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ALFREDO LORENZI

A mixed boundary value problem for the Laplace equation in an angle (2nd part)

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A Mixed Boundary Value Problem for the Laplace Equation in an Angle.

(2nd part)

ALFREDO LORENZI (*)

SUMMARY - We give a proof of the regularity theorem stated in the first part of this work (1). Such a theorem deals with the smoothness near the corner of functions harmonic in an angle and verifying mixed boundary conditions.

Introduction to the second part.

In this paper we are going to prove the regularity theorem stated in the first part of this work, which has already appeared in this journal (1). We make use of the same notations as in the first part; in particular, numbers in square brackets are referred to the bibliography quoted there. Moreover, since this paper starts with formula (7.1), formulas of the type (5.22) are obviously contained in the first part of this work.

7. – Proof of theorem 2.

Since the proof is rather long and complex, we premise it with a plan divided into two parts.

(*) Indirizzo dell'A.: Istituto Matematico « F. Enriquez » - Via Saldini, 50 - 20133 Milano.

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(1) See Rend. Sem. Mat. Padova, **54** (1975).

I) The first part consists in showing that the derivatives up to the order $s-1$ of the traces u_θ of u, defined by formula (5.1), belong to $L^p(0, +\infty)$ and in showing that the differential quotients of the derivatives of order $s-1$ belong to $L^p((0, +\infty)\times(0, +\infty))$. These properties can be synthetized by the following

PROPOSITION. *$u'_\theta \in W^{s-1-1/p,p}(0, +\infty)$ for every $\theta\in[0, \alpha]$, if, and only if, the conditions listed in graphs* 1, 2, 3 *contained in the statement of theorem* 2 *are verified.*

Such a proposition is a consequence of lemmas 7.1, 7.2 and 7.3 stated below.

II) In the second part we infer the smoothness of u from that of u_θ.

Like in theorem 1 we consider separately the two cases $\alpha\in[\pi, 2\pi)$ and $\alpha\in(0, \pi)$. If $\alpha\in[\pi, 2\pi)$ we show that the derivatives up to the order $s-1$ of the functions h_θ, defined by (5.22), belong to $L^p(-\infty, +\infty)$ and that the differential quotients of the derivatives of order $s-1$ belong to $L^p(R^2)$. Such properties are shown in lemma 7.4.

After establishing this fact, it is immediate to realize that u has the required regularity (i.e. $Du\in W^{s-1,p}(\Omega_\alpha)$), since u coincides with the Poisson integrals of h_0 and $h_{\alpha-\pi}$ in the intersection of the domains (see section 5, 1st case).

Finally, if $\alpha\in(0,\pi)$, the regularity of u is a consequence of the fact that it is a solution to the Dirichlet problem (5.27) with data a and u_0. An application of theorem 4 in section 6 and of lemma 7.5, stated below, concludes the proof of the theorem.

LEMMA 7.1. *Suppose that $a\in W^{1/p',p}(0, +\infty)$, $b\in L^p(0, +\infty)$ and that a and b possess properties* (1.7). *Suppose also that (p, α, ω) is such that*

$$\left(1-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\notin Z\,, \tag{7.1}$$

$$\frac{\alpha}{p'\pi}-\frac{\omega}{\pi}\notin Z\,. \tag{7.2}$$

Then the first derivatives of the functions u_θ belong to $L^p(0, +\infty)$ for every $\theta\in[0,\alpha]$, if, and only if, the conditions listed in graph 1 *in theorem* 2 *are verified.*

If such conditions are fulfilled, then the following formula holds:

$$(7.3)\qquad u'_\theta(r) = $$

$$= \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H_1(r,\theta,t)\,a'(t)\,dt + \dfrac{1}{\alpha}\int_0^{+\infty} K_1(r,\theta,t)\,b(t)\,dt + b(r)\,\chi(\theta)\cos\omega & \theta\in[0,\alpha) \\ a'(r) & \theta=\alpha \end{cases}$$

where χ, H_1 *and* K_1 *are defined by the equations:*

$$(7.4)\qquad \chi(\theta) = \begin{cases} 1 & \theta = 0 \\ 0 & \theta\neq 0 \end{cases}$$

$$(7.5)\qquad \begin{cases} H_1(r,\theta,t) = (-1)^{N_1+1}\cdot \\ \qquad\cdot\dfrac{r^{\sigma_1-1}t^{2\beta-\sigma_1}\{r^{2\beta}\sin[(\sigma_1-2\beta)\theta-\omega]+t^{2\beta}\sin[(\sigma_1\theta-\omega)]\}}{r^{4\beta}+2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}} \\ K_1(r,\theta,t) = \\ \qquad = -\dfrac{r^{\sigma_1-1}t^{2\beta-\sigma_1}\{r^{2\beta}\sin[(\sigma_1-2\beta)\theta-\omega]-t^{2\beta}\sin[(\sigma_1\theta-\omega)]\}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}} \end{cases}$$

N_1 *and* σ_1 *being defined respectively by* (3.5) *and* (3.12).
Moreover the trace of u_θ *at* $r=0$ *is given by*

$$u_\theta(0) = a(0)\,,\qquad \theta\in[0,\alpha]\,.$$

LEMMA 7.2. *Suppose that* $s\geqslant 2$ *is a given integer, that*

$$a'\in W^{s-2,p}(0,+\infty),\ b\in W^{s-2,p}(0,+\infty)$$

and that (p,α,ω) *is such that*

$$(7.6)\qquad \left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\notin Z\qquad j=1,\dots,s-1$$

Then the function u'_θ, *defined by* (7.3), *belongs to* $W^{s-2,p}(0,+\infty)$ *for every* $\theta\in[0,\alpha]$, *if, and only if, the compatibility conditions listed in graphs* 2 *in theorem* 2 *are verified.*
Moreover, if such conditions are satisfied, the derivatives of u_θ *are*

given by the formulas

$$(7.7)\qquad u_\theta^{(j)}(r)=$$

$$=\begin{cases}\dfrac{1}{\alpha}\displaystyle\int_0^{+\infty}H_j(r,\theta,t)\,a^{(j)}(t)\,dt+\dfrac{1}{\alpha}\int_0^{+\infty}K_j(r,\theta,t)\,b^{(j-1)}(t)\,dt+b^{(j-1)}(r)\,\chi(\theta)\cos\omega & \theta\in[0,\alpha)\\ a^{(j)}(r) & \theta=\alpha\end{cases}$$
$$j=1,\dots,s-1$$

and their traces at $r=0$ *are given by the formulas*:

i) *if either* $n_j=0$, *or* $n_j=1$ *and* $\sigma_j\neq 1$, *or* $n_j=2$ *and neither* σ_j *nor* τ_j *equals* 1, *then*

$$(7.8)\qquad u_\theta^{(j)}(0)=\frac{\sin(j\theta-\omega)}{\sin(j\alpha-\omega)}\,a^{(j)}(0)+\frac{\sin[j(\alpha-\theta)]}{\sin(j\alpha-\omega)}\,b^{(j-1)}(0)\qquad j=1,\dots,s-2;$$

ii) *if either* $n_j=1$ *and* $\sigma_j=1$ *or* $n_j=2$ *and* $\sigma_j=1$, *then*

$$(7.9)\qquad u_\theta^{(j)}(0)=b^{(j-1)}(0)\cos(j\theta-\omega)-$$
$$-\frac{1}{\alpha}\sin(j\theta-\omega)\int_0^{+\infty}[(-1)^{N_j}a^{(j)}(t)-b^{(j-1)}(t)]\,\frac{dt}{t}\qquad j=1,\dots,s-2;$$

iii) *if* $n_j=2$ *and* $\tau_j=1$, *then*

$$(7.10)\qquad u_\theta^{(j)}(0)=b^{(j-1)}(0)\cos(j\theta-\omega)+$$
$$+\frac{1}{\alpha}\sin(j\theta-\omega)\int_0^{+\infty}[(-1)^{N_j}a^{(j)}(t)+b^{(j-1)}(t)]\,\frac{dt}{t}\qquad j=1,\dots,s-2\,.$$

The kernels H_j *and* K_j *are defined as follows*:

$$(7.11)\qquad\begin{cases}H_j(r,\theta,t)=(-1)^{N_j+1}\cdot\\ \cdot\dfrac{r^{\sigma_j-1}t^{2\beta-\sigma_j}\{r^{2\beta}\sin[(\sigma_j+j-1-2\beta)\theta-\omega]+t^{2\beta}\sin[(\sigma_j+j-1)\theta-\omega]\}}{r^{4\beta}+2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\\[2ex] K_j(r,\theta,t)=\\ =-\dfrac{r^{\sigma_j-1}t^{2\beta-\sigma_j}\{r^{2\beta}\sin[(\sigma_j+j-1-2\beta)\theta-\omega]-t^{2\beta}\sin[(\sigma_j+j-1)\theta-\omega]\}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\end{cases}$$

N_j *and* σ_j *being defined respectively by* (3.5) *and* (3.12).

REMARK TO LEMMA 7.2. We observe that, taking into account equation (3.7), it is easy to realize that $(j\alpha-\omega)/\pi\in Z$, if, and only if, either $n_j=1$ and $\sigma_j=1$, or $n_j=2$ and either $\sigma_j=1$ or $\tau_j=1$.

LEMMA 7.3. *Suppose that $s\geqslant 2$ is a given integer, that*

$$a^{(s-1)},\ b^{(s-2)}\in W^{1/p',p}(0,+\infty)$$

and that (p,α,ω) satisfies (7.6) *with $j=s-1$ and*

$$\left(s-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\notin Z\,. \tag{7.12}$$

Then the function $u_\theta^{(s-1)}$, defined by (7.7) *with $j=s-1$, belongs to $W^{1/p',p}(0,+\infty)$ for every $\theta\in[0,\alpha]$, if, and only if, the compatibility conditions listed in graph* 3 *in theorem* 2 *are verified. Moreover, if such conditions are satisfied and $p>2$, the traces of $u_\theta^{(s-1)}$ at $r=0$ are given by either formula* (7.8) *or* (7.9) *with j replaced by $s-1$.*

REMARK TO LEMMA 7.3. Formula (7.10) is omitted on account of remark 3.3.

LEMMA 7.4. *Suppose that $s\geqslant 2$ is a given integer, that (p,α,ω) satisfies* (7.1); (7.6), (7.12) *and also the following condition*

$$\alpha\in[\pi,2\pi)\,. \tag{7.13}$$

Then, if a and b possess properties (1.6) *and* (1.7), *a necessary and sufficient condition in order that $h_\theta'\in W^{s-1-1/p,p}(0,+\infty)$ for every $\theta\in[0,\alpha-\pi]$ is that the conditions listed in graphs* 1, 2, 3 *in theorem* 2 *are satisfied.*

LEMMA 7.5. *Suppose that the hypotheses of lemma* 7.4, *except* (7.13), *are verified and that also the following ones hold:*

$$\alpha\in(0,\pi)$$

$$\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}\notin\frac{1}{2}N\qquad j=1,\ldots,s-1\ \text{(2)} \tag{7.14}$$

$$\left(s-\frac{2}{p}\right)\frac{\alpha}{\pi}\notin\frac{1}{2}N\,. \tag{7.15}$$

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(2) N denotes the set of all positive integers.

Then the pair (a, u_0), u_0 *being defined by* (5.1), *satisfies the conditions listed in graphs* 1, 2, 3 *in theorem* 4, *if the pair* (a, b) *satisfies the conditions listed in graphs* 1, 2, 3 *in theorem* 2.

