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Abelian Groups with Many Automorphisms.

JUTTA HAUSEN - JOHNNY A. JOHNSON (*)

1. - The results.

Given a group G , its automorphism group $\text{Aut } G$ acts as a group of permutations on the set of all subgroups of G . Problem 91 of Fuchs [2] proposes to determine the equivalence classes (orbits) of subgroups of G determined by this action of $\text{Aut } G$. In this note we characterize the class of all abelian groups G for which these equivalence classes are precisely the isomorphism classes of subgroups of G . These are the T_1 -groups in the sense of Polimeni [4]: G is a T_1 -group if, given two subgroups H and K of G which are isomorphic, there exists α in $\text{Aut } G$ such that $\alpha(H) = K$. This definition is weaker than the definition of a T'_1 -group: G is a T'_1 -group if every isomorphism between any two of its subgroups can be extended to an automorphism of G . We shall prove the following theorem which shows among other things that every abelian T_1 -group actually is a T'_1 -group.

(1.1) THEOREM. For an abelian group G the following statements are equivalent.

- (i) $\text{Aut } G$ operates transitively on the isomorphism classes of subgroups of G (i.e. G is T_1).
- (ii) Either G is a torsion group all of whose p -components are homo-cocyclic of finite rank, or G is a divisible group whose torsion-free rank and p -ranks are all finite.

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- (iii) G is a characteristic subgroup of a divisible group whose torsion-free rank and p -ranks are all finite.
- (iv) Every isomorphism between any two subgroups of G is induced by some automorphism of G (i.e., G is T_1').

Let \mathcal{A} be the class of all abelian groups G such that every automorphism of any subgroup of G can be extended to an automorphism of G . Clearly every abelian T_1' -group (and hence, in view of (1.1), every abelian T_1 -group) is contained in \mathcal{A} . In [3] Mishina has shown that an abelian torsion group belongs to \mathcal{A} if and only if all of its p -components are homo-cocyclic. This result together with (1.1) implies the following theorem.

(1.2) **THEOREM.** Let G be an abelian torsion group all of whose p -ranks are finite. Then the following conditions are equivalent.

- (i) $\text{Aut } G$ operates transitively on the isomorphism classes of subgroups of G .
- (ii) All p -components of G are homo-cocyclic.
- (iii) G is a characteristic subgroup of a divisible torsion group.
- (iv) Every automorphism of any subgroup of G can be extended to an automorphism of G .
- (v) Every isomorphism between any two subgroups of G can be extended to an automorphism of G .

2. - The proof.

All groups, except groups of automorphisms, are assumed to be abelian and the notation and terminology will be that of Fuchs [1, 2]. We will make use of the following auxiliary results. Some proofs are left to the reader.

(2.1) **LEMMA.** Characteristic subgroups of T_1 -groups are T_1 -groups.

(2.2) **LEMMA.** Characteristic subgroups of T_1' -groups are T_1' -groups.

(2.3) **LEMMA.** If G is a T_1 -group, then G is not isomorphic to a proper subgroup of itself.

(2.4) **COROLLARY.** If G is a T_1 -group and $G = A \oplus \bigoplus_{i \in I} B_i$, where $0 \neq B_i \simeq B_j$, for all i, j in I , then I is finite.

(2.5) PROPOSITION. Let G be a T_1 -group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or G is a p -group, then there exists α in $\text{Aut } G$ such that $\alpha(x) = y$.

PROOF. Since $\langle x \rangle \simeq \langle y \rangle$, there exists β in $\text{Aut } G$ such that

$$\langle y \rangle = \beta(\langle x \rangle) = \langle \beta(x) \rangle$$

and consequently y generates $\langle \beta(x) \rangle$. If $o(x)$ is infinite, $y = k \cdot \beta(x)$, where $k = \pm 1$; if G is a p -group, then $y = k \cdot \beta(x)$, for some integer k relatively prime to p . In either case, the mapping γ such that $g \rightarrow kg$, for all $g \in G$, is an automorphism of G . Hence $\alpha = \gamma \circ \beta \in \text{Aut } G$ and

$$\alpha(x) = (\gamma \circ \beta)(x) = \gamma(\beta(x)) = k \cdot \beta(x) = y$$

as desired.

(2.6) COROLLARY. Let G be a T_1 -group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or G is a primary group, then, for all primes p , $h_p(x) = h_p(y)$.

A group is called homo-cocyclic if it is the direct sum of pairwise isomorphic cocyclic groups (cf. [1], p. 16).

(2.7) PROPOSITION. If G is a p -primary T_1 -group, then G is homo-cocyclic of finite rank.

