RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Compactness methods for quasi-linear evolution-equations

Rendiconti del Seminario Matematico della Università di Padova, tome 55 (1976), p. 151-166

<http://www.numdam.org/item?id=RSMUP_1976__55__151_0>

Rend. Sem. Mat. Univ. Padova, Vol. 55 (1976)

Compactness Methods for Quasi-Linear Evolution-Equations.

Andrea Schiaffino (*)

Introduction.

Let X be a complex Banach space with norm $|\cdot|$ and let $-A$ be the infinitesimal generator of the strongly-continuous semigroup $\{\exp[-tA];\, t\geqslant 0\}$.

In this paper we consider the existence of a solution to the integral equation

$$\text{(PB1)}\quad u(t) = \exp[-tA]x_0 - \int_0^t \exp[-(t-s)A]F(u(s))\,ds\,, \qquad t\geqslant 0\,,$$

where F is a continuous function from $K\subset X$ into X.

A solution of (PB1) is called a « mild » solution to the abstract Cauchy problem

$$\text{(PB2)}\qquad u'(t) + Au(t) + F(u(t)) = 0 \qquad u(0) = x_0\,.$$

A solution of (PB2) is called a « strict » solution; it is well known that a strict solution is also a mild solution and that a mild solution is strict if it is differentiable.

In [11] are given some techniques to set up approximate solutions to (PB1) and in [7], [8], [9] and [11] are given criteria for the exi-

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stence of solutions; these criteria use hypotheses on F; we will study sufficient conditions for A in the case that F verify only the hypothesis (considered in [9])

$$\text{(HP1)} \qquad \lim{}' t^{-1} d(x - tF(x), K) = 0$$

where $d(x, K) = \text{g.l.b.}\{|y - x|;\ y \in K\}$.

Moreover we consider the following hypothesis

$$\text{(HP2)} \qquad \exp[-tA]K \subset K\,, \qquad t \geqslant 0\,.$$

The main result of this paper is the following theorem

THEOREM 1. Let us suppose

i) K is convex and locally closed; $x_0 \in K$.

ii) $-A$ si the infinitesimal generator of an analytical semigroup.

iii) $\exp[-tA]$ is compact for every $t > 0$.

iv) (HP1) and (HP2) hold.

Then a local solution to (PB1) exists. Moreover a global solution to (PB1) exists if $F(K)$ is bounded. The solution is strict if $x_0 \in K \cap D(A)$ and F is locally Hölder-continuous.

To prove this theorem we construct approximate continuous solutions $u_\varepsilon \in K$ to (PB1), such that

$$u(t) = \exp[-tA]x_0 - \int_0^t \exp[-(t-s)A]F(u_\varepsilon(s))\,ds + \int_0^t \exp[-(t-s)A]\,v_\varepsilon(s)\,ds$$

where v_ε are piecewise-continuous functions satisfying $|v_\varepsilon(t)| < \varepsilon$.

The construction of $u_\varepsilon(t)$ is given in section 2 in which we use some lemmas proved in section 1. Our construction is different by the one given in [11] because we suppose that K is a convex set; this hypothesis is necessary, in our case, to construct $u_\varepsilon(t)$ in K.

The proof of theorem 1 is given in section 3; in section 4 we give some examples concerning non-linear perturbation of heat equation.

1. – Preliminar results.

In this section we prove some technical lemmas, in order to construct approximate solutions to (PB1). Throughout this paper we suppose that (HP1) and (HP2) hold and that K is a convex locally-closed set.

LEMMA 1. For every $\varepsilon > 0$ the function

$$h(\varepsilon, x) = \text{l.u.b.}\{h > 0 : d(x - hF(x), K) < h\varepsilon\} \tag{1}$$

is lower semicontinuous (lsc.).

PROOF. We first remark that the application $h \to h^{-1} d(x - hF(x), K)$ is increasing, due to the convexity of K, for every $u \in K$. Let $x_0 \in K$, $h \in]0, h(\varepsilon, x_0)[$ and

$$\varepsilon' = h^{-1} d(x_0 - hF(x_0), K) < \varepsilon \; ;$$

let x belong to K, then

$$h^{-1} d(x - hF(x), K) \leqslant h^{-1} |x - x_0| + |F(x) - F(x_0)| + \varepsilon' < \varepsilon$$

if $h^{-1}|x - x_0| + |F(x) - F(x_0)| < \varepsilon - \varepsilon'$; then, for the continuity of F, the lemma follows.