In conclusion we observe that the proof of theorem 2 ends with the proofs of lemmas 7.1, 7.2, 7.3, 7.4, 7.5.

PROOF OF LEMMA 7.1. We observe that the proof rests essentially on the following identities

$$(7.16)\quad \begin{cases} H(r,\theta,t) = (-1)^{M_1+1}\sum_{j=0}^{m-1}(-1)^j r^{\nu-1+2j\beta}t^{-\nu-2j\beta}\cdot \\ \qquad\qquad \cdot\sin[(\nu-1+2j\beta)\theta-\omega] + \dfrac{r}{t}H_1(r,\theta,t) \; (^3) \\ K(r,\theta,t) = \sum_{j=0}^{m-1} r^{\nu-1+2j\beta}t^{-\nu-2j\beta}\sin[(\nu-1+2j\beta)\theta-\omega] + \dfrac{r}{t}K_1(r,\theta,t) \end{cases}$$

where H and K, H_1 and K_1, β, M_1, ν, m are defined respectively by (2.16), (7.5), (2.5), (3.4), (3.10), (3.1).

For simplicity's sake we limit ourselves to proving the lemma only when $m = 2$ and (ν, ϱ) belongs to set 4 (4), ϱ being defined by (3.11) (5). Recalling remark 3.1 it follows that in our case $p \in (1, 2)$.

We begin by observing that from formulas (5.3) and (7.16) we can derive the identity

$$(7.17)\quad u_\theta(r) = -\frac{1}{\alpha}r^{\nu-1}\sin[(\nu-1)\theta-\omega]\int_0^{+\infty}[(-1)^{M_1}a(t) - B_p(t)]t^{-\nu}dt +$$

$$+\frac{1}{\alpha}r^{\varrho-1}\sin[(\varrho-1)\theta-\omega]\int_0^{+\infty}\{(-1)^{M_1}[a(t)-a(0)] + B_p(t) - B_p(0)\}\,t^{-\varrho}\,dt +$$

$$+\frac{1}{\alpha}a(0)\int_0^{+\infty}\tilde{H}(r,\theta,t)\,dt + \frac{1}{\alpha}B_p(0)\int_0^{+\infty}\tilde{K}(r,\theta,t)\,dt + \frac{1}{\alpha}A_\theta(r) + \frac{1}{\alpha}b_\theta(r) +$$

$$+\chi(\theta)B_p(r)\cos\omega$$

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(3) When $m = 0$, $\sum_{j=0}^{-1} c_j$ is, by definition, equal to 0.

(4) See pictures in section 3.

(5) The remaining cases can be proved in an analogous way.

where χ is defined by (7.4) and the functions $\tilde{H}$ and $\tilde{K}$ can be obtained from H and K by substituting $\varrho = \nu + 2\beta$ for ν: therefore it points out that they are homogeneous of degree -1 and integrable [6] over $(0, +\infty)$. Moreover the functions A_θ and b_θ are defined by the formulas

$$(7.18)\qquad \begin{cases} A_\theta(r) = \displaystyle\int_0^{+\infty} \frac{r}{t} H_1(r, \theta, t)[a(t) - a(0)]\, dt \\ b_\theta(r) = \displaystyle\int_0^{+\infty} \frac{r}{t} K_1(r, \theta, t)[B_p(t) - B_p(0)] dt. \end{cases}$$

We observe that the first two integrals in the right hand side in (7.17) converge on account of property (2.17) of the hypothesis $(\nu, \varrho) \in \text{set } 4$, of corollary 5.4, lemma 5.2 and lemma 7.6, stated below, whose proof we postpone.

Lemma 7.6. *Let f be a function such that $r^{-\gamma} f \in L^p(0, +\infty)$ for some $\gamma \in (0, 1)$ and $f' \in L^p(0, +\infty)$. Then the following estimates hold:*

$$(7.18\,bis)\qquad \begin{cases} \displaystyle\int_0^{+\infty} |f(r)|\, r^{-\delta}\, dr \leqslant C_4(\|r^{-\gamma} f\|_{L^p(0,+\infty)} + \|f'\|_{L^p(0,+\infty)}) \\ \qquad\qquad \text{if } \gamma \in \left(0, \frac{1}{p}\right),\ \delta \in \left(\gamma + \frac{1}{p'}, 1\right) \\ \displaystyle\int_0^{+\infty} |f(r) - f(0)|\, r^{-\delta}\, dr \leqslant C_4(\|r^{-\gamma} f\|_{L^p(0,+\infty)} + \|f'\|_{L^p(0,+\infty)}) \\ \qquad\qquad \text{if } \delta \in \left(\max\left(1, \gamma + \frac{1}{p'}\right), 1 + \frac{1}{p'}\right) \end{cases}$$

where C_4 is a positive constant depending only on (p, γ, δ).

Now we go on with the proof of lemma 7.1: differentiating formula

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[6] « Integrable » means here Lebesgue integrable, except for the function $t \to \tilde{K}(r, 0, t)$, for which it means, as usual, integrable in the Cauchy principal value sense.

(7.17) with respect to r, we obtain the equation

$$(7.19)\qquad u_\theta'(r) = -\frac{\nu-1}{\alpha} r^{\nu-2}\sin[(\nu-1)\theta-\omega]\int_0^{+\infty}[(-1)^{M_1}a(t)-B_p(t)]t^{-\nu}dt +$$
$$+\frac{\varrho-1}{\alpha} r^{\varrho-2}\sin[(\varrho-1)\theta-\omega]\int_0^{+\infty}\{(-1)^{M_1}[a(t)-a(0)]+B_p(t)-B_p(0)\}t^{-\varrho}dt +$$
$$+\frac{1}{\alpha}A_{\theta,1}(r)+\frac{1}{\alpha}b_{\theta,1}(r)+\chi(\theta)\,b(r)\cos\omega$$

where

$$(7.20)\qquad \begin{cases} A_{\theta,1}(r) = \displaystyle\int_0^{+\infty} H_1(r,\theta,t)\,a'(t)\,dt\,, \\ b_{\theta,1}(r) = \displaystyle\int_0^{+\infty} K_1(r,\theta,t)\,b(t)\,dt\,. \end{cases}$$

Formula (7.19) is a consequence of the fact that the integrals of $\tilde{H}$ and $\tilde{K}$ with respect to t do not depend on r, since $\tilde{H}$ and $\tilde{K}$ are functions homogeneous of degree -1 in (r, t), and of the following estimates:

$$(7.21)\qquad \begin{cases} \|A_\theta\|_{L^p(0,R)} \leqslant C(R)\big[\|r^{-1/p'}a\|_{L^p(0,+\infty)} + \|a'\|_{L^p(0,+\infty)}\big]\,, \\ \qquad\qquad\qquad\qquad \theta\in[0,\alpha)\,,\ R\in(0,+\infty) \\ \|b_\theta\|_{L^p(0,R)} \leqslant C(R)\big[\|r^{-1/p'}B_p\|_{L^p(0,+\infty)} + \|b\|_{L^p(0,+\infty)}\big]\,, \end{cases}$$

where $C(R)$ is a positive constant depending on R and also on $(\theta, p, \alpha, \omega)$.

Estimates (7.21) can be proved easily using identity (7.16), estimates (7.18 bis) with δ equal respectively to ν and ϱ, and γ equal to $1/p'$, recalling the hypotheses on a and b and corollary 5.4 and lemma 5.2.

Now we observe that the equations $A_\theta' = A_{\theta,1}$ and $b_\theta' = b_{\theta,1}$ hold, when a and b are replaced by functions in $C_0^\infty([0,+\infty))$, and we notice that, according to lemma 5.5 and property (7.2), the linear mappings $a' \to A_{\theta,1}$ and $b \to b_{\theta,1}$ are continuous from $L^p(0,+\infty)$ into itself; therefore, taking (7.21) into account, we infer that (7.19) holds when $\theta \in (0, \alpha]$, the case $\theta = \alpha$ being trivial.

Finally we observe that, when $\theta = 0$ the function b_0 is defined by a singular integral: the following lemma 7.7 gives a formula to differentiate b_0 that enables us to conclude that (7.19) holds for every $\theta \in [0, \alpha]$, recalling that, on account of inequality (3.17) and property (7.2), $\sigma_1 \in (1/p', 1/p' + 2\beta)$.

LEMMA 7.7. *Suppose that f is a function such that $f' \in L^p(0, +\infty)$ and consider the functions F_1 and F_2 so defined*

$$F_1(r) = \int_0^{+\infty} \frac{r^{\gamma_1} t^{2\beta - \gamma_1 - 1}}{r^{2\beta} - t^{2\beta}} f(t)\, dt \tag{7.22}$$

$$F_2(r) = \int_0^{+\infty} \frac{r^{\gamma_2} t^{2\beta - \gamma_2 - 1}}{r^{2\beta} - t^{2\beta}} [f(t) - f(0)]\, dt \tag{7.23}$$

where $\beta \in (\frac{1}{4}, +\infty)$, $\gamma_1 \in (1/p', 2\beta)$, $\gamma_2 \in (1/p', 1/p' + 2\beta)$ and the integrals are taken in the Cauchy principal value sense. Then the first derivative of F_j $(j = 1, 2)$ belongs to $L^p(0, +\infty)$ and is given by the formula

$$F_j'(r) = \int_0^{+\infty} \frac{r^{\gamma_j - 1} t^{2\beta - \gamma_j}}{r^{2\beta} - t^{2\beta}} f'(t)\, dt \qquad (j = 1, 2)\,. \tag{7.24}$$

Moreover the linear mapping $f' \to F_j'$ is continuous from $L^p(0, +\infty)$ into itself.

If, in addition to the previous properties, $\gamma_2 < 2\beta$, then the functions F_j $(j = 1, 2)$ possess traces at $r = 0$ which are given by the formulas

$$F_1(0) = -\alpha f(0) \operatorname{cotg}\left(\frac{\gamma_1}{2\beta}\pi\right) \tag{7.25}$$

$$F_2(0) = 0\,. \tag{7.26}$$

Now it is easy to realize, using formula (7.19), that $u_\theta' \in L^p(0, +\infty)$

for every $\theta \in [0, \alpha]$, if, and only if,

$$(7.27) \qquad \int_0^{+\infty} [(-1)^{M_1} a(t) - B_p(t)]\, t^{-\nu}\, dt =$$

$$= \int_0^{+\infty} \{(-1)^{M_1}[a(t) - a(0)] + B_p(t) - B_p(0)\}\, t^{-\varrho}\, dt = 0\,.$$

Observe, then, that an integration by parts yields the equations

$$\int_0^{+\infty} B_p(t)\, t^{-\nu}\, dt = \frac{1}{\nu - 1} \int_0^{+\infty} b(t)\, t^{1-\nu}\, dt$$

$$\int_0^{+\infty} [B_p(t) - B_p(0)]\, t^{-\varrho}\, dt = \frac{1}{\varrho - 1} \int_0^{+\infty} b(t)\, t^{1-\varrho}\, dt\,,$$

since, on account of estimates (3.15), (3.16) and of the property $(1 + r)^{1/p} b \in L^p(0, +\infty)$, the integrated terms vanish and the functions $t \to b(t)\, t^{1-\nu}$ and $t \to b(t)\, t^{1-\varrho}$ belong to $L^1(0, +\infty)$. Therefore conditions (7.27) are equivalent to conditions C1 and C4 listed in graph 1 in theorem 2.

