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Consistency Property and Model Existence Theorem for Second Order Negative Languages with Conjunctions and Quantifications over Sets of Cardinality Smaller than a Strong Limit Cardinal of Denumerable Cofinality.

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SUMMARY - In a second order negative infinitary language $L^{2-}_{k,k}$, where k is a strong limit cardinal of cofinality ω, we extend Karp's notions of ω-chain of models and ω-satisfiability; then we introduce an adequate notion of consistency property and we prove a model existence theorem to the effect that any set in a consistency property is ω-satisfiable.

SOMMARIO - In un linguaggio infinitario del secondo ordine negativo $L^{2-}_{k,k}$ con k cardinale limite forte di cofinalità ω, si estendono le nozioni di Karp di ω-catena di modelli e di ω-soddisfacibilità. Quindi si introduce una adeguata nozione di proprietà di consistenza e si dimostra un teorema di esistenza di modelli affermante che ogni insieme in una proprietà di consistenza è ω-soddisfacibile.

Introduction.

Second order positive languages, L^{2+}, i.e. second order languages where the second order variables are quantified only universally, look rather interesting from the point of view of interpolation theorems

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and definability, since these problems are mainly concerned with predicates which can be considered universally quantified second order variables. Indeed this approach was taken by Chang in his interpolation theorem in [2], and, more explicitly by Maehara and Takeuti in their interpolation theorems in [9].

Another promising language for interpolation and definability theorems is $L_{k,k}$, i.e. the infinitary language where we allow conjunctions over sets of less than k formulas and quantifications over sets of less than k variables, where k is a strong limit cardinal of cofinality ω. Indeed Karp in [7] was able to extend to this language Craig's interpolation theorem overcoming, in a sense, Malitz's limit to interpolation theorems for infinitary languages, see [11]. Of course, to do this, Karp could not use only the usual notion of satisfiability, but she took advantage of ω-chains of models and ω-satisfiability which were introduced by Karp herself in [6].

The main tool to prove Katp's interpolation theorem in [7] is an adequate notion of consistency property. Consistency property for first order languages are well explained by Smullyan in [13], and are one of the main tools in Keisler's book [8] on the model theory of $L_{\omega_1,\omega}$.

In this paper we will consider the language $L^{2-}_{k,k}$, a combination of the previous mentioned languages, and we will develop adequate tools in this language in order to be able to prove new interpolation theorems in a following paper. Namely we will extend the notions of ω-chain of structures and ω-satisfiability to $L^{2-}_{k,k}$, then we will introduce adequate notions of consistency properties and prove the related model existence theorems.

Chapter I

PRELIMINARIES

I.1. – Set theoretic preliminaries.

The development of this paper will be informal, and we will use informally the basic notion of a set theory. This will be one with classes, axiom of regularity and axiom of choice, for instance the set theory in [12].

We will use $\in$, $\subseteq$, $\subset$ to denote the relations of membership, inclusion and proper inclusion, and $\notin$, $\nsubseteq$, $\not\subset$ to denote their negations.

$\{x: Q(x)\}$ will denote the class of the sets x such that $Q(x)$. If $\{x: Q(x)\}\subseteq y$ where y is a set, then also $\{x: Q(x)\}$ is a set. Most of the time that we will use the notation $\{x: Q(x)\}$ it will be easy to show that there is a set containing $\{x: Q(x)\}$; this will be so often and so clear that we will not even mention it.

If A and B are classes, $A-B$ will denote the class $\{x: x\in A$ and $x\notin B\}$, and A^c will denote the class $\{x: x\notin A\}$.

Functions are classes of ordered pairs $\langle x, y\rangle$ such that if $\langle x, y\rangle$ and $\langle x, z\rangle$ belong to the function then $y=z$. dom f and rng f will denote the domain and the range of the function f. $f\urcorner A$ will denote the restriction of the function f to the class A, that is $f\urcorner A=\{\langle x, y\rangle$: $\langle x, y\rangle\in f$ and $x\in A\}$. ${}^AB$ will denote the set of all functions whose domain is A and whose range is in B.

$\cup A$ will denote $\{x: \exists y\in A$ and $x\in y\}$. If $A=\{A_1, A_2, \ldots, A_n\}$ we will write $A_1\cup A_2\cup\ldots\cup A_n$ instead of $\cup A$.

Ordinals and cardinals will be defined as usual. $|A|$ will denote the cardinality of the set A.

If α is a cardinal, α^+ will denote the cardinal successor of α. A limit ordinal is one which is not equal to $\alpha+1$ for any ordinal α. A limit cardinal is one which is not equal to α^+ for any cardinal α.

The cofinality of a cardinal α is the least ordinal δ such that there are cardinals $a_i<\alpha$ for $i\in\delta$ and $\cup\{a_i: i\in\delta\}=\alpha$.

A limit cardinal α is called a strong limit cardinal if $2^\delta<\alpha$ for $\delta<\alpha$.