LEMMA 2. If we define

$$\sigma'(\varepsilon, x) = \\ = \text{l.u.b.}\{T > 0 : \min\{h(\varepsilon, \exp[-sA]x) - 2T ; s \in [0, T]\} > 0\} \tag{2}$$

then, for every $\varepsilon > 0$, the function $\sigma'(\varepsilon, \cdot)$ is lsc. on K.

PROOF. Let $x_0 \in K$ and $T \in]0, \sigma'(\varepsilon, x_0)[$; we have

$$h(\varepsilon, \exp[-sA]x_0) > 2T \; , \qquad s \in [0, T] \, .$$

It is obvious that $h(\varepsilon, \exp[-sA]x)$ is lsc. in $K \times \bar{R}_+$ and, because the compactness of $[0, T]$, it exists a covering $\{]t_i - \delta_i, t_i + \delta_i[\}_{i=1,\dots,n}$

of $[0, T]$ such that

$$h(\varepsilon, \exp[-sA]x) > 2T\,, \qquad |s - t_i| < \delta_i\,, \qquad |x - x_0| < \delta_i\,;$$

let $\delta_0 = \min(\delta_1, \ldots, \delta_n)$ and $x \in K \cap B(x_0, \delta_0)$, so we have

$$h(\varepsilon, \exp[-sA]x) > 2T$$

and the lemma is proved.

LEMMA 3. Let M and M' be two metric spaces and let d and d' denote their respective metrics. Let $G\colon M \to M'$ be continuous; we define

$$\varrho(\varepsilon, x) = \text{l.u.b.}\{\varrho > 0 : \omega(G, B(x, \varrho)) < \varepsilon\}\,, \tag{3}$$

where

$$\omega(G, B(x, \varrho)) = \text{l.u.b.}\{d'(G(y), G(z)),\; y, z \in B(x, \varrho)\}.$$

Let $\sigma\colon M \to R_+$ be lsc.; then the function $x \to \varrho(\sigma(x), x)$ is lsc..

PROOF. Let $x_0 \in M$, $\varrho_0 = \varrho(\sigma(x_0), x_0)$ and $\varrho' \in]0, \varrho_0[$. We have

$$\sigma' = \omega\big(G, B(x_0, (\varrho_0 + \varrho')/2)\big) < \sigma(x_0)$$

therefore there exists $r \in]0, (\varrho_0 - \varrho')/2[$ such that $\sigma(x) > \sigma'$, $\forall x \in B\cdot$ $\cdot(x_0, r)$.

If $x \in B(x_0, r)$ we have: $B(x, \varrho') \subset B(x_0, (\varrho_0 + \varrho')/2)$, therefore $\omega(G, B(x, \varrho')) \leqslant \sigma' < \sigma(x)$; consequently $\varrho(\sigma(x), x) \geqslant \varrho'$ and the assertion of lemma follows.

In the following we use the functions

$\sigma''(\varepsilon, x)$ defined by (3) in the case

$$M = K \qquad M' = X \qquad G(x) = F(x)\,,$$

$\varrho'(\varepsilon; x, t)$ defined by (3) in the case

$$M = K \times \bar{R}_+ \qquad M' = X \qquad G(x, t) = \exp[-tA]x\,,$$

$\varrho''(\varepsilon; x, t)$ defined by (3) in the case

$$M = K \times \bar{R}_+ \qquad M' = X \qquad G(x, t) = F(\exp[-tA]x)\,.$$

Let us remark that all the functions

$$h(\varepsilon, x)\,, \qquad \sigma'(\varepsilon, x)\,, \qquad \sigma''(\varepsilon, x)\,, \qquad \varrho'(\varepsilon; x, t)\,, \qquad \varrho''(\varepsilon; x, t)$$

are lsc. for every $\varepsilon > 0$.

LEMMA 4. Let x belong to K and $T > 0$; let moreover suppose

$$T \leqslant \min\{\sigma'(\varepsilon, x);\ \varrho'(\varrho''(\varepsilon; x, 0); x, 0)\}$$

then it exists a Lipschitz-continuous function $y(s)\colon [0, T] \to K$ such that

$$(4) \qquad |\exp[-sA]x - TF(x) - y(s)| < 2\varepsilon T \qquad s \in [0, T]\,.$$

PROOF. Let us remark that

$$d(\exp[-sA]x - TF(x), K) \leqslant d(\exp[-sA]x - TF(\exp[-sA]x), K) + \\ + T|F(\exp[-sA]x) - F(x)| < 2\varepsilon T\,.$$

Let $c_0 = 2\varepsilon T - \max_{[0,T]} d(\exp[-sA]x - TF(x), K) > 0$; let us choose $0 = t_0 < t_1 < \ldots < t_n = T$ in such a way ahat

$$\omega(\exp[-sA]x - TF(x), [t_{i-1}, t_i]) < c_0/4\,.$$

Let $x_i = \exp[-t_i A]x - TF(x)$ and $y_i \in K$ in such way that $|x_i - y_i| < 2\varepsilon T - c_0/4$ and, finally, we can define

$$y(s) = y_{i-1} + \frac{s - t_{i-1}}{t_i - t_{i-1}}(y_i - y_{i-1}) \qquad s \in [t_{i-1}, t_i]\,.$$

Then we have

$$|\exp[-sA]x - TF(x) - y(s)| < 2\varepsilon T - c_0/2 + \\ + \left|\exp[-sA]x - TF(x) - x_i - \frac{s - t_{i-1}}{t_i - t_{i-1}}\right| < 2\varepsilon T$$

and the lemma follows.

2. – Approximate solutions.

If x_0 belongs to K, there exist three positive numbers r, M, N such that $K \cap B(x_0, 2r)$ is closed and

$$|F(x)| \leqslant M \qquad x \in K \cap B(x_0, 2r)\,,$$

$$\|\exp[-tA]\| \leqslant N \qquad 0 \leqslant t \leqslant r/M\,.$$

Now we can consider the function $\sigma'''(\varepsilon, x) = \min\big(r, \sigma''(\varepsilon, x)\big)$ and define the lsc. function $\big(\varepsilon > 0,\ x \in K \cap B(x_0, r)\big)$:

$$T(\varepsilon, x) = \min\big\{\varrho'\big(\varrho''(\varepsilon; x, 0); x, 0\big); \sigma'(\varepsilon, x); \\ \varrho'\big(\sigma'''(\varepsilon, x)/2; x, 0\big); \sigma'''(\varepsilon, x)/N(M+\varepsilon)\big\}\,.$$

LEMMA 5. If $x \in K \cap B(x_0, r)$ it exists $u_\varepsilon(t) \in C^0[0, T(\varepsilon, x); X]$ such that $u_\varepsilon(t) \in K \cap B(x_0, 2r)$ and

$$u_\varepsilon(t) = \exp[-tA]x - \int_0^t \exp[-(t-s)A]F\big(u_\varepsilon(s)\big)\,ds + \\ + \int_0^t \exp[-(t-s)A]\,v_\varepsilon(s)\,ds \tag{1}$$

where v_ε is a continuous function verifying $|v_\varepsilon(t)| < 2\varepsilon$.

PROOF. Let us write $T = T(\varepsilon, x)$ and give

$$u_\varepsilon(t) = \exp[-tA]x - T^{-1}\int_0^t \exp[-(t-s)A]\big(\exp[-sA]x - y(s)\big)\,ds$$

where $y(s)$ is given by lemma 4. We have

$$u_\varepsilon(t) = t^{-1}\int_0^t \exp[-(t-s)A]\left[\exp[-sA]x + \frac{t}{T}\big(y(s) - \exp[-sA]x\big)\right]ds$$

therefore $u_\varepsilon(t) \in K$ because, being K a convex set, the mean value theorem holds. Now we have

$$|u_\varepsilon(t) - x| \leqslant |\exp[-tA]x - x| + tN(M+\varepsilon) \leqslant \sigma'''(\varepsilon, x)$$

therefore $|F(u_\varepsilon(t)) - F(x)| \leqslant \varepsilon$ because $\sigma'''(\varepsilon, x) \leqslant \sigma''(\varepsilon, x)$.