Now we have to prove that $u_\theta(0) = a(0)$ for every $\theta \in [0, \alpha]$. For simplicity's sake we limit ourselves to considering the previous case $m = 2$ and $(\nu, \varrho) \in \text{set } 4$.

Taking into account the compatibility conditions (7.27) and the homogeneity of the kernels $\tilde{H}$, $\tilde{K}$, H_1, K_1, formula (7.17) can be rewritten as follows:

$$u_\theta(r) = \frac{1}{\alpha}\, a(0) \int_0^{+\infty} \tilde{H}(1, \theta, t)\, dt + \frac{1}{\alpha}\, B_p(0) \int_0^{+\infty} \tilde{K}(1, \theta, t)\, dt +$$

$$+ \frac{1}{\alpha} \int_0^{+\infty} H_1(1, \theta, t)[a(rt) - a(0)]\, t^{-1}\, dt + \frac{1}{\alpha} \int_0^{+\infty} K_1(1, \theta, t)[B_p(rt) - B_p(0)]\, t^{-1}\, dt +$$

$$+ \chi(\theta)\, B_p(r) \cos \omega\,.$$

The wanted equation $u_\theta(0) = a(0)$ is a consequence of the following equations

$$\frac{1}{\alpha}\int_0^{+\infty}\tilde{H}(1,\theta,t)dt=1 \qquad\qquad \theta\in[0,\alpha)$$

$$\frac{1}{\alpha}\int_0^{+\infty}\tilde{K}(1,\theta,t)\,\mathrm{d}t=\begin{cases}-\cos\omega & \theta=0\\ 0 & \theta\in(0,\alpha)\,,\end{cases}$$

of the fact that the functions a and B_p are Hölder continuous with exponent $1/p'$, of the fact that the functions $t\to H(1,\theta,t)t^{-1/p}$ and $t\to K(1,\theta,t)t^{-1/p}$ belong to $L^1(0,+\infty)$ respectively for every $\theta\in[0,\alpha)$ and for every $\theta\in(0,\alpha)$, and, finally, of formula (7.23) in lemma 7.7.

This concludes the proof of lemma 7.1.

PROOF OF LEMMA 7.6. For simplicity's sake we limit ourselves to proving the latter inequality in (7.18 bis), since the former can be proved likewise. Observe that under our hypotheses $f\in W^{1,p}(0,1)$: then from the classical equation

$$f(r)-f(0)=\int_0^r f'(t)\,dt\,,$$

using Hölder and Hardy's inequalities, it is easy to derive the following chain of inequalities

$$\int_0^{+\infty}|f(r)-f(0)|\,r^{-\delta}\,dr\leqslant\int_0^1 r^{1-\delta}\Big(r^{-1}\int_0^r|f'(t)|dt\Big)dr+\int_1^{+\infty}|f(r)|\,r^{-\delta}\,dr+$$
$$+|f(0)|\int_1^{+\infty}r^{-\delta}\,dr\leqslant\Big(\int_0^1 r^{(1-\delta)p'}dr\Big)^{1/p'}\Big(\int_0^{+\infty}\Big|r^{-1}\int_0^r|f'(t)|\,dt\Big|^p dr\Big)^{1/p}+$$
$$+\Big(\int_1^{+\infty}r^{(\gamma-\delta)p'}\,dr\Big)^{1/p'}\Big(\int_0^{+\infty}|f(r)|^p\,r^{-\gamma p}\,dr\Big)^{1/p}+\frac{|f(0)|}{\delta-1}=$$
$$=\frac{p}{[(1-\delta)p'+1]^{1/p'}(p-1)}\,\|f'\|_{L^p(0,+\infty)}+$$
$$+\frac{1}{[(\delta-\gamma)p'+1]^{1/p'}}\,\|r^{-\gamma}f\|_{L^p(0,+\infty)}+\frac{|f(0)|}{\delta-1}\,.$$

Now we remark that, since $f \in W^{1,p}(0,1)$, there exists a positive constant C, independent of f, such that

$$|f(0)| \leqslant C\left[\|f\|_{L^p(0,1)} + \|f'\|_{L^p(0,1)}\right];$$

from such an estimate it is easy to derive the following one

$$|f(0)| \leqslant C\left[\|r^{-\gamma} f\|_{L^p(0,+\infty)} + \|f'\|_{L^p(0,+\infty)}\right]$$

which, together with the previous chain of inequalities, proves the assertion.

PROOF OF LEMMA 7.7. We limit ourselves to proving the assertion for the function F_2, that we denote by F. We drop also the subscript to γ_2: then we have

$$\gamma \in \left(\frac{1}{p'}, \frac{1}{p'} + 2\beta\right). \tag{7.28}$$

In order to calculate the first derivative of F (7), let φ be any function in $C_0^\infty(0, +\infty)$; moreover define the function $\Phi \in C_0^\infty(0, +\infty)$ as follows:

$$\Phi(r) = r^{\gamma/\beta - 1}\varphi(r^{1/\beta}).$$

Now, using simple changes of variables in the integrals, the Poincaré-Bertrand formula (see [10]) and well-known properties of the Hilbert transformation, we get the chain of equations:

$$\int_0^{+\infty} \varphi'(r) F(r)\, dr = \frac{1}{\beta}\int_0^{+\infty} \varphi'(r)\, dr \int_0^{+\infty} \frac{r^\gamma t^{1-\gamma/\beta}}{r^{2\beta} - t^2}[f(t^{1/\beta}) - f(0)]\, dt =$$

$$= \frac{1}{\beta^2}\int_0^{+\infty} \varphi'(r^{1/\beta}) r^{1/\beta - 1}\, dr \int_0^{+\infty} \frac{r^{\gamma/\beta} t^{1-\gamma/\beta}}{r^2 - t^2}[f(t^{1/\beta}) - f(0)]\, dt =$$

$$= \frac{1}{\beta}\int_0^{+\infty} [f(t^{1/\beta}) - f(0)]\, t^{1-\gamma/\beta}\cdot$$

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(7) Such a function is well defined, since f is Hölder continuous with exponent $1/p'$.

$$\cdot\left\{\int_0^{+\infty}\frac{r}{r^2-t^2}\,\frac{d}{dr}\left[r^{\gamma/\beta-1}\varphi(r^{1/\beta})\,dr-\left(\frac{\gamma}{\beta}-1\right)\right]\int_0^{+\infty}\frac{r^{\gamma/\beta-1}}{r^2-t^2}\varphi(r^{1/\beta})\,dr\right\}dt=$$

$$=\frac{1}{\beta}\int_0^{+\infty}[f(t^{1/\beta})-f(0)]\,t^{1-\gamma/\beta}\left\{\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\Phi'(|r|)}{r-t}\,dr+\left(1-\frac{\gamma}{\beta}\right)\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr\right\}dt=$$

$$=\frac{1}{\beta}\int_0^{+\infty}[f(t^{1/\beta})-f(0)]\,t^{1-\gamma/\beta}\cdot$$

$$\cdot\left\{\frac{1}{2}\,\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{\Phi(|r|)\,\operatorname{sgn} r}{r-t}\,dr+\left(1-\frac{\gamma}{\beta}\right)\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr\right\}dt=$$

$$=\frac{1}{\beta}\int_0^{+\infty}[f(t^{1/\beta})-f(0)]\,t^{1-\gamma/\beta}\cdot$$

$$\cdot\left\{\frac{d}{dt}\left[t\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr\right]+\left(1-\frac{\gamma}{\beta}\right)\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr\right\}dt=$$

$$=\frac{1}{\beta}\int_0^{+\infty}[f(t^{1/\beta})-f(0)]\,\frac{d}{dt}\left[t^{2-\gamma/\beta}\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr\right]dt=$$

$$=\frac{1}{2\beta}\int_0^{+\infty}[f(t^{1/\beta})-f(0)]\,\frac{d}{dt}\left[t^{1-\gamma/\beta}\int_{-\infty}^{+\infty}\frac{\Phi(|r|)\,\operatorname{sgn} r}{r-t}\,dr\right]dt\,.$$

If we denote by Ψ the Hilbert transform of the function $r\to\Phi(|r|)\operatorname{sgn} r$, since the latter function is in $C_0^\infty(-\infty,+\infty)$, then, according to Privalov's theorem (see [15]), Ψ is Hölder continuous in $(-\infty,+\infty)$ with any exponent $\lambda\in(0,1)$. Moreover from the identity

$$t\Psi(t)=\int_{-\infty}^{+\infty}\frac{|r|\,\Phi(|r|)}{\pi(t-r)}\,dr$$

it follows that $t\Psi\in L^\infty(-\infty,+\infty)$, recalling that every function in $W^{1,p}(-\infty,+\infty)$ is bounded.

Integrating by parts and taking advantage of property (7.28), it is easy to realize that the integrated terms vanish, so that we obtain the chain of equations:

$$\int_0^{+\infty}\varphi'(r)F(r)\,dr=-\frac{1}{\beta}\int_0^{+\infty}f'(t^{1/\beta})\,t^{(1-\gamma)/\beta+1}\,dt\int_0^{+\infty}\frac{\Phi(r)}{r^2-t^2}\,dr=$$

$$=-\int_0^{+\infty}f'(t)\,t^{2\beta-\gamma}\,dt\int_0^{+\infty}\frac{r^{\gamma-1}\varphi(r)}{r^{2\beta}-t^{2\beta}}\,dr=-\int_0^{+\infty}\varphi(r)\,dr\int_0^{+\infty}\frac{r^{\gamma-1}t^{2\beta-\gamma}}{r^{2\beta}-t^{2\beta}}f'(t)\,dt$$

that proves formula (7.24). Moreover an application of lemma 5.1 shows that the linear mapping $f'\to F'$ is continuous from $L^p(0,+\infty)$ into itself.

Let us compute the trace of F at $r=0$ under the additional hypothesis $\gamma<2\beta$. Using the formula

$$\int_0^{+\infty}\frac{r^{\gamma}t^{2\beta-\gamma-1}}{r^{2\beta}-t^{2\beta}}\,dt=-\alpha\operatorname{cotg}\left(\frac{\gamma\pi}{2\beta}\right),$$

where the integral is taken in the Cauchy principal value sense, we get the identity

$$F(r)=\int_0^{+\infty}\frac{r^{\gamma}t^{2\beta-\gamma-1}}{r^{2\beta}-t^{2\beta}}[f(t)-f(r)]\,dt-\alpha[f(r)-f(0)]\operatorname{cotg}\left(\frac{\gamma\pi}{2\beta}\right).$$

We observe that the integral appearing in the right hand side is an ordinary Lebesgue integral, since f is Hölder continuous with exponent $1/p'$ and it converges to 0 as $r\to 0$ on account of the estimate

$$\int_0^{+\infty}\frac{r^{\gamma}t^{2\beta-\gamma-1}}{|r^{2\beta}-t^{2\beta}|}|f(t)-f(r)|\,dt\leqslant\|f'\|_{L^p(0,+\infty)}\int_0^{+\infty}\frac{r^{\gamma}t^{2\beta-\gamma-1}|t-r|^{1/p'}}{|r^{2\beta}-t^{2\beta}|}\,dt=$$

$$=r^{1/p'}\|f'\|_{L^p(0,+\infty)}\int_0^{+\infty}\frac{t^{2\beta-\gamma-1}|t-1|^{1/p'}}{|t^{2\beta}-1|}\,dt.$$

This concludes the proof of the lemma.