We will reserve the letter k to denote a strong limit cardinal of cofinality ω. As usual ω denotes the first infinite ordinal.

$\emptyset$ is the symbol that we reserve for the empty set, i.e. the set $\{x: x\in A$ and $x\notin A\}$ for any A.

$\cdot$ is the symbol that we reserve for the operation of composition of two functions f_1 and f_2, i.e. $f_1\cdot f_2$ is the function $\{\langle x, z\rangle$: there is y such that $\langle x, y\rangle\in f_1$ and $\langle y, z\rangle\in f_2\}$.

$$\times\{A_i: i\in I\}=\{\{(i, a_i): i\in I\}: a_i\in A_i\}$$

I.2. – The language.

The *languages* $L^{2+}_{\alpha,\alpha}$, $L^{2-}_{\alpha,\alpha}$, $L^2_{\alpha,\alpha}$ will consist of the following symbols:

(*a*) α individual variables: v_i for $i\in\alpha$ (0-placed variables), (*b*) α p-placed predicate variables for each $p \in \omega$ and $p \neq 0$: V_i^p for $i \in \alpha$,

(*c*) connectives: — and &,

(*d*) quantifier: $\forall$,

(*e*) truth symbol: t,

(*f*) identity symbol: $=$,

(*g*) auxiliary symbols: , and (and).

There is no loss in generality assuming that $L^{2+}_{\alpha,\alpha}$, $L^{2-}_{\alpha,\alpha}$, $L^{2}_{\alpha,\alpha}$ do not have individual and predicate constants, since these can be regarded as specific variables that we decide not to quantify. Indeed it is considering constants in this way that the notion of satisfaction for a language including constants could be extended to that of ω-satisfaction for a language including constants (see section 1.3).

The *formulas* of the language $L^{2}_{\alpha,\alpha}$ are defined as follows:

(i) $V_j^p(v_{i_1}, \ldots, v_{i_p})$ is an atomic formula for all $p \in \omega$, $j \in \alpha$ and $i_1, \ldots, i_p \in \alpha$. t is an atomic formula. $v_{i_1} = v_{i_2}$ is an atomic formula for all $i_1, i_2 \in \alpha$. Atomic formulas are formules.

(ii) If F is a formula, then $-F$ is a formula.

(iii) If $\overline{F}$ is a non empty set of less than α formulas, then $\&\overline{F}$ is a formula.

(iv) If $\bar{v}$ is a set of less than α individual variables, and F is a formula, then $\forall\bar{v}F$ is also a formula.

(v) If $\overline{V}$ is a set of less than α predicate variables, and F is a formula, then $\forall\overline{V}F$ is also a formula.

(vi) Nothing else is a formula.

The *scope* of an occurrence of the connective — in a formula F is the formula G which is the second element of the ordered pair $-G$ where — is the given occurrence of —.

The scope of an occurrence of the connective & in a formula F is the set of formulas $\overline{G}$ which is the second element of the ordered pair $\&\overline{G}$ where & is the given occurrence of &.

The scope of an occurrence of the quantifier $\forall$ in a formula F is the formula G which is the third element of the ordered triple $\forall\bar{v}G$ or of the ordered triple $\forall\overline{V}G$ where $\forall$ is the given occurrence of $\forall$.

An occurrence of $\forall$ is called *first order* if $\forall$ is followed by a set of individual variables; while it is called *second order* if it is followed by a set of predicate variables.

A *first order formula* (a formula in the language $L_{\alpha,\alpha}$) is one in which all occurrences of $\forall$ are first order.

An occurrence of a variable in a formula is *bound* (*free*) if it is (is not) within the scope of a quantifier followed by a set containing the given variable.

If F is a formula and $\bar{v}$ is a set of variables each one of them in a set of variables following a quantifier in F and f is a $1-1$ function that preserves the type of the variables from $\bar{v}$ onto a set $\bar{v}'$ of variables that do not occur in F then the result of substituting $f(v)$ for each occurrence of the variable $v \in \bar{v}$ in some set of variables following a quantifier or in the scope of the same quantifier is still a formula. We will call this procedure to go from one formula to another a *change of bound variables*.

Clearly substituting any variables v' for a variable v of the same type in a formula F we obtain another formula F', but it may happen that an occurrence of v is free (bound) in F and the corresponding occurrence of v' in F' is bound (free). If this happen we will speak of *capture of variables*.

When performing a substitution of free variables, to *avoid a capture of variables* is to perform a change of bound variables before the substitution such that the range of the function in the change of bound variables is disjoint from the set of the variables introduced with the substitution.

Immediate subformula of a formula F is the formula:

G if F is $-G$;
G if F is $\&\bar{G}$ and $G \in \bar{G}$;
$G(\bar{v}/f)$ if F is $\forall\bar{v}\, G$ for all functions f from $\bar{v}$ into the variables preserving the type of the variables, where $G(\bar{v}/f)$ stands for the formula obtained from G by substituting the variables $f(v)$ for each variable $v \in \bar{v}$ and avoiding the capture of variables.