Moreover

$$|u_\varepsilon(t) - x_0| \leqslant |x - x_0| + \sigma'''(\varepsilon, x) \leqslant 2r$$

because $\sigma'''(\varepsilon, x) \leqslant r$.

Finally let us define

$$v_\varepsilon(t) = F(u_\varepsilon(t)) - T^{-1}(\exp[-tA]x - y(t)) ;$$

then

$$|v_\varepsilon(t)| \leqslant |F(u_\varepsilon(t)) - F(x)| + |F(x) - T^{-1}(\exp[-tA]x - y(t))| < 2\varepsilon$$

and the lemma follows.

An analogous statement of this lemma is the following: for every $x \in K \cap B(x_0, r)$ there exist $T(\varepsilon, x)$ and $u_\varepsilon(t)$ verifying:

i) $T(\varepsilon, x) > 0$ is lsc. in $K \cap B(x_0, r)$,

ii) $u_\varepsilon(t) \in C^0(0, T(\varepsilon, x); X)$ and $u_\varepsilon(t) \in K$,

iii) $u_\varepsilon(t)$ verifys (1).

We can now prove the following

THEOREM 2. Let (HP1) and (HP2) hold; then for every $x_0 \in K$ there exist $T = T(x_0) > 0$ and $u_\varepsilon \in C^0(0, T; X)$ verifying (1) with v_ε piecewise-continuous.

PROOF. Let us use the symbols of previous lemma and pose $T = r/M$.

For $x \in K$ such that $|x - x_0| \leqslant r$ let $u_\varepsilon(t, x)$ be the function introduced by lemma 5.

If there exist $t_1, \ldots, t_n$ and $x_1, \ldots, x_n$ such that

$$\text{(2)} \qquad \begin{cases} t_1 = T(\varepsilon, x_0)\,, \\ x_{i+1} = u_\varepsilon(T(\varepsilon, x_i), x_i)\ ,\ t_{i+1} = t_i + T(\varepsilon, x_i)\ ,\ i = 0, \ldots, n-1 \end{cases}$$

and $t_{n-1} < r/M \leqslant t_n$, we can define

$$u_\varepsilon(t) = u_\varepsilon(t - t_{i-1}, x_{i-1}) \qquad t \in [t_{i-1}, t_i]\,,$$

and the thesis follows.

Now let us assume that a finite sequence as above cannot be found. Then the (2) define two sequences $\{t_n\}$ and $\{x_n\}$ where $\{t_n\}$ is increasing and $t_n \to t_0 \leqslant r/M$; the sequence $\{x_n\}$ verifys

$$x_{n+1} = \exp[-(t_{n+1} - t_n)A]\,x_n + \int_{t_n}^{t_{n+1}} \exp[-(t_{n+1} - s)A]\,H(s)\,ds$$

where $H(s)$ is piecewise-continuous and bounded by $M + \varepsilon$.

By induction

$$x_n = \exp[-t_n A]\,x_0 + \int_0^{t_n} \exp[-(t_n - s)A]\,H(s)\,ds\,.$$

Now we can evaluate $|x_{n+p} - x_n|$;

$$|x_{n+p} - x_n| \leqslant |\exp[-t_{n+p}A]\,x_0 - \exp[-t_n A]\,x_0| + \\ + \left|\int_{t_n}^{t_{n+p}} \exp[-(t_{n+p} - s)A]\,H(s)\,ds\right| + \\ + \left|\int_0^{t_n} \exp[-(t_{n+p} - t_n)A] - \exp[-(t_n - s)A]\,H(s)\,ds\right|.$$

The first two terms go to zero as n and $n + p$ diverge; the third term goes also to zero for the Lebesgue convergence theorem; thus $\{x_n\}$ converges. Let x be its limit, then

$$0 < T(\varepsilon, x) \leqslant \lim{}' \, T(\varepsilon, x_n) = \lim{}' \, (t_{n+1} - t_n) = 0$$

which is unpossible and the theorem follows.