PROOF OF LEMMA 7.2. We divide the proof of the lemma into two parts; in the first one we prove by induction on s that the functions u'_θ belong to $W^{s-2,p}(0,+\infty)$ for every $\theta\in[0,\alpha]$ and that their derivatives are given by formulas (7.7), if, and only if, the conditions listed in graph 2 in theorem 2 are verified. In the second part we prove that the traces of the derivatives of u_θ are given by one of formulas (7.8), (7.9), (7.10).

Now we shall prove the first part of the lemma: that is, we shall suppose that the first part of the lemma holds for s and we shall prove that it holds also for $s+1$, after observing that it obviously holds for $s=1$ on account of lemma 7.1. Actually we have to show only that the function

$$u_\theta^{(s-1)}(r)=\frac{1}{\alpha}\int_0^{+\infty}H_{s-1}(r,\theta,t)\,a^{(s-1)}(t)dt+\frac{1}{\alpha}\int_0^{+\infty}K_{s-1}(r,\theta,t)b^{(s-2)}(t)dt+ \\ +\chi(\theta)\,b^{(s-2)}(r)\cos\omega \text{ [8]} \tag{7.29}$$

belongs to $L^p(0,+\infty)$, if, and only if, the conditions listed in graph 2 corresponding to $j=s-1$ are verified. Such a proof depends essentially on lemmas 7.8 and 7.9 stated below (that we shall prove later on): however, for simplicity's sake we shall limit ourselves to considering the case $n_{s-1}=2$, $\sigma_{s-1}=1$ and (consequently) $\tau_{s-1}\in(-1,1+1/p')$ [9], since the remaining cases can be treated in a similar way.

LEMMA 7.8. *The functions H_j and K_j, defined by* (7.11), *verify the following recursive relations*:

$$\left\{\begin{aligned} H_j(r,\theta,t)&=\sum_{v=0}^{n_j-1}(-1)^{N_j+1+v}r^{\sigma_j-1+2v\beta}t^{-\sigma_j-2v\beta}\cdot\\ &\qquad\cdot\sin[(\sigma_j+j-1+2v\beta)\theta-\omega]+\frac{r}{t}H_{j+1}(r,\theta,t)\\ K_j(r,\theta,t)&=\sum_{v=0}^{n_j-1}r^{\sigma_j-1+2v\beta}t^{-\sigma_j-2v\beta}\cdot\\ &\qquad\cdot\sin[(\sigma_j+j-1+2v\beta)\theta-\omega]+\frac{r}{t}K_{j+1}(r,\theta,t)\,. \end{aligned}\right. \tag{7.30}$$

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[8] We recall that the function χ is defined by (7.4).

[9] See definitions (3.2), (3.12), (3.13) and inequalities (3.17), (3.19) with $j=s-1$.

Moreover, if (p, α, ω) *satisfies* (7.6), *the following properties hold*:

i) *The functions* $t \to H_j(1, \theta, t)\, t^{-1/p}$ *and* $t \to K_j(1, \theta, t)\, t^{-1/p}$ *belong to* $L^1(0, +\infty)$ *respectively for every* $\theta \in [0, \alpha)$ *and for every* $\theta \in (0, \alpha)$;

ii) *the function* $(r, t) \to K_j(r, 0, t)$ *verifies the hypotheses of lemma* 5.1;

iii) *the functions* $t \to H_j(1, \theta, t)$ *and* $t \to K_j(1, \theta, t)$ *belong to* $L^1(0, +\infty)$ *respectively for every* $\theta \in [0, \alpha)$ *and for every* $\theta \in (0, \alpha)$, *if, and only if,*

$$\sigma_j \in (1, 1 + 2\beta)\ ; \tag{7.31}$$

iv) *the function* $t \to K_j(1, 0, t)$ *is integrable over* $(0, +\infty)$ *in the Cauchy principal value sense, if, and only if,* (7.31) *is verified*;

v) *if* (7.31) *is verified, then the following equations hold*:

$$\begin{cases} \displaystyle\int_0^{+\infty} H_j(1, \theta, t)\, dt = \alpha \frac{\sin(j\theta - \omega)}{\sin(j\alpha - \omega)} & \theta \in [0, \alpha) \\ \displaystyle\int_0^{+\infty} K_j(1, \theta, t)\, dt = \begin{cases} \alpha \operatorname{cotg}(j\alpha - \omega) \sin \omega & \theta = 0 \\ \alpha \dfrac{\sin[j(\alpha - \theta)]}{\sin(j\alpha - \omega)} & \theta \in (0, \alpha); \end{cases} \end{cases} \tag{7.32}$$

vi) *the functions* $t \to H_{j+1}(1, \theta, t)\, t^{-1}$ *and* $t \to K_{j+1}(1, \theta, t)\, t^{-1}$ *belong to* $L^1(0, +\infty)$ *respectively for every* $\theta \in [0, \alpha)$ *and for every* $\theta \in (0, \alpha)$ *if, and only if,*

$$\sigma_{j+1} \in (0, 2\beta)\ ; \tag{7.33}$$

vii) *the function* $t \to K_{j+1}(1, 0, t)\, t^{-1}$ *is integrable over* $(0, +\infty)$ *in the Cauchy principal value sense, if, and only if,* (7.33) *is verified*;

viii) *if* (7.33) *is verified, then the following equations hold*:

$$\begin{cases} \displaystyle\int_0^{+\infty} H_{j+1}(1, \theta, t)\, t^{-1}\, dt = \alpha \frac{\sin(j\theta - \omega)}{\sin(j\alpha - \omega)} & \theta \in [0, \alpha) \\ \displaystyle\int_0^{+\infty} K_{j+1}(1, \theta, t)\, t^{-1}\, dt = \begin{cases} \alpha \operatorname{cotg}(j\alpha - \omega) \sin \omega & \theta = 0 \\ \alpha \dfrac{\sin[j(\alpha - \theta)]}{\sin(j\alpha - \omega)} & \theta \in (0, \alpha). \end{cases} \end{cases} \tag{7.34}$$

LEMMA 7.9. *Let $f \in W^{\gamma,p}(0, +\infty)$ and $\gamma \in (0, 1]$. Then the following estimates hold:*

$$(7.35)\qquad \begin{cases} \displaystyle\int_0^{+\infty} |f(t)|\, t^{-\delta}\, dt \leqslant C_1 \|f\|_{W^{\gamma,p}(0,+\infty)} & \text{if } \delta \in \left(\dfrac{1}{p'}, \min\left(\dfrac{1}{p'} + \gamma, 1\right)\right) \\ \displaystyle\int_0^{+\infty} |f(t) - f(0)|\, t^{-\delta}\, dt \leqslant C_2 \|f\|_{W^{\gamma,p}(0,+\infty)} & \\ & \text{if } \gamma \in \left(\dfrac{1}{p}, 1\right],\ \delta \in \left(1, \dfrac{1}{p'} + \gamma\right) \end{cases}$$

where C_1 and C_2 are positive constants depending only on (p, γ, δ).

Now we go on with the proof of the lemma. We begin by observing that the functions $t \to H_{s-1}(r, \theta, t)$ and $t \to K_{s-1}(r, \theta, t)$ are integrable [10] over bounded intervals in $(0, +\infty)$ for every $r \in (0, +\infty)$ and $\theta \in [0, \alpha)$.

Suppose now that φ is a function in $C_0^\infty(-\infty, +\infty)$ with the following properties:

$$(7.36)\qquad \begin{cases} 0 \leqslant \varphi(r) \leqslant 1 & 0 \leqslant r\,, \\ \varphi(r) = 1 & 0 \leqslant r \leqslant 1\,, \\ \varphi(r) = 0\,, & 2 \leqslant r\,, \end{cases}$$

and define φ_ε for every $\varepsilon > 0$ as follows:

$$\varphi_\varepsilon(t) = \varphi(\varepsilon t)\,.$$

Applying identities (7.30) and lemma 7.9, with $\gamma = 1$, and recalling that in our case $n_{s-1} = 2$, $\sigma_{s-1} = 1$ and $\tau_{s-1} = 1 + 2\beta \in (1, 1 + 1/p')$, we

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([10]) « Integrable » means here Lebesgue integrable, except for the functions $t \to K_{s-1}(r, 0, t)$ for which it means, as usual, integrable in the Cauchy principal value sense.

get the identity

$$(7.37)\qquad u_\theta^{(s-1)}(r)=a^{(s-1)}(0)\int_0^{+\infty}H_{s-1}(r,\theta,t)\,\varphi_\varepsilon(t)\,dt+b^{(s-2)}(0)\cdot$$

$$\cdot\int_0^{+\infty}K_{s-1}(r,\theta,t)\,\varphi_\varepsilon(t)\,dt-\frac{1}{\alpha}\sum_{v=0}^{1}r^{2v\beta}\sin\left[(s-1+2v\beta)\theta-\omega\right]\cdot$$

$$\cdot\int_0^{+\infty}\{(-1)^{N_{s-1}+v}[a^{(s-1)}(t)-a^{(s-1)}(0)\,\varphi_\varepsilon(t)]-[b^{(s-2)}(t)-b^{(s-2)}(0)\,\varphi_\varepsilon(t)]\}\cdot$$

$$\cdot\frac{dt}{t^{1+2v\beta}}+\frac{1}{\alpha}A_\theta(r)+\frac{1}{\alpha}b_\theta(r)+\chi(\theta)\,b^{(s-2)}(r)\cos\omega$$

where

$$A_\theta(r)=\int_0^{+\infty}\frac{r}{t}H_s(r,\theta,t)[a^{(s-1)}(t)-a^{(s-1)}(0)\,\varphi_\varepsilon(t)]\,dt$$

$$b_\theta(r)=\int_0^{+\infty}\frac{r}{t}K_s(r,\theta,t)[b^{(s-2)}(t)-b^{(s-2)}(0)\,\varphi_\varepsilon(t)]\,dt\,.$$

Taking again into account identities (7.30) and recalling i) and ii) in lemma 7.8 and lemmas 5.1, 5.5, 7.9 it is easy to realize that the functions A_θ and b_θ satisfy the following estimate for every $R\in(0,+\infty)$ and $\theta\in[0,\alpha)$:

$$\max\{\|A_\theta\|_{L^p(0,R)},\ \|b_\theta\|_{L^p(0,R)}\}\leqslant C(R)[\|a^{(s-1)}\|_{W^{1,p}(0,+\infty)}+\|b^{(s-2)}\|_{W^{1,p}(0,+\infty)}]$$

where $C(R)$ is a positive constant depending on R and also on $(\theta, p, \alpha, \omega, \varepsilon)$. Repeating the arguments used in the proof of lemma 7.1, when $\theta\in(0,\alpha)$, and applying also lemma 7.7, when $\theta=0$ ([11]), we may differentiate (7.37) and we obtain the identity

$$(7.38)\qquad u_\theta^{(s)}(r)=I_\theta(r)+\frac{1}{\alpha}\int_0^{+\infty}H_s(r,\theta,t)[a^{(s)}(t)-a^{(s-1)}(0)\,\varphi'_\varepsilon(t)]\,dt+$$

$$+\frac{1}{\alpha}\int_0^{+\infty}K_s(r,\theta,t)[b^{(s-1)}(t)-b^{(s-2)}(0)\,\varphi'_\varepsilon(t)]\,dt+\chi(\theta)\,b^{(s-1)}(r)\cos\omega$$

([11]) In fact, on account of inequality (3.17), $\sigma_s\in(1/p',\,1/p'+2\beta)$.

