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An Approximation Property for Abstract Differential Equations

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1. – Let X be a reflexive Banach and A be a closed linear operator with domain D_A dense in X. Let A^* be its adjoint with domain $D_{A^*} \subset X^*$, the dual space of X. Let also $[a, b] \subset R$ be an interval; R represents the real line. By $\mathfrak{D}_{[a,b]}(X^*)$ we mean the space of all infinitely differentiable (X^*-valued) functions defined on $[a, b]$ with compact support and $\mathfrak{D}'_{[a,b]}(X)$ the space of X-valued distributions on $[a, b]$. Similarly we define $\mathfrak{D}'_R(X)$ and $\mathfrak{D}_R(D_{A^*})$. Note that D_A and D_{A^*} are also Banach spaces under their graph norms.

Consider a homogeneous abstract differential equation

$$\frac{1}{i}\frac{du}{dt} - Au = 0 . \tag{1}$$

For convenience, we write

$$L = \frac{1}{i}\frac{d}{dt} - A \qquad \text{and} \qquad L^* = \frac{1}{i}\frac{d}{dt} - A^*.$$

DEFINITION 1. By $V_{[a,b]}$ we mean the set of all those $u \in \mathfrak{D}'_{[a,b]}(X)$ which are weak solutions of (1) on $a \leqslant t \leqslant b$ i.e. $\langle u, L^*\varphi\rangle = 0$ for all $\varphi \in \mathfrak{D}'_{[a,q]}(D_{A^*})$.

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DEFINITION 2. By $N_{[a,b]}$ we mean the set of all those $u \in \mathfrak{D}'_{[a,b]}(D_A)$ which are solutions of (1) i.e. $Lu = 0$ on $a \leqslant t \leqslant b$.

Similarly, we define V_R and N_R.

2. – In this paper we prove the following.

THEOREM. *If the abstract differential operator L satisfies*

Hyp. I. *Let $\Delta > 0$ be fixed. If $\varphi \in \mathfrak{D}_R(D_{A^*})$ and* $\operatorname{supp} L^*\varphi \subset [a, b]$, *then* $\operatorname{supp} \varphi \subset [a - \Delta, b + \Delta]$,

Hyp. $\underline{\overline{\text{II}}}$. *For all $\varphi \subset \mathfrak{D}_R(D_{A^*})$*

$$|\varphi^{(j)}(t)|_{X^*} \leqslant \text{const} \sup |L^*(\varphi)^{(j)}(t)|_{X^*}$$

$j = 0, 1, 2, \ldots$. The « const » depends on j and support of φ. Then V_R is dense in $V_{[a,b]}$ under the topology of $\mathfrak{D}'_{[a+\Delta,b-\Delta]}(X)$.

REMARK. This kind of results has been studied by S. Zaidman [5] for $L^2_{\text{loc}}(H)$ where Hyp. I holds weakly with $\Delta = 0$; H is a Hilbert Space. These results are related to the problem of existence of global solution of $Lu = f$. In [4] Zaidman proved an approximation property to ensure the existence of a global solution whereas in [2] and [3] the author used Hahn-Banach theorem after establishing Hyp. I and $\underline{\overline{\text{II}}}$ under suitable condition on the resolvent of A^*.

In section 3, we present a variation of an example of Agmon and Nirenberg [1] to show that the result is the best possible.

PROOF OF THE THEOREM. Let $\varphi \in \mathfrak{D}_{[a+\Delta,b-\Delta]}(X^*)$ such that

$$\langle x, \varphi \rangle = 0 \tag{2}$$

for all $x \in V_R$. To prove the Theorem, it is enough to show $\langle h, \varphi \rangle = 0$ for all $h \in V_{[a,b]}$. Extend $\varphi = 0$ outside the interval $[a + \Delta, b - \Delta]$. We first observe that $\varphi \in \overline{L^*(\mathfrak{D}_R(D_{A^*}))}$ where the closure is being taken in $\mathfrak{D}_R(X^*)$. In fact, if $\varphi \notin \overline{L^*(\mathfrak{D}_R(D_{A^*}))}$ there exists $U \in \mathfrak{D}_R(X)$, (recall X is reflexive) such that

$$\langle U, L^* k \rangle = 0 \tag{3}$$

for all $k \in \mathfrak{D}_R(D_{A^*})$ and

$$\langle U, \varphi \rangle = 1\,. \tag{4}$$

From (3) and the definition of V_R, one has $U \in V_R$ but then (4) contradicts (2) and so the choice of φ. Thus $\varphi \in \overline{L^*(\mathfrak{D}_R(D_{A^*}))}$.