3. – The proof of the existence theorem.

Throughout this section we assume that the hypotheses of theorem 1 hold and we use the notations introduced in theorem 2.

LEMMA 6. Let $\sigma \in]0, T[$ and $C \subset X$ a bounded set; then the set

$$E_\sigma = \bigcup_{t \in [\sigma, T]} \exp[-tA] C$$

is relatively compact.

PROOF. Let $x_n = \exp[-t_n A] c_n$ $(t_n \in [\sigma, T], c_n \in C)$ be a sequence in E_σ; we can suppose $t_n \to t \in [\sigma, T]$ and $\exp[-tA] c_n \to x \in X$. Then

$$|x_n - x| \leqslant |\exp[-t_n A] c_n - \exp[-tA] c_n| + |\exp[-tA] c_n - x| \leqslant$$
$$\leqslant \|\exp[-t_n A] - \exp[-tA]\| \, |c_n| + |\exp[-tA] c_n - x|$$

which goes to zero because the semigroup is analytical and $\{c_n\}$ is bounded.

LEMMA 7. Let us define

$$w_\varepsilon(t) = \int_0^t \exp[-(t-s)A] F(u_\varepsilon(s)) \, ds ;$$

then it exists a sequence $\varepsilon_n \to 0$ such that $w_{\varepsilon_n}(t)$ is is uniformly convergent.

PROOF. Let $\sigma \in]0, T[$ and define

$$w'_{\varepsilon,\sigma}(t) = \begin{cases} w_\varepsilon(t) & t \leqslant \sigma \\ \displaystyle\int_{t-\sigma}^t \exp[-(t-s)A] F(u_\varepsilon(s)) \, ds & t > \sigma \end{cases}$$

$$w''_{\varepsilon,\sigma}(t) = \begin{cases} 0 & t \leqslant \sigma \\ \displaystyle\int_0^{t-\sigma} \exp[-(t-s)A] F(u_\varepsilon(s)) \, ds & t > \sigma . \end{cases}$$

The functions $w'_{\varepsilon,\sigma}$ and $w''_{\varepsilon,\sigma}$ are continuous and their sum is w_ε. Let us consider the set E_σ introduced in lemma 6 in the case $C = F(B(x_0, r))$ and the closed convex hull D_σ of the set

$$\bigcup_{\tau\in[0,T]} \tau E_\sigma .$$

It is obvious that D_σ is compact and $w''_{\varepsilon,\sigma}(t) = 0 \in D_\sigma$ for $t \leqslant \sigma$; if $t > \sigma$

$$w''_{\varepsilon,\sigma}(t) = \int_0^{t-\sigma} \exp[-(t-s)A]\, F(u_\sigma(s))\, ds \in D_\sigma$$

for the convexity of D_σ and the mean value theorem.

To apply Ascoli's theorem we remark that

$$\frac{d}{dt} w''_{\varepsilon,\sigma}(t) = \exp[-\sigma A]\, F(u_\varepsilon(t-\sigma)) - \int_0^{t-\sigma} A \exp[-(t-s)A]\, F(u_\varepsilon(s))\, ds$$

and

$$\left|\frac{d}{dt} w''_{\varepsilon,\sigma}(t)\right| \leqslant N(M+\varepsilon) + T\frac{N}{\sigma}(M+\varepsilon)$$

therefore, for fixed σ, $w''_{\varepsilon,\sigma}$ describes a compact set in $C^0(0, T; X)$.

Let us now consider a sequence $\sigma_k \to 0$; by the diagonal method we can constract a subsequence of $\{\varepsilon_n\}$, let us call it still $\{\varepsilon_n\}$, such that $w''_{\varepsilon_n,\sigma_k}$ is uniformly convergent in $[0, T]$ for every k.