where

$$(7.39)\qquad I_\theta(r)=\frac{a^{(s-1)}(0)}{r}\int_0^{+\infty}H_{s-1}(r,\theta,t)\,\varphi_\varepsilon'(t)\,t\,dt+$$

$$+\frac{b^{(s-2)}(0)}{r}\int_0^{+\infty}K_{s-1}(r,\theta,t)\,\varphi_\varepsilon'(t)\,t\,dt+\frac{2\beta}{\alpha}r^{2\beta-1}\sin[(s-1+2\beta)\theta-\omega]\cdot$$

$$\cdot\int_0^{+\infty}\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)\,\varphi_\varepsilon(t)]+b^{(s-2)}(t)-b^{(s-2)}(0)\,\varphi_\varepsilon(t)\}\frac{dt}{t^{1+2\beta}}\,.$$

In order to differentiate the first two integrals in the right hand side of (7.37) we have taken advantage of the fact that the kernels H_{s-1} and K_{s-1} are homogeneous of degree -1 and integrable over bounded intervals. Now, using properties i) and ii) in lemma 7.8 and recalling lemmas 5.1 and 5.5, we infer that a necessary condition in order that $u_\theta^{(s)}\in L^p(0,+\infty)$ for every $\theta\in[0,\alpha)$ is that I_θ possesses such a property. From the equations (that hold for every $\theta\in(0,\alpha)$)

$$(-1)^{N_{s-1}+1}\lim_{r\to0}\int_0^{+\infty}H_{s-1}(r,\theta,t)\,\varphi_\varepsilon'(t)\,t\,dt=$$

$$=\lim_{r\to0}\int_0^{+\infty}K_{s-1}(r,\theta,t)\,\varphi_\varepsilon'(t)\,t\,dt=\sin[(s-1)\theta-\omega]$$

and from (7.39) it follows easily that, if I_θ belongs to $L^p(0,+\infty)$ for every $\theta\in[0,\alpha)$, then the following conditions are to be satisfied:

$$(7.40)\qquad\left\{\begin{aligned}&(-1)^{N_{s-1}}a^{(s-1)}(0)-b^{(s-2)}(0)=0\\&\int_0^{+\infty}\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)\,\varphi_\varepsilon(t)]+\\&\qquad\qquad+b^{(s-2)}(t)-b^{(s-2)}(0)\,\varphi_\varepsilon(t)\}\frac{dt}{t^{1+2\beta}}=0\,.\end{aligned}\right.$$

Since lemma 7.9 assures that the function

$$t\to\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)]+b^{(s-2)}(t)-b^{(s-2)}(0)\}t^{-1-2\beta}$$

belongs to $L^1(0, +\infty)$ and

$$\lim_{\varepsilon\to 0}\int_0^{+\infty}[1-\varphi_\varepsilon(t)]\frac{dt}{t^{1+2\beta}}=0$$

we can replace in (7.40) φ_ε with 1, so that conditions (7.40) become the wanted conditions $C_{s-1}5$ and $C_{s-1}4$.

In order to prove the sufficiency of $C_{s-1}4$ and $C_{s-1}5$ observe that they imply the identity

$$(7.41)\qquad u_\theta^{(s-1)}(r)=b^{(s-2)}(0)\int_0^{+\infty}[(-1)^{N_{s-1}}H_{s-1}(r,\theta,t)+K_{s-1}(r,\theta,t)]\,dt-$$
$$-\frac{1}{\alpha}\sin[(s-1)\theta-\omega]\int_0^{+\infty}[(-1)^{N_{s-1}}a^{(s-1)}(t)-b^{(s-2)}(t)]\frac{dt}{t}+\frac{1}{\alpha}A_\theta(r)+$$
$$+\frac{1}{\alpha}b_\theta(r)+\chi(\theta)\,b^{(s-2)}(r)\cos\omega$$

where, now, the functions A_θ and b_θ are so defined:

$$(7.42)\qquad\begin{cases} A_\theta(r)=\displaystyle\int_0^{+\infty}\frac{r}{t}H_s(r,\theta,t)[a^{(s-1)}(t)-a^{(s-1)}(0)]\,dt\\[2ex] b_\theta(r)\;=\displaystyle\int_0^{+\infty}\frac{r}{t}K_s(r,\theta,t)[b^{(s-2)}(t)-b^{(s-2)}(0)]\,dt;\end{cases}$$

we remark that they have properties analogous to the ones of the functions A_θ and b_θ defined by (7.18). Finally we observe that the function

$$(-1)^{N_{s-1}}H_{s-1}(r,\theta,t)+K_{s-1}(r,\theta,t)$$
$$=\frac{2r^{2\beta}t^{2\beta-1}\{t^{4\beta}\sin[(s-1+2\beta)\theta-\omega]-r^{4\beta}\sin[(s-1-2\beta)\theta-\omega]\}}{r^{8\beta}-2r^{4\beta}t^{4\beta}\cos(4\beta\theta)+t^{8\beta}}$$

is integrable over $(0,+\infty)$ for every $\theta\in[0,\alpha)$.

Differentiating (7.41) and recalling that the first integral in (7.41) does not depend on r, since the kernel is a function homogeneous of degree -1 in (r, t), we get that for every $\theta \in [0, \alpha)$

$$u_\theta^{(s)}(r) = \frac{1}{\alpha}\int_0^{+\infty} H_s(r,\ \theta,\ t)\, a^{(s)}(t)\, dt + \\ + \frac{1}{\alpha}\int_0^{+\infty} K_s(r,\ \theta,\ t)\, b^{(s-1)}(t)\, dt + \chi(\theta)\ b^{(s-1)}(r)\cos\omega$$

that assures that $u_\theta^{(s)} \in L^p(0, +\infty)$ for every $\theta \in [0, \alpha]$ and it is represented by formula (7.7).

This concludes the proof by induction.

Now we have to prove the formula for the traces of $u_\theta^{(j)}$. We limit ourselves to observing that they can be proved by taking advantage of equations (7.7), identities (7.30), formulas (7.32), (7.34) and compatibility conditions listed in graph 2. For instance, we shall calculate the trace at $r = 0$ of the function $u_\theta^{(j)}$ in the case $n_j = 2$, $\sigma_j = 1$, similar to the one that we have just considered. From formula (7.7) it follows easily, after a simple change of variable, recalling that the functions $a^{(j)}$ and $b^{(j-1)}$ are Hölder continuous with exponent $1/p'$ and that i) and ii) in lemma 7.8 hold, that

$$\lim_{r\to 0} A_\theta(r) = \lim_{r\to 0} b_\theta(r) = 0 \qquad \theta \in [0, \alpha)\,.$$

Therefore, formula (7.9) is a consequence of the equations

$$(7.43)\quad \begin{cases} \lim\limits_{r\to 0} u_\theta^{(j)}(r) = \dfrac{1}{\alpha}\, b^{(j-1)}(0)\displaystyle\int_0^{+\infty} [(-1)^{N_j} H_j(1, \theta, t) + K_j(1, \theta, t)]\, dt - \\ \quad - \dfrac{1}{\alpha}\sin(j\theta - \omega)\displaystyle\int_0^{+\infty} [(-1)^{N_j} a^{(j)}(t) - b^{(j-1)}(t)]\,\dfrac{dt}{t} + \chi(\theta)\, b^{(j-1)}(0)\cos\omega \\ \displaystyle\int_0^{+\infty} [(-1)^{N_j} H_j(1, \theta, t) + K_j(1, \theta, t)]\, dt = \\ \qquad = \begin{cases} 0 & \theta = 0 \\ \alpha\cos(j\theta - \omega) & \theta \in (0, \alpha)\,. \end{cases} \end{cases}$$

This concludes the proof of lemma 7.2.

PROOF OF LEMMA 7.8. It follows easily from definitions (7.11), (3.12), inequality (3.17), equation (2.7) and from the formula

$$\int_0^{+\infty} \frac{t^{2\beta-\sigma_j}}{1-t^{2\beta}}\,dt = \alpha \operatorname{cotg}(j\alpha-\omega) \qquad \sigma_j \in (1, 1+2\beta)\,,$$

where the integral is taken in the Cauchy principal value sense.

PROOF OF LEMMA 7.9. We observe that the first of estimates (7.35), when $\gamma \in (0, 1/p)$, is an immediate consequence of lemma 5.3 and Hölder's inequality. While, if $\gamma \in (1/p, 1)$, such an estimate follows from Hölder's inequality and from the fact that under such a hypothesis f is Hölder continuous with exponent $\gamma - 1/p$. Finally, if $\gamma = 1/p$ and δ is any (fixed) number in $(1/p', 1)$, we may choose $\gamma' \in (0, 1/p)$ so that $\delta \in (1/p', 1/p' + \gamma')$. Hence, for the previous results, the first estimate in (7.35) holds with γ replaced by γ'. Since $W^{\gamma',p}(0, +\infty) \supseteq W^{\gamma,p}(0, +\infty)$ with a continuous embedding, the assertion follows.

As far as the latter estimate in (7.35) is concerned, we observe that it is an immediate consequence of Hölder's inequality and Hölder continuity of f with exponent $\gamma - 1/p$.

PROOF OF LEMMA 7.3. We have to show that the function $u_\theta^{(s-1)}$, defined by (7.7), belongs to $W^{1/p',p}(0, +\infty)$, if, and only if, the conditions listed in graph 3 in theorem 2 are satisfied. The proof depends essentially on lemma 7.10 stated below: for simplicity's sake we shall limit ourselves to considering the case $q_s = 2$, $\sigma_{s-1} = 1$, $\tau_{s-1} \in (1, 2/p')$ [12], that is the most complex, observing that the remaining cases can be treated in a similar way.

LEMMA 7.10. *If H_s^* and K_s^* are defined by formulas* (7.11), *with $j = s$ and (N_s, σ_s) replaced by (M_s, σ_s^*)* [13], *then the following identities*

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[12] For the definitions of q_s, σ_{s-1}, τ_{s-1} see formulas (3.3), (3.12), (3.13).
[13] For the definitions of M_s, σ_s^* see formulas (3.4), (3.14).

hold:

$$(7.44)\quad \begin{cases} H_{s-1}(r,\theta,t) = \sum_{v=0}^{q_s-1}(-1)^{N_{s-1}+1+v} r^{\sigma_{s-1}-1+2v\beta} t^{-\sigma_{s-1}-2v\beta} \cdot \\ \qquad \cdot \sin[(\sigma_{s-1}+s-2+2v\beta)\theta-\omega] + \dfrac{r}{t} H_s^*(r,\theta,t) \\ K_{s-1}(r,\theta,t) = \sum_{v=0}^{q_s-1} r^{\sigma_{s-1}-1+2v\beta} t^{-\sigma_{s-1}-2v\beta} \cdot \\ \qquad \cdot \sin[(\sigma_{s-1}+s-2+2v\beta)\theta-\omega] + \dfrac{r}{t} K_s^*(r,\theta,t)\,. \end{cases}$$