PROOF. Since $G[p]$ is a characteristic subgroup of G , (2.1) and (2.4) imply that $G[p]$ is finite and G has finite rank k . By (2.6) any two non-zero elements in $G[p]$ have equal height in G . Hence $G = \bigoplus_{i=1}^k Z(p^n)$, for some $1 \leq n \leq \infty$, as claimed.

(2.8) COROLLARY. If G is a torsion T_1 -group, then every p -component of G is homo-cocyclic of finite rank.

(2.9) PROPOSITION. If G is a non-torsion T_1 -group, then G is a divisible group whose torsion-free and p -ranks are all finite.

PROOF. In view of (2.4) and the structure theorem for divisible groups (cf. [1], p. 104) it suffices to show that G is divisible, which is equivalent to $h_p(x) = \infty$, for all $x \in G$, and all primes p . Let $x \in G$ and let p be a prime. If $o(x) = \infty$, then, for all integers $n \geq 1$,

$$o(x) = o(p^n x) \quad \text{and} \quad h_p(x) = h_p(p^n x) \geq n$$

by (2.6). Hence, $h_p(x) = \infty$ whenever $x \in G$ has infinite order. Suppose $o(x) < \infty$. Since G is not torsion there exists $y \in G$ such that $o(y) = \infty$. Hence

$$o(x + y) = \infty \quad \text{and} \quad h_p(x + y) = h_p(y) = \infty.$$

By ([1], p. 6, exercise 11) $h_p(x)$ cannot be finite and $h_p(x) = \infty$ as claimed.

(2.10) PROPOSITION. If D is a divisible group whose torsion-free rank and p -ranks are all finite, then D is a T'_1 -group.

PROOF. Let $\varphi: A_1 \rightarrow A_2$ be an isomorphism between the two subgroups A_1 and A_2 of D . Let \bar{D}_i be the divisible hull of A_i in D , $i = 1, 2$. Since $\bar{D}_2 \geq A_2$ is injective, there exists a homomorphism $\sigma: \bar{D}_1 \rightarrow \bar{D}_2$ such that $\sigma|_{A_1} = \varphi$. The fact that \bar{D}_1 is an essential extension of A_1 and φ is monic implies σ is monic (cf. [1], p. 106 f). Also, σ is epic since $\sigma(\bar{D}_1)$ is divisible and

$$A_2 = \varphi(A_1) = \sigma(A_1) \leq \sigma(\bar{D}_1) \leq \bar{D}_2.$$

Hence $\sigma: \bar{D}_1 \rightarrow \bar{D}_2$ is an isomorphism extending φ . There exists $D_i \leq D$ such that

$$\bar{D}_1 \oplus D_1 = D = \bar{D}_2 \oplus D_2.$$

Since the torsion-free rank and all p -ranks of D are finite, $\bar{D}_1 \simeq \bar{D}_2$ implies $D_1 \simeq D_2$: Let $\tau: D_1 \rightarrow D_2$ be an isomorphism and define $\alpha: D \rightarrow D$ by $\alpha|_{\bar{D}_1} = \sigma$ and $\alpha|_{D_1} = \tau$. Then $\alpha \in \text{Aut } D$ and $\alpha|_{A_1} = \sigma|_{A_1} = \varphi$, completing the proof.

PROOF OF (1.1). The proof is cyclic. (ii) follows from (i) by (2.8) and (2.9). Assume the validity of (ii). Clearly, we may assume that G is torsion. If the p -component G_p of G is not divisible, then $G_p = \bigoplus_{i=1}^k Z(p^n)$, for some integer n and $G_p = D_p[p^n]$, where $D_p = \bigoplus_{i=1}^k Z(p^\infty)$. Hence, for all p , G_p is a characteristic subgroup of a divisible p -group D_p of finite rank, and $G = \bigoplus_p G_p$ is characteristic in $D = \bigoplus_p D_p$, as stated in (iii). Using (2.2) and (2.10), (iv) follows from (iii). The last implication is trivial.

REFERENCES

- [1] L. FUCHS, *Infinite Abelian Groups*, vol. I, Academic Press, New York, 1970.
- [2] L. FUCHS, *Infinite Abelian Groups*, vol. II, Academic Press, New York, 1973.
- [3] A. P. MISHINA, *On automorphisms and endomorphisms of abelian groups*, Vestnik, Moscow Univ. Ser. Matem. i Mekh., no. 4 (1962), 39-43 [Russian].
- [4] A. D. POLIMENI, *Groups in which $\text{Aut}(G)$ is transitive on the isomorphism classes of G* , Pacific J. Math., **48** (1973), 473-480.

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