For every k we have

$$|w_{\varepsilon_n}(t) - w_{\varepsilon_m}(t)| \leqslant |w'_{\varepsilon_n,\sigma_k}(t) - w'_{\varepsilon_m,\sigma_k}(t)| + |w''_{\varepsilon_n,\sigma_k}(t) - w''_{\varepsilon_m,\sigma_k}(t)| \leqslant$$
$$\leqslant 2\sigma_k M^2 + |w''_{\varepsilon_n,\sigma_k}(t) - w''_{\varepsilon_m,\sigma_k}(t)|$$

and

$$\lim_{n,m\to\infty}{}'' \; |w_{\varepsilon_n}(t) - w_{\varepsilon_m}(t)| \leqslant 2\sigma_k M^2$$

uniformly in t and for every k.

Because we can choose σ_k arbitrarely small, the lemma follows.

Proof of theorem 1. By lemma 7 the sequence $u_{\varepsilon_n}(t)$ converges uniformly to

$$u(t) = \exp[-tA]x_0 - w(t)$$

where $w(t) = \lim w_{\varepsilon_n}(t)$; now

$$w(t) = \lim\int_0^t \exp[-(t-s)A]\,F\big(u_{\varepsilon_n}(s)\big)\,ds\ ;$$

let us note that $F\big(u_{\varepsilon_n}(s)\big) \to F\big(u(s)\big)$ pointwise and $|F\big(u_{\varepsilon_n}(s)\big)| \leqslant M$; by dominated convergence theorem

$$w(t) = \exp[-tA]x_0 - \int_0^t \exp[-(t-s)A]\,F\big(u(s)\big)\,ds$$

and theorem 1 follows.

Remark 1. If $F(K)$ is bounded we can choose $r > 0$ arbitrarely large, so a maximal solution of (PB1) is defined for every $t \geqslant 0$.

Remark 2. Because the analiticity of $\exp[-tA]$, $u(t)$ is Hölder continuous, see [3]. If F is locally Hölder continuous, also $F\big(u(t)\big)$ is Hölder continuous. Therefore (see [3]), if $x_0 \in K \cap D(A)$, $u(t)$ is a classical solution of (PB1) and du/dt, Au are Hölder continuous.

4. – The case of quasi-linear heat equation.

In the following we denote by Ω a bounded open set in R^n whose boundary $\partial\Omega$ is regular and by $\alpha(x)$ and $\beta(x)$ two real continuous functions defined in $\bar{\Omega}$ such that $\alpha(x) < \beta(x)$. Let us consider the compact domain in R^{n+1}

$$D = \big\{(x, u) \in \bar{\Omega} \times R\colon \alpha(x) \leqslant u \leqslant \beta(x)\big\}$$

and the convex sets

$$K\ = \big\{u \in C^0(\bar{\Omega})\colon \alpha(x) \leqslant u(x) \leqslant \beta(x)\big\},$$

$$K_p = \big\{u \in L^p(\Omega)\colon \alpha(x) \leqslant u(x) \leqslant \beta(x)\ \text{a.e.}\big\}.$$

Let us consider a real (necessary bounded) continuous function $f(x, u)$ defined on D and the function

$$(1) \qquad (Fu)(x) = f(x, u(x))$$

defined on K or K_p.

LEMMA 8. Let $X = C^0(\bar{\Omega})$, the function $F\colon K \to X$ defined by (1) is continuous; moreover F veriflys the condition (HP1) iff

$$(2) \qquad f(x, \alpha(x)) \leqslant 0 , \qquad f(x, \beta(x)) \geqslant 0 .$$

PROOF. It is obvious that F is continuous. Let us first note that

$$d(v, K) = \max_{x \in \bar{\Omega}} |v(x) - v_K(x)|$$

where

$$(3) \qquad v_K(x) = \begin{cases} \alpha(x) & \alpha(x) \geqslant v(x) \\ v(x) & \alpha(x) \leqslant v(x) \leqslant \beta(x) \\ \beta(x) & v(x) \leqslant \beta(x) . \end{cases}$$

The condition (2) *is necessary.* Let us suppose $f(x_0, \alpha(x_0)) > 0$, $x_0 \in \bar{\Omega}$. In the case $v(x) = \alpha(x) - tf(x, \alpha(x))$ $(t > 0)$, we have $v(x_0) < \alpha(x_0)$ and $v_K(x_0) = \alpha(x_0)$. Then

$$d(\alpha - tF\alpha, K) = |v - v_K| \geqslant v_K(x_0) - v(x_0) = tf(x_0, \alpha(x_0))$$

and

$$\lim t^{-1} d(\alpha - tF\alpha, K) \geqslant f(x_0, \alpha(x_0)) > 0$$

and (HP1) doesn't hold.