Moreover, if (p,α,ω) *satisfies* (7.6), *with* $j = s-1$, *and* (7.12), *the following properties hold*:

i) *the functions* $t \to H_s^*(1,\theta,t)\,t^{-2/p}$ *and* $t \to K_s^*(1,\theta,t)\,t^{-2/p}$ *belong to* $L^1(0,+\infty)$ *respectively for every* $\theta \in [0,\alpha)$ *and for every* $\theta \in (0,\alpha)$;

ii) *the function* $(r,t) \to K_s^*(r,0,t)$ *verifies the hypotheses of lemma* 5.1;

iii) *the functions* $t \to H_s^*(1,\theta,t)\,t^{-1}$ *and* $t \to K_s^*(1,\theta,t)\,t^{-1}$ *belong to* $L^1(0,+\infty)$ *respectively for every* $\theta \in [0,\alpha)$ *and for every* $\theta \in (0,\alpha)$, *if, and only if,*

$$(7.45)\qquad \sigma_s^* \in (0, 2\beta)\;;$$

iv) *the function* $t \to K_s^*(1,0,t)\,t^{-1}$ *is integrable over* $(0,+\infty)$ *in the Cauchy principal value sense, if, and only if,* (7.45) *is verified*;

v) *if* (7.45) *is verified then the following equations hold*:

$$(7.46)\quad \begin{cases} \displaystyle\int_0^{+\infty} H_s^*(1,\theta,t)\,t^{-1}\,dt = \alpha\,\frac{\sin[(s-1)\theta-\omega]}{\sin[(s-1)\alpha-\omega]} & \theta \in [0,\alpha) \\ \displaystyle\int_0^{+\infty} K_s^*(1,\theta,t)\,t^{-1}\,dt = \begin{cases} \alpha \operatorname{cotg}[(s-1)\alpha-\omega]\sin\omega & \theta = 0 \\ \alpha\,\dfrac{\sin[(s-1)(\alpha-\theta)]}{\sin[(s-1)(\alpha-\omega)]} & \theta \in (\theta,\alpha)\,. \end{cases} \end{cases}$$

PROOF. It follows easily from definition (3.14), inequality (3.21), equation (2.7) and from the following formula

$$\int_0^{+\infty} \frac{t^{2\beta-\sigma_s^*}}{1-t^{2\beta}}\,dt = \alpha \operatorname{cotg}[(s-1)\alpha-\omega]\,,$$

where the integral is taken in the Cauchy principal value sense.

Now we go on with the proof of lemma 7.3. We observe that in our case $p \in (2, +\infty)$. Hence the functions $a^{(s-1)}$ and $b^{(s-2)}$, which belong to $W^{1/p',p}(0, +\infty)$, are Hölder continuous with exponent $1/p' - 1/p$ and, therefore, they possess traces at $r = 0$. Then, if φ is a function in $C_0^\infty(-\infty, +\infty)$ with properties (7.36), from lemmas 7.10 and 7.9, recalling that $\sigma_{s-1} = 1$ and that $\tau_{s-1} = \sigma_{s-1} + 2\beta \in (1, 2/p')$, we infer the identity, valid for every $\theta \in [0, \alpha)$:

$$u_\theta^{(s-1)}(r) = -\frac{\sin[(s-1)\theta - \omega]}{\alpha} \cdot$$

$$\cdot \int_0^{+\infty} \{(-1)^{N_{s-1}}[a^{(s-1)}(t) - a^{(s-1)}(0)\varphi(t)] - [b^{(s-2)}(t) - b^{(s-2)}(0)\varphi(t)]\} \frac{dt}{t} +$$

$$+ \frac{1}{\alpha} r^{\tau_{s-1}-1} \sin[(s-1+2\beta)\theta - \omega] \cdot$$

$$\cdot \int_0^{+\infty} \{(-1)^{N_{s-1}}[a^{(s-1)}(t) - a^{(s-1)}(0)\varphi(t)] + b^{(s-2)}(t) - b^{(s-2)}(0)\varphi(t)\} \frac{dt}{t^{\tau_{s-1}}} +$$

$$+ \frac{1}{\alpha} A_\theta(r) + \frac{1}{\alpha} b_\theta(r) + \frac{1}{\alpha} a^{(s-1)}(0) \int_0^{+\infty} H_{s-1}(r, \theta, t)\varphi(t)\, dt +$$

$$+ \frac{1}{\alpha} b^{(s-2)}(0) \int_0^{+\infty} K_{s-1}(r, \theta, t)\varphi(t)\, dt + \chi(\theta)\, b^{(s-2)}(r) \cos\omega\,,$$

where the functions A_θ and b_θ are defined by the following equations

$$A_\theta(r) = \int_0^{+\infty} \frac{r}{t} H_s^*(r, \theta, t)[a^{(s-1)}(t) - a^{(s-1)}(0)\varphi(t)]\, dt$$

$$b_\theta(r) = \int_0^{+\infty} \frac{r}{t} K_s^*(r, \theta, t)[b^{(s-2)}(t) - b^{(s-2)}(0)\varphi(t)]\, dt$$

and χ is defined by (7.4).

Taking advantage of lemmas 7.10, 5.5 and 5.6, we infer that $u_\theta^{(s-1)} \in W^{1/p',p}(0, +\infty)$ for every $\theta \in [0, \alpha]$, if, and only if, the function

$$(7.47)\qquad I_\theta(r) = \frac{1}{\alpha} r^{\tau_{s-1}-1} \sin[(s-1+2\beta)\theta-\omega]\cdot$$
$$\cdot\int_0^{+\infty}\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)\varphi(t)]+b^{(s-2)}(t)-b^{(s-2)}(0)\varphi(t)\}\frac{dt}{t^{\tau_{s-1}}}+$$
$$+\frac{1}{\alpha}a^{(s-1)}(0)\int_0^{+\infty}[H_{s-1}(r,\theta,t)+(-1)^{N_{s-1}}K_{s-1}(r,\theta,t)]\varphi(t)\,\mathrm{d}t+$$
$$+\frac{1}{\alpha}[b^{(s-2)}(0)-(-1)^{N_{s-1}}a^{(s-1)}(0)]\int_0^{+\infty}K_{s-1}(r,\theta,t)\varphi(t)\,dt$$

has its own differential quotient in $L^p((0,+\infty)\times(0,+\infty))$. Observe that $I_\theta\in L^p_{\mathrm{loc}}([0,+\infty))$ for every $\theta\in[0,\alpha)$: hence, if it possesses the aforementioned property, then it admits a trace at $r=0$. Since the function $t\to H_{s-1}(1,\theta,t)+(-1)^{N_{s-1}}K_{s-1}(1,\theta,t)$ is integrable over $(0,+\infty)$ for every $\theta\in[0,\alpha)$, we get easily the formula

$$\lim_{r\to 0}\int_0^{+\infty}[H_{s-1}(r,\theta,t)+(-1)^{N_{s-1}}K_{s-1}(r,\theta,t)]\varphi(t)\,dt=$$
$$=\int_0^{+\infty}[H_{s-1}(1,\theta,t)+(-1)^{N_{s-1}}K_{s-1}(1,\theta,t)]\,dt$$

by using properties (7.36). Moreover observe that, except (at most) a finite number of θ's belonging to $(0,\alpha)$, we get the relation

$$(7.48)\qquad \lim_{r\to 0}\int_0^{+\infty}K_{s-1}(r,\theta,t)\varphi(t)\,dt=\infty\,.$$

In fact from the equation

$$\int_0^{+\infty}K_{s-1}(r,\theta,t)\varphi(t)\,dt=\sin[(s-1)\theta-\omega]\cdot$$
$$\cdot\int_0^{+\infty}\frac{t^{4\beta-1}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\varphi(t)\,dt-\sin[(s-1-2\beta)\theta-\omega]\cdot$$
$$\cdot\int_0^{+\infty}\frac{r^{2\beta}t^{2\beta-1}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\varphi(t)\,dt$$

and from the estimates

$$(1-|\cos(2\beta\theta)|)(r^{4\beta}+t^{4\beta})\leqslant r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}\leqslant$$
$$\leqslant(1+|\cos(2\beta\theta)|)(r^{4\beta}+t^{4\beta}),$$

recalling properties (7.36), we can derive the estimates

$$\int_0^{+\infty}\frac{t^{4\beta-1}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\varphi(t)\,dt\geqslant\frac{1}{1+|\cos(2\beta\theta)|}\int_0^1\frac{t^{4\beta-1}}{r^{4\beta}+t^{4\beta}}\,dt=$$
$$=\frac{1}{4\beta(1+|\cos(2\beta\theta)|)}\ln(1+r^{-4\beta})$$

$$\int_0^{+\infty}\frac{r^{2\beta}t^{2\beta-1}}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}}\varphi(t)\,dt\leqslant\frac{1}{1-|\cos(2\beta\theta)|}\cdot$$
$$\cdot\int_0^{+\infty}\frac{r^{2\beta}t^{2\beta-1}}{r^{4\beta}+t^{4\beta}}\,dt=\frac{\pi}{4\beta(1-|\cos(2\beta\theta)|)}$$

which prove (7.48).

Recalling, now, that $\tau_{s-1}>1$, a necessary condition for $\lim_{r\to0}I_\theta(r)$ to exist for every $\theta\in(0,\alpha)$ is that

$$b^{(s-2)}(0)-(-1)^{N_{s-1}}a^{(s-1)}(0)=0\,. \tag{7.49}$$

Taking advantage of this condition, we may write a new identity for $u_\theta^{(s-1)}$, that is

$$u_\theta^{(s-1)}(r)=-\frac{1}{\alpha}\sin[(s-1)\theta-\omega]\cdot \tag{7.50}$$
$$\cdot\int_0^{+\infty}[(-1)^{N_{s-1}}a^{(s-1)}(t)-b^{(s-2)}(t)]\frac{dt}{t}+\frac{1}{\alpha}r^{\tau_{s-1}-1}\sin[(s-1+2\beta)\theta-\omega]\cdot$$
$$\cdot\int_0^{+\infty}\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)]+b^{(s-2)}(t)-b^{(s-2)}(0)\}\frac{dt}{t^{\tau_{s-1}}}+\frac{1}{\alpha}\cdot$$

$$\cdot\int_0^{+\infty} \frac{r}{t} H_s^*(r,\theta,t)[a^{(s-1)}(t)-a^{(s-1)}(0)]\,dt+\frac{1}{\alpha}\cdot$$

$$\cdot\int_0^{+\infty} \frac{r}{t} K_s^*(r,\theta,t)[b^{(s-2)}(t)-b^{(s-2)}(0)]\,dt+\frac{1}{\alpha}a^{(s-1)}(0)\cdot$$

$$\cdot\int_0^{+\infty}[H_{s-1}(r,\theta,t)+(-1)^{N_{s-1}}K_{s-1}(r,\theta,t)]\,dt+\chi(\theta)\,b^{(s-2)}(r)\cos\omega\,.$$

Observing that the last integral in the right hand side does not depend on r, since the integrand is homogeneous of degree -1 in (r,t), and that each term, but the second one, has its own differential quotient in $L^p((0,+\infty)\times(0,+\infty))$, it follows that $u_\theta^{(s-1)}\in W^{1/p',p}(0,+\infty)$, if, and only if,

$$\int_0^{+\infty}\{(-1)^{N_{s-1}}[a^{(s-1)}(t)-a^{(s-1)}(0)]+b^{(s-2)}(t)-b^{(s-2)}(0)\}\frac{dt}{t^{\tau_{s-1}}}=0\,. \tag{7.51}$$

In conclusion we have proved that conditions (7.49) and (7.51) (i.e. conditions $C_{s-1}5$ and $C_{s-1}4$ in graph 3) are necessary and sufficient in order that $u_\theta^{(s-1)}\in W^{1/p',p}(0,+\infty)$.