Now consider a sequence $k_n \in \mathfrak{D}_R(D_{A^*})$ such that $L^* k_n \to \varphi$ in $\mathfrak{D}_R(X^*)$. From Hyp. I and II, k_n is a Cauchy sequence in $\mathfrak{D}_R(X^*)$. Since $\mathfrak{D}_R(X^*)$ is complete, there exists k such that $k_n \to k$ in $\mathfrak{D}_R(X^*)$. It is easy to verify that A^* and so $L^* = (1/i)(d/dt) - A^*$ is a closed linear operator with domain $\mathfrak{D}_R(D_{A^*})$ dense in $\mathfrak{D}_R(X^*)$. Thus $L^* k_n \to L^* k = \varphi$ and from Hyp. I supp $k \subset [a, b]$. Consequently $k \in \mathfrak{D}_{[a,b]}(D_{A^*})$ and $L^* k = \varphi$. Thus for an arbitrary choice of $h \in V_{[a,b]}$ one has

$$\langle h, \varphi \rangle = \langle h, L^* k \rangle = 0 . \tag{5}$$

This completes the proof.

If $u \in \mathfrak{D}'_{[a,b]}(X)$ is a weak solution of (1), then $u \in \mathfrak{D}'_{[a,b]}(D_A) = \mathfrak{L}(\mathfrak{D}_{[a,b]}(R); D_A)$. In fact, consider $\varphi = \psi \otimes x$ where $\psi \in \mathfrak{D}_{[a,b]}(R)$ and $x \in D_{A^*}$. As u is a weak solution of (1)

$$\left\langle u, \frac{1}{i} \frac{d\psi \otimes x}{dt} - A^* \psi \otimes x \right\rangle = 0 \tag{6}$$

from where

$$\left(\left\langle \frac{1}{i} \frac{du}{dt}, \psi \right\rangle, x \right) = (\langle u, \psi \rangle, A^* x) \tag{7}$$

$\forall x \in D_{A^*}$. $(\ ,\ )$ represents the duality between X and X^*. Hence $\langle u, \psi \rangle \in D_{A^{**}} = D_A$ ($A^{**} = A$ as the space is reflexive) and so

$$\left\langle \frac{1}{i} \frac{du}{dt}, \psi \right\rangle = A \langle u, \psi \rangle = \langle Au, \psi \rangle \tag{8}$$

for all $\psi \in \mathfrak{D}_{[a,b]}(R)$.

To conclude that $u \in \mathfrak{D}'_{[a,b]}(D_A)$ we observe if $\langle u, \psi_n \rangle$ converges in X, in view of (8) $\langle Au, \psi_n \rangle$ also converges in X.

It clearly implies that $V_{[a,b]} \simeq N_{[a,b]}$. So we have proved the following.

COROLLARY. *Under the Hyp. I and II of the Theorem, N_R is dense in $N_{[a,b]}$ in the topology of $\mathfrak{D}'_{[a+\Delta, b-\Delta]}(X)$.*

3. – An example. Let X be the Banach space consisting of all continuous complex valued functions defined on $0 \leqslant x \leqslant 1$ and vanishing at the origin. Define $A = (i/\Delta)(d/dx)$ the closed linear operator on X with domain D_A consisting of all c'-functions in X; $\Delta > 0$. Consider the equation

$$\frac{1}{i}\frac{du}{dt} - Au = \frac{1}{i}\left(\frac{\partial}{\partial t}u + \frac{1}{\Delta}\frac{\partial}{\partial x}u\right) = 0 . \tag{9}$$

Let $u \in \mathfrak{D}'_{[a,b]}(D_A)$ is a solution of (9) on $a \leqslant t \leqslant b$. It is obvious that $(u*\alpha)(t, x)$ is also a solution of (9) and so $(u*\alpha)(t, x)$ is constant along the direction $(\Delta, 1)$ as its directional derivative along that direction is zero. Since $(u*\alpha)(t, x) = 0$, one has $(u*\alpha)(t, x) = 0$ for $t \geqslant a + \Delta$. As α is arbitrary we conclude that $\operatorname{supp} u \subset [a, a + \Delta]$. By using a similar argument one can show that if $u \in \mathfrak{D}'_R(D_A)$ is a solution of (9) then $u \equiv 0$. Thus, both N_R and $N_{[a,b]}$ when restricted to $[a + \Delta, b - \Delta]$ are identical. In fact both vanish.

REFERENCES

[1] S. Agmon - L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math., **16** (1963), pp. 121-239.

[2] M. A. Malik, *Existence of solutions of abstract differential equations in a local space*, Canad. Math. Bull., **16** (1973), pp. 239-244.

[3] M. A. Malik, *Weak generalized solutions of abstract differential equations*, J. Math. Anal. Math., **40** (1972), pp. 763-768.

[4] S. Zaidman, *Un teorema i esistenza globale per alcune equazioni differenziali astratte*, Ricerche di Matematica, **13** (1964), pp. 57-69.

[5] S. Zaidman, *On a certain approximation property for first-order abstract differential equations*, Rendiconti del Seminario Matematico dell'Università di Padova, **46** (1971), pp. 191-198.

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