The condition (2) *is sufficient.* On the contrary there exists $\varepsilon > 0$ $u \in K$ and a sequence $t_n \to 0$, such that $d(u - t_n Fu, K) \geqslant \varepsilon t_n$.

Let us pose $v_n = (u - t_n Fu)_K$, then

$$(4) \qquad |u - t_n Fu - v_n| > \varepsilon t_n .$$

Therefore $v_n(x_n) = \alpha(x_n)$ or $v_n(x_n) = \beta(x_n)$ so we can suppose, evenctually keeping in mind a subsequence, that $v_n(x_n) = \alpha(x_n)$ and $x_n \to$

$\to x \in \bar{\Omega}$. By (4) we have

$$\alpha(x_n) - u(x_n) + t_n f(x_n, \alpha(x_n)) > \varepsilon t_n \tag{4'}$$

and

$$\alpha(x_n) \leqslant u(x_n) < \alpha(x_n) + t_n f(x_n, \alpha(x_n)) - \varepsilon t_n$$

therefore $\alpha(x) = u(x)$.

From (4′)

$$0 \geqslant t_n^{-1}(\alpha(x_n) - u(x_n)) \geqslant \varepsilon - f(x_n, \alpha(x_n))$$

that is unpossible because $f(x_n, \alpha(x_n)) \to f(x, \alpha(x)) \leqslant 0$; the lemma follows.

LEMMA 9. Let $X = L^p(\bar{\Omega})$, $1 \leqslant p < +\infty$; the function $F\colon K_p \to X$ defined by (1) is continuous; moreover F verify the condition (HP1) iff (2) holds.

PROOF. The function F is continuous because of the Lebesgue convergence theorem. Let us first note that, also in this case, $d(v, K_p) = |v - v_K|$ where v_K is defined by (3).

The condition (2) *is necessary.* Let u belong to K_p. Let us consider the functions

$$\psi_t^+(x) = \begin{cases} 0 & u(x) - tf(x, u(x)) \leqslant \beta(x) \\ 1 & u(x) - tf(x, u(x)) > \beta(x) \end{cases}$$

$$\psi_t^-(x) = \begin{cases} 0 & u(x) - tf(x, u(x)) \geqslant \alpha(x) \\ 1 & u(x) - tf(x, u(x)) < \alpha(x)\,. \end{cases}$$

Now we have

$$|d(u - tFu, K)|^p =$$
$$= \int_\Omega \{\psi_t^-(x)|u(x) - tf(x, u(x)) - \alpha(x)|^p + \psi_t^+(x)|u(x) - tf(x, u(x)) - \beta(x)|^p\} dx$$

and

$$\left|\frac{d(u - tF(u), K)}{t}\right|^p = \int_\Omega \left\{\psi_t^-(x)\left|\frac{u(x) - \alpha(x)}{t} - f(x, u(x))\right|^{p,x} + \right. \tag{5}$$
$$\left. + \psi_t^+(x)\left|\frac{\beta(x) - u(x)}{t} + f(x, u(x))\right|^p\right\} dx\,.$$

Let us consider $u(x) = \alpha(x)$ and $E = \{x \in \bar{\Omega}: f(x, \alpha(x)) > 0\}$; now $\psi_t^-(x) = 1$ on E and

$$|t^{-1} d(u - tF(u), K)|^p \geqslant \int_E |f(x, \alpha(x))|^p dx$$

therefore $\text{mis}\, E = 0$ and the thesis follows.