Now, we prove that, supposing $p>2$, the trace of $u_\theta^{(s-1)}$ is given by formula either (7.8) or (7.9). We observe that such formulas can be obtained by taking advantage of equation (7.30), identities (7.44), equations (7.32), (7.46) and compatibility conditions in graph 3. We shall limit ourselves to calculating the trace of the function $u_\theta^{(s-1)}$ in the case $q_s=1$, $\sigma_{s-1}\in(1,2/p')$: then we get the identity

$$u_\theta^{(s-1)}(r)=\frac{1}{\alpha}\int_0^{+\infty}\frac{r}{t}H_s^*(r,\theta,t)[a^{(s-1)}(t)-a^{(s-1)}(0)]\,dt+ \tag{7.52}$$

$$+\frac{1}{\alpha}\int_0^{+\infty}\frac{r}{t}K_s^*(r,\theta,t)[b^{(s-2)}(t)-b^{(s-2)}(0)]\,dt+\frac{1}{\alpha}a^{(s-1)}(0)\int_0^{+\infty}H_{s-1}(r,\theta,t)\,dt+$$

$$+\frac{1}{\alpha}b^{(s-2)}(0)\int_0^{+\infty}K_{s-1}(r,\theta,t)\,dt+\chi(\theta)\,b^{(s-2)}(r)\cos\omega\,.$$

In fact the functions $t \to H_{s-1}(1, \theta, t)$ and $t \to K_{s-1}(1, \theta, t)$ are integrable over $(0, +\infty)$ on account of estimate (3.18) and property (7.31). Taking advantage of properties iii) and iv) in lemma 7.8, of the homogeneity of the kernels, of properties i) and ii) in lemma 7.10 and, finally, of the Hölder continuity with exponent $1/p' - 1/p$ of $a^{(s-1)}$ and $b^{(s-2)}$, taking the limit in (7.52) as $r \to 0$ and recalling equations (7.32), we get that

$$u_\theta^{(s-1)}(0) = \frac{\sin[(s-1)\theta - \omega]}{\sin[(s-1)\alpha - \omega]} a^{(s-1)}(0) + \frac{\sin[(s-1)(\alpha - \theta)]}{\sin[(s-1)\alpha - \omega]} b^{(s-2)}(0)$$

that is just formula (7.8).

This concludes the proof of the lemma.

PROOF OF LEMMA 7.4. In order to prove that $h'_\theta \in W^{s-1-1/p,p}\cdot(-\infty, +\infty)$ for every $\theta \in [0, \alpha - \pi]$ we consider first the case $p \neq 2$, then the case $p = 2$. If $p \neq 2$, from (5.22) it is easy to infer that h'_θ possesses the aforementioned properties, if, and only if, $u'_\theta \in W^{s-1-1/p,p}\cdot(0, +\infty)$ for every $\theta \in [0, \alpha]$ and the following equations hold

$$(-1)^j u_{\theta+\pi}^{(j)}(0) = u_\theta^{(j)}(0) \qquad \theta \in [0, \alpha - \pi]\,,\ j = 0, 1, \ldots, s-2\,,$$

and, only when $p > 2$,

$$(-1)^{s-1} u_{\theta+\pi}^{(s-1)}(0) = u_\theta^{(s-1)}(0) \qquad \theta \in [0, \alpha - \pi]\,.$$

A direct application of lemmas 7.1, 7.2, 7.3 shows that $h'_\theta \in W^{s-1-1/p,p}\cdot(-\infty, +\infty)$ for every $\theta \in [0, \alpha - \pi]$, if, and only if, the conditions listed in graphs 1, 2, 3 in theorem 2, are verified: this proves the lemma in the case $p \neq 2$.

Suppose, now, that $p = 2$: the same argument as before shows that $h'_\theta \in W^{s-2,2}(-\infty, +\infty)$ for every $\theta \in [0, \alpha - \pi]$. It remains to show that $h_\theta^{(s-1)} \in W^{\frac{1}{2},2}(-\infty, +\infty)$ for every $\theta \in [0, \alpha - \pi]$; we observe that it is well-known that proving such a property is equivalent to proving that

$$r^{-\frac{1}{2}}[u_\theta^{(s-1)} - (-1)^{s-1} u_{\theta+\pi}^{(s-1)}] \in L^2(0, +\infty) \qquad \text{for every } \theta \in [0, \alpha - \pi]\,.$$

We remark that, on account of formula (3.8), $0 \leqslant q_s \leqslant 1$: moreover, when $q_s = 1$, from estimate (3.18) and property (7.12) we infer that $\sigma_{s-1} \in (1/2, 1)$. Hence the compatibility conditions listed in graph 3 imply that, whatever q_s is, $u_\theta^{(s-1)}$ can be represented as follows:

$$u_\theta^{(s-1)}(r) = \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} H_s^*(r, \theta, t)\, a^{(s-1)}(t)\, dt +$$

$$+ \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} K_s^*(r, \theta, t)\, b^{(s-2)}(t)\, dt + \chi(\theta)\, b^{(s-2)}(r) \cos\omega \qquad \theta \in [0, \alpha)$$

where χ is defined by (7.4) and H_s^*, K_s^* are defined in the statement of lemma 7.10.

From (3.21) and (7.12) with $p = 2$ we get that $\sigma_s^* \in (0, 2\beta)$: therefore, applying iii), iv) and v) in lemma 7.10 and a simple change of variable in the integrals, we infer easily the identity valid for every $\theta \in [0, \alpha - \pi)$

$$(7.53) \qquad u_\theta^{(s-1)}(r) - (-1)^{s-1} u_{\theta+\pi}^{(s-1)}(r) = \frac{1}{\alpha} \cdot$$

$$\cdot \int_0^{+\infty} H_s^*(1, \theta, t)[a^{(s-1)}(rt) - a^{(s-1)}(r)] \frac{dt}{t} + (-1)^s \frac{1}{\alpha} \cdot$$

$$\cdot \int_0^{+\infty} H_s^*(1, \theta + \pi, t)[a^{(s-1)}(rt) - a^{(s-1)}(r)] \frac{dt}{t} + \frac{1}{\alpha} \cdot$$

$$\cdot \int_0^{+\infty} K_s^*(1, \theta, t)[b^{(s-2)}(rt) - b^{(s-2)}(r)] \frac{dt}{t} + \frac{1}{\alpha}(-1)^s \cdot$$

$$\cdot \int_0^{+\infty} K_s^*(1, \theta + \pi, t)[b^{(s-2)}(rt) - b^{(s-2)}(r)] \frac{dt}{t}.$$

When $\theta = \alpha - \pi$, identity (7.53) is to be replaced by the following one:

$$u_{\alpha-\pi}^{(s-1)}(r) - (-1)^{s-1} a^{(s-1)}(r) = \frac{1}{\alpha}\int_0^{+\infty} H_s(1, \alpha - \pi, t)[a^{(s-1)}(rt) - a^{(s-1)}(r)] \frac{dt}{t} +$$

$$+ \frac{1}{\alpha}\int_0^{+\infty} K_s^*(1, \theta, t)[b^{(s-2)}(rt) - b^{(s-2)}(r)] \frac{dt}{t}$$

which is a consequence of equations (7.46) with $\theta = \alpha - \pi$.

In order to prove that $r^{-\frac{1}{2}}[u_\theta^{(s-1)}-(-1)^{s-1}u_{\theta+\pi}^{(s-1)}]\in L^2(0,+\infty)$, it suffices to show that the functions

$$r\to r^{-\frac{1}{2}}\int\limits_0^{+\infty} H_s^*(1,\theta,t)[a^{(s-1)}(rt)-a^{(s-1)}(r)]\,\frac{dt}{t}$$

$$r\to r^{-\frac{1}{2}}\int\limits_0^{+\infty} K_s^*(1,\theta,t)[b^{(s-2)}(rt)-b^{(s-2)}(r)]\,\frac{dt}{t}$$

belong to $L^2(0,+\infty)$ for every $\theta\in[0,\alpha-\pi]$.

We limit ourselves to proving such a property for the latter function, since a similar procedure may be carried out for the former.

Consider the function φ so defined

$$\varphi(t)=\left(\int\limits_0^{+\infty}|b^{(s-2)}(rt)-b^{(s-2)}(r)|^2\,\frac{dr}{r}\right)^{\frac{1}{2}};$$

since $b^{(s-2)}\in W^{\frac{1}{2},2}(0,+\infty)$, from the definition of the seminorm $|\cdot|_{W^{\frac{1}{2},2}(0,+\infty)}$ it is easy to infer the following equation

$$\int\limits_0^{+\infty}\left|\frac{\varphi(t)}{1-t}\right|^2 dt=|b^{(s-2)}|^2_{W^{\frac{1}{2},2}(0,+\infty)}\,. \tag{7.54}$$

Moreover an application of lemma 5.7 shows that

$$\varphi(t)\leqslant 2^{-\frac{1}{2}}\,|b^{(s-2)}|_{W^{\frac{1}{2},2}(0,+\infty)}(2+|\ln t|)\qquad t\in(0,+\infty)\,. \tag{7.55}$$

Now, observing that the function $t\to K_s(1,\theta,t)(2+|\ln t|)\,t^{-1}$ is Lebesgue integrable for every $\theta\in[0,\alpha-\pi]$ over $(0,\frac{1}{2})\cup(\frac{3}{2},+\infty)$ and that the function $t\to K_s^*(1,\theta,t)(t-1)\,t^{-1}$ is continuous in $(0,\infty)$ for every $\theta\in[0,\alpha-\pi]$, applying Minkowski's inequality for integrals and taking advantage of (7.54), (7.55), we get the chain of inequalities

$$\left\{\int_0^{+\infty}\left(\int_0^{+\infty}|K_s^*(1,\theta,t)[b^{(s-2)}(rt)-b^{(s-2)}(r)]|\frac{dt}{t}\right)^2\frac{dr}{r}\right\}^{\frac{1}{2}}\leqslant$$

$$\leqslant\int_0^{+\infty}|K_s^*(1,\theta,t)|\varphi(t)\frac{dt}{t}=\int_{|t-1|<\frac{1}{2}}|K_s^*(1,\theta,t)|\varphi(t)\frac{dt}{t}+$$

$$+\int_{\{t>0:|t-1|>\frac{1}{2}\}}|K_s^*(1,\theta,t)|\varphi(t)\frac{dt}{t}\leqslant$$

$$\leqslant\left(\int_{|t-1|<\frac{1}{2}}|K_s^*(1,\theta,t)(t-1)|^2\frac{dt}{t^2}\right)^{\frac{1}{2}}\left(\int_{|t-1|<\frac{1}{2}}\left|\frac{\varphi(t)}{t-1}\right|^2dt\right)^{\frac{1}{2}}+$$

$$+2^{-\frac{1}{2}}|b^{(s-2)}|_{W^{\frac{1}{2},2}(0,+\infty)}\int_{\{t>0:|t-1|>\frac{1}{2}\}}|K_s^*(1,\theta,t)|(2+|\ln t|)\,dt\leqslant$$

$$\leqslant|b^{(s-2)}|_{W^{\frac{1}{2},2}(0,+\infty)}\left\{\left(\int_{|t-1|<\frac{1}{2}}|K_s^*(1,\theta,t)(t-1)|^2\frac{dt}{t^2}\right)^{\frac{1}{2}}+\right.$$

$$\left.+\,2^{-\frac{1}{2}}\int_{\{t>0:|t-1|>\frac{1}{2}\}}|K_s^*(1,\theta,t)|(2+|\ln t|)\,dt\right\}$$

which proves the assertion.

Proof of Lemma 7.5. We have to show that the conditions listed in graphs 1, 2, 3 in theorem 2 for the pair (a, b) imply the conditions listed in graphs 1, 2, 3 in theorem 4 (see section 6) for the pair (a, u_0).