A *subformula* of a formula is either the formula itself or an immediate subformula of a formula which was already proved to be a subformula of the given formula.

A *weak subformula* is either a subformula or the negation of a subformula.

The *depth* of an occurrence of a subformula in a formula is the number of connectives and quantifiers in the scope of which the occurrence of the subformula is.

REMARK. The depth of an occurrence of a subformula in a formula is a finite number.

A *formula* is *negative* (*positive*) if all second order quantifiers in it are within the scope of an odd (even) number of negation symbols.

The formulas of $L^{2-}_{\alpha,\alpha}$ ($L^{2+}_{\alpha,\alpha}$) are the negative (positive) formulas of $L^{2}_{\alpha,\alpha}$.

A *quantifier* is *universal* (*existential*) if it is in the scope of an even (odd) number of negation symbols.

An *occurrence of a subformula* in a formula is *negative* (*positive*) if it is in the scope of an odd (even) number of negation symbols.

The *rank* of a formula is defined as follows:

if F is an atomic formula then its rank is 0;

if F is $-G$ then rank $F = (\text{rank } G) + 1$;

if F is $\&\bar{G}$ then rank $F = \text{Max}\,(\text{rank } G) + 1 \colon G \in \bar{G}$;

if F is $\forall \bar{v} G$ then rank $F = (\text{rank } G) + 1$ for any set of variables $\bar{v}$.

A *sentence*, or closed formula, is a formula without free variables.

It is clear that a variable v may occur free and bound in the same formula F. In this case we may assume that there are always enough variables and so we can perform a change of bound variables replacing the bound occurrences of v with a variable that does not occur in F. And we may assume that either all occurrences of a variables are free in a formula or all occurrences of a variable are bound in a formula. Furthermore, it may be that a bound variable occurs in the sets of variables following two or more different occurrences of the quantifier. Then again under the assumption that there are always enough variables, we can perform a change of bound variables in such a way that no bound variable occurs in more than one set following a quantifier, and we may assume this least clause.

The same can be said of the sets s of formulas, considering how a variable occurs in $\&s$.

The symbols δ_i^j with $i \in \alpha$ and $j \in \omega$ are called metavariables.

If used instead of a j-placed variable in a formula, they give rise to *metaformulas*.

It is clear that substituting variables for all metavariables of the same number of places in a metaformula, we obtain a formula. Every time we do the above we do it preserving the number of places.

At this point it is clear what we mean by a *submetaformula.*

I.3. – ω-chains of structures and ω-satisfaction.

From now on, when we will consider the languages $L^2_{k,k}$, $L^{2+}_{k,k}$, $L^{2-}_{k,k}$, k will always be a strong limit cardinal of cofinality ω.

The notions of structure, type of a structure adequate to a given language, substructure, and satisfaction are defined as usual.

An *ω-chain of structures* $\mathcal{M}$ is a sequence $\langle M_n \colon n \in \omega \rangle$ of sets M_n such that for all $n \in \omega$ $M_n \subseteq M_{n+1}$.

A *bounded assignment* $\boldsymbol{a}$ *in* $\mathcal{M}$ *to a set of variables* is a function that maps each p-placed variable in the given set into a set of p-tuples of elements of $\cup \{M_n \colon n \in \omega\}$ for $p \neq 0$, and for some fixed $n \in \omega$ each individual variable into M_n.

The ω-chain of structures $\mathcal{M}$ *ω-satisfies* a formula F of $L^2_{k,k}$ ($L^{2+}_{k,k}$, $L^{2-}_{k,k}$) (or F is ω-satisfied by $\mathcal{M}$) under the bounded assignment $\boldsymbol{a}$ to the variables free in F, $\mathcal{M}, \boldsymbol{a} \models^\omega F$, if one of the following cases holds:

(i) F is t,

(ii) F is $V^p_i(v_{i_1}, \ldots, v_{i_p})$ and $\langle \boldsymbol{a}(v_{i_1}), \ldots, \boldsymbol{a}(v_{i_p}) \rangle \in \boldsymbol{a}(V^p_i)$ where $i, i_1, \ldots, i_p \in k$,

(iii) F is $-G$ and not $\mathcal{M}, \boldsymbol{a} \models^\omega G$,

(iv) F is $\&\overline{G}$ and for all $G \in \overline{G}$, $\mathcal{M}, \boldsymbol{a} \models^\omega G$,

(v) F is $\forall \overline{v} G$ with $\overline{v}$ a set of individual variables and for all bounded assignments $\boldsymbol{b}$ to $\overline{v}$, $\mathcal{M}$, $(\boldsymbol{a} - \boldsymbol{a} \urcorner \overline{v}) \cup \boldsymbol{b} