mean equality for $n = 1$. Let $u(z) \in \mathcal{U}_C$ and let $u'(z)$ be the corresponding tempered ultra-distribution as in (42). Because of the construction of Hasumi [3, pp. 99-101] and the construction in this paper leading to the mapping BV defined in (31), it is now clear that the element $U \in \mathcal{H}'$ in the duality (39) corresponding to $u(z) \in \mathcal{V}_C$ and the element U in the duality of [3, Proposition 5, p. 101] corresponding to the tempered ultra-distribution $u'(z)$ such that $u(z) \leftrightarrow u'(z)$ as in (42) are exactly the same element of $\mathcal{H}'$ (and hence are exactly the same element of the subspace $\mathcal{H}'_C$, $C = \bigcup_\sigma C_\sigma$, defined in (30).) We thus have the following theorem which relates the space $\mathcal{U}_C$ with the tempered ultra-distributions $\mathcal{U}$.

THEOREM 6. *Let* $C = \bigcup_\sigma C_\sigma$. *The correspondence* (42) *is a* $1-1$ *correspondence between the elements of* $\mathcal{U}_C$ *and the subspace of the tempered ultra-distributions generated by* $\mathcal{A}^*_{\omega,C}$. *Further , if* $u(z) \in \mathcal{U}_C$ *and* $u'(z) \in \mathcal{U}$ *such that* $u(z) \leftrightarrow u'(z)$ *as in* (42), *then the element* $U \in \mathcal{H}'$ *generated from* $u(z)$ *by* (39) *and the element* $U \in \mathcal{H}'$ *generated from* $u'(z)$ *by* [3, Proposition 5, p. 101] *are identical*; *and hence* $U \in \mathcal{H}'_C$.

We emphasize again that if $n = 1$ in Theorem 6, the correspondence (42) is actual equality. Finally, we note the following. Because of the properties of the space $\mathcal{A}_\omega$ [3, p. 100] it is now obvious that the only tempered ultra-distributions that will have representatives which have distributional boundary values in $\mathcal{H}'$ are those tempered ultra-distributions $u'(z)$ that are generated by elements $f(z) \in \mathcal{A}^*_{\omega,C} \subset \mathcal{A}_\omega$. If the dimension $n = 1$, then all representatives of such a tempered ultra-distribution $u'(z)$ will have $\mathcal{H}'$ boundary values. But if $n \geqslant 2$, then the only representatives of such a $u'(z) \in \mathcal{U}$ that will have $\mathcal{H}'$ boundary values are those representatives of the form $(f(z) + p(z))$ where $f(z) \in \mathcal{A}^*_{\omega,C}$ generates $u'(z)$ and $p(z)$ ranges over Π_C. But there are other representatives of $u'(z)$ of the form $(f(z) + q(z))$ where $q(z) \in \Pi$ such that $q(z) \notin \Pi_C$. Thus if the dimension $n \geqslant 2$, no tempered ultra-distribution has the property that all of its representatives have $\mathcal{H}'$ boundary values.

REFERENCES

[1] R. D. CARMICHAEL, *Distributional boundary values of functions analytic in tubular radial domains*, Indiana Univ. Math. J., **20** (1971), pp. 843-853.

[2] R. D. CARMICHAEL, *Distributions of exponential growth and their Fourier transforms*, Duke Math. J., **40** (1973), pp. 765-783.

[3] M. HASUMI, *Note on the n-dimensional tempered ultra-distributions*, Tôhoku Math. J., **13** (1961), pp. 94-104.

[4] L. HÖRMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[5] L. SCHWARTZ, *Théorie des Distributions*, Hermann, Paris, 1966.

[6] L. SCHWARTZ, *Mathematics for the Physical Sciences*, Addison-Wesley, Reading, Mass., 1966.

[7] J. SEBASTIÃO E SILVA, *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel*, Math. Ann., **136** (1958), pp. 58-96.

[8] J. SEBASTIÃO E SILVA, *Les séries de multipôles des physiciens et la théorie des ultradistributions*, Math. Ann., **174** (1967), pp. 109-142.

[9] V. S. VLADIMIROV, *Methods of the Theory of Functions of Several Complex Variables*, M.I.T. Press, Cambridge, Mass., 1966.

[10] K. YOSHINAGA, *On spaces distributions of exponential growth*, Bull. Kyushu Inst. Tech. Math. Natur. Sci., **6** (1960), pp. 1-16.

[11] Z. ZIELEŹNY, *On the space of convolution operators in* $\mathcal{K}_1'$, Studia Math., **31** (1968), pp. 111-124.

Manoscritto pervenuto in Redazione il 6 febbraio 1976.