For the reader's convenience we denote by $m(\omega)$, $n_j(\omega)$, $q_s(\omega)$ and so on the quantities related to problem (1.3) and by $m(0)$, $n_j(0)$, $q_s(0)$ and so on the quantities related to the Dirichlet problem (6.1) in section 6. Firstly we observe that the hypothesis $\alpha\in(0,\pi)$ implies, on account of formulas (3.6), (3.7), (3.8) that

$$0\leqslant m(0)\leqslant 1\,,\qquad 0\leqslant m(\omega)\leqslant 1\,,$$

$$0\leqslant n_j(0)\leqslant 1\,,\qquad 0\leqslant n_j(\omega)\leqslant 1\,,$$

$$0\leqslant q_s(0)\leqslant 1\,,\qquad 0\leqslant q_s(\omega)\leqslant 1\,.$$

Obviously we may suppose that none of the integers $m(0)$, $n_j(0)$, $q_s(0)$ equals 0: in fact, if some is 0, e.g. if $n_j(0)=0$, from theorem 4 we infer that there are no compatibility conditions on $a^{(j)}$ and $u_0^{(j)}$, that is no condition on $a^{(j)}$, $b^{(j-1)}$.

Moreover we observe that the proof of the lemma rests essentially on lemma 7.11 whose proof we postpone.

LEMMA 7.11. *Suppose $\alpha, \omega \in (0, \pi)$. Then the following propositions hold*:

i) *if* $m(0)=1$, *either* $\nu(\omega)-\nu(0)$ *or* $1+\sigma_1(\omega)-\nu(0)$ *belongs to* $(0, 2\beta)$;

ii) *if* $n_j(0)=1$, *either* $\sigma_j(\omega)-\sigma_j(0)$ *or* $1+\sigma_{j+1}(\omega)-\sigma_j(0)$ *belongs to* $(0, 2\beta)$;

iii) *if* $q_s(0)=1$, *either* $\sigma_{s-1}(\omega)-\sigma_{s-1}(0)$ *or* $1+\sigma_s^*(\omega)-\sigma_{s-1}(0)$ *belongs to* $(0, 2\beta)$.

For simplicity's sake we shall limit ourselves to showing that the conditions in graph 2, theorem 4, related to $n_j(0)=1$, are verified. More explicitly they are:

$$u_0^{(j)}(0)-(-1)^{N_j(0)}a^{(j)}(0)=0 \qquad \text{if } \sigma_j(0)=1 \tag{7.56}$$

$$\int_0^{+\infty}[u_0^{(j)}(r)-(-1)^{N_j(0)}a^{(j)}(r)]\,r^{-\sigma_j(0)}\,dr=0 \qquad \text{if } \sigma_j(0)\in\left(\frac{1}{p'}, 1\right) \tag{7.57}$$

$$\int_0^{+\infty}\{u_0^{(j)}(r)-u_0^{(j)}(0)-(-1)^{N_j(0)}[a^{(j)}(r)-a^{(j)}(0)]\}\,r^{-\sigma_j(0)}dr=0 \qquad \text{if } \sigma_j(0)\in\left(1, 1+\frac{1}{p'}\right). \tag{7.58}$$

We begin by observing that condition (7.56) is satisfied for the following reason: from formula (3.12) it is easy to infer that $\sigma_j(0)=1$ implies $j\alpha=N_j(0)\pi$ and $\sigma_j(\omega)\neq 1$, since $\omega\in(0,\pi)$. Hence the trace of $u_0^{(j)}$ at $r=0$ is given by formula (7.8), that in our case reads:

$$u_0^{(j)}(0)=(-1)^{N_j(0)}a^{(j)}(0)\,.$$

This proves condition (7.56). As far as conditions (7.57) and (7.58) are concerned, we recall that $u_0^{(j)}$ can be represented as follows:

$$u_0^{(j)}(r) = \frac{1}{\alpha}\int_0^{+\infty} H_j(r, 0, t)\, a^{(j)}(t)\, dt + \frac{1}{\alpha}\int_0^{+\infty} K_j(r, 0, t)\, b^{(j-1)}(t)\, dt + b^{(j-1)}(r)\cos\omega$$

$$\text{when } \sigma_j \in \left(\frac{1}{p'}, 1 + \frac{1}{p'}\right)$$

$$u_0^{(j)}(r) = \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} H_{j+1}(r, 0, t)\, a^{(j)}(t)\, dt +$$

$$+ \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} K_{j+1}(r, 0, t)\, b^{(j-1)}(t)\, dt + b^{(j-1)}(r)\cos\omega \qquad \text{when } \sigma_j \in \left(\frac{1}{p'}, 1\right)$$

$$u_0^{(j)}(r) = u_0^{(j)}(0) + \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} H_{j+1}(r, 0, t)[a^{(j)}(t) - a^{(j)}(0)]\, dt +$$

$$+ \frac{1}{\alpha}\int_0^{+\infty} \frac{r}{t} K_{j+1}(r, 0, t)[b^{(j-2)}(t) - b^{(j-2)}(0)]\, dt \qquad \text{when } \sigma_j \in \left(1, 1 + \frac{1}{p'}\right)$$

where the last two formulas can be obtained from the first one, taking advantage of the compatibility conditions listed in graph 2, theorem 2. Now from formulas (3.12) and (7.11) we infer easily the equations:

$$(7.59)\quad \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H_j(r, 0, t)\, r^{-\sigma_j(0)} dr = (-1)^{N_j(0)} t^{-\sigma_j(0)} \text{ if } \sigma_j(\omega) - \sigma_j(0) \in (0, 2\beta) \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K_j(r, 0, t)\, r^{-\sigma_j(0)} dr = -t^{-\sigma_j(0)} \cos\omega \quad \text{if } \sigma_j(\omega) - \sigma_j(0) \in (0, 2\beta) \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H_{j+1}(r, 0, t)\, r^{1-\sigma_j(0)} dr = (-1)^{N_j(0)} t^{1-\sigma_j(0)} \\ \qquad\qquad \text{if } \sigma_{j+1}(\omega) - \sigma_j(0) + 1 \in (0, 2\beta) \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K_{j+1}(r, 0, t)\, r^{1-\sigma_j(0)} dr = -t^{1-\sigma_j(0)} \cos\omega \\ \qquad\qquad \text{if } \sigma_{j+1}(\omega) - \sigma_j(0) + 1 \in (0, 2\beta) . \end{cases}$$

Substituting a suitable representation of $u_0^{(j)}$ in the left hand sides of formulas (7.57) and (7.58), interchanging the integrations with the aid of the Poincaré-Bertrand formula, it is easy to verify equations (7.57) and (7.58). Thus the lemma is fully proved.

PROOF OF LEMMA 7.11. Since the proofs of i), ii) and iii) are analogous, for brevity's sake we limit ourselves to proving ii). Observe, now, that from the hypothesis $\omega \in (0, \pi)$ and from equations (3.5), (3.2) it is easy to derive the following inequalities that hold for every $j \in N$:

$$\left\{\begin{array}{l} -1 \leqslant N_j(\omega) - N_j(0) \leqslant 0 \,, \\ -1 \leqslant n_j(\omega) \;\; - n_j(0) \; \leqslant 1 \,. \end{array}\right. \tag{7.60}$$

Suppose, now, $n_j(0) = 1$ and $n_j(\omega) = 0$: then from (7.60), (3.2) and (3.5) we get the inequalities

$$0 \geqslant N_j(\omega) - N_j(0) = N_{j+1}(\omega) - N_{j+1}(0) + 1 \geqslant 0 \,,$$

that imply $N_j(\omega) = N_j(0)$. Therefore, from formula (3.12) we infer that

$$\sigma_j(\omega) - \sigma_j(0) = 2\left(\frac{\omega}{\pi} + N_j(\omega) - N_j(0)\right)\beta = 2\,\frac{\omega}{\pi}\beta \in (0, 2\beta) \,,$$

that is the assertion.

Suppose, now, $n_j(0) = n_j(\omega) = 1$: from equation (3.7) recalling that $\alpha \in (0, \pi)$, we obtain the inequalities

$$\left\{\begin{array}{l} N_j(\omega) + 1 - \dfrac{\alpha}{\pi} \leqslant \left(j - \dfrac{1}{p}\right)\dfrac{\alpha}{\pi} + 1 - \dfrac{\omega}{\pi} < N_j(\omega) + 1 \\ N_j(0) \; + 1 - \dfrac{\alpha}{\pi} \leqslant \left(j - \dfrac{1}{p}\right)\dfrac{\alpha}{\pi} + 1 < N_j(0) + 1 \,. \end{array}\right. \tag{7.61}$$

They imply the following one:

$$N_j(\omega) - N_j(0) > -\,\frac{\alpha + \omega}{\pi} \,.$$

If we suppose $\alpha+\omega\in(0,\pi]$, then from (7.60), recalling that the N_j's are integers, we infer that $N_j(\omega)=N_j(0)$: this equation, in turn, implies $\sigma_j(\omega)-\sigma_j(0)=2(\omega/\pi)\beta\in(0,2\beta)$.

Finally, suppose $\alpha+\omega\in(\pi,2\pi)$: consider the equation

$$
(7.62)\qquad n_j(0)-n_j(\omega)=
$$

$$
=\begin{cases}
1 & \text{if } N_j(\omega)\leqslant\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}+1-\frac{\omega}{\pi}<N_j(\omega)+1-\max\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)\\
0 & \text{if } N_j(\omega)+1-\max\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)\leqslant\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}+1-\frac{\omega}{\pi}<\\
& \qquad <N_j(\omega)+1-\min\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)\\
-1 & \text{if } N_j(\omega)+1-\min\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)\leqslant\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}+1-\frac{\omega}{\pi}<\\
& \qquad <N_j(\omega)+2-\frac{\alpha+\omega}{\pi}\\
0 & \text{if } N_j(\omega)+2-\frac{\alpha+\omega}{\pi}\leqslant\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}+1-\frac{\omega}{\pi}<N_j(\omega)+1\,.
\end{cases}
$$

We have, by hypothesis, $n_j(0)-n_j(\omega)=0$: if we suppose that the second chain of inequalities in the right member of (7.62) holds, then, recalling also (7.61), we infer that

$$
N_j(\omega)-N_j(0)>\min\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)-\frac{\alpha}{\pi}-\frac{\omega}{\pi}=-\max\left(\frac{\alpha}{\pi},\frac{\omega}{\pi}\right)>-1\,.
$$

Hence (7.60) implies that $N_j(\omega)=N_j(0)$: therefore $\sigma_j(\omega)-\sigma_j(0)=2\cdot$ $\cdot(\omega/\pi)\beta\in(0,2\beta)$.

Suppose, now, that the third chain of inequalities in (7.62) holds: taking into account also (7.60) and recalling that $n_j(\omega)=1$, we get that

$$
0\leqslant 1+N_j(\omega)-N_j(0)=N_{j+1}(\omega)-N_j(0)<\frac{\alpha}{\pi}<1;
$$

this implies $N_{j+1}(\omega) = N_j(0)$: hence

$$1 + \sigma_{j+1}(\omega) - \sigma_j(0) = 2\left(\frac{\omega}{\pi} + N_{j+1}(\omega) - N_j(0)\right)\beta = 2\,\frac{\omega}{\pi}\beta \in (0, 2\beta)\,.$$

This concludes the proof of the lemma.

End of the proof of theorem 2.

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