The condition is sufficient. Let u belong to K_p. Let us prove that for every x, it exists $t_x > 0$ such that $\psi_t^-(x) = \psi_t^+(x) = 0$. In fact if $u(x) = \beta(x)$, by (3): $u(x) - tf(x, u(x)) = \beta(x) - tf(x, \beta(x)) \leqslant \beta(x)$ and $\psi_t^+(x) = 0$; if $u(x) < \beta(x)$ and t is small $u(x) - tf(x, u(x)) \leqslant \beta(x)$; analougsly we procede for $\psi_t^-(x)$. Then the integrand function in (5) goes to zero punctually.

In order to use Lebesgue's convergence theorem we must prove, for instance, that

$$\psi_t^-(x) \left| \frac{u(x) - \alpha(x)}{t} - f(x, u(x)) \right|^p$$

is bounded by $|f(x, u(x))|^p$.

If $\psi_t^-(x) = 0$ we have nothing to prove; if $\psi_t^-(x) = 1$ we have $u(x) - tf(x, u(x)) < \alpha(x)$, and, by (3), $u(x) > \alpha(x)$; then

$$0 < \frac{u(x) - \alpha(x)}{t} < f(x, u(x))$$

therefore $f(x, u(x)) - t^{-1}(u(x) - \alpha(x)) < f(x, u(x))$; the lemma follows.

THEOREM 3. Let (3) hold; moreover

i) $\alpha, \beta \in W^{1,1}(\Omega)$,

ii) $\Delta\alpha \leqslant 0$, $\Delta\beta \geqslant 0$,

iii) $\alpha(x) \leqslant 0 \leqslant \beta(x)$, $x \in \partial\Omega$.

Let us consider a measurable (necessarely bounded) function $u_0(x)$ verifying $\alpha(x) \leqslant u_0(x) \leqslant \beta(x)$ a.e., that is $u_0 \in K_p$ $(p > 1)$. Then, in every $L^p(\Omega)$ a global strict solution to the quasi-linear heat equation exists

$$(6) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(x, u(t, x)) = 0 & x \in \Omega, \quad t \geqslant 0 \\ u(t, x) = 0 & x \in \partial\Omega, \ t \geqslant 0 \\ u(0, x) = u_0(x) & x \in \Omega\,. \end{cases}$$

PROOF. Let $X = L^p(\Omega)$ and $A = -\Delta\big(D(\Delta) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\big)$; we remark that (HP2) holds because of the maximum principle; thus for theorem 1 and lemma 9 we can conclude that a local mild solution to equation (6) exists. We note that $F(K_p)$ is bounded in L^∞-norm, therefore also in L^p-norm; then we conclude that a global solution to equation (6) exists.

After let $v(t) = -f(x, u(t, x)) \in C^0(0, \infty; L^p(\Omega)) \subset L^p_{\mathrm{loc}}(0, \infty; L^p(\Omega))$; thus u is a mild solution to the problem

$$u(0) = u_0 , \qquad u'(t) - \Delta u(t) = v(t)$$

and, for a well-known result by Aganovic-Vishik (see [1]), u is a strict solution.

REMARK. If, in addition to the hypotheses of theorem 3, we suppose

$$|f(x, u_2) - f(x, u_1)| \leqslant L|u_2 - u_1|^\alpha ,$$
$$(x, u_i) \in D \;\; (i = 1, 2) ; \qquad L > 0 ; \qquad \alpha \in]0; 1]$$

we have $\partial u/\partial t$, $\Delta u \in C^\infty(0, \infty; L^p(\Omega))$.

PROOF. The function F is holder-continuous in K_p; in fact, if $u_1, u_2 \in K_p$

$$|F(u_2) - F(u_1)|^p_{L^p(\Omega)} = \int_\Omega |f(x, u_2(x)) - f(x, u_1(x))|^p \, dx \leqslant$$
$$\leqslant L^p \int_\Omega |u_2(x) - u_1(x)|^{p\alpha} dx \leqslant L^p (\operatorname{mis} \Omega)^{1-\alpha} \Big(\int_\Omega |u_2(x) - u_1(x)|^p \, dx \Big)^p .$$

The thesis follows from remark 2 of theorem 1.

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Manoscritto pervenuto in redazione il 5 luglio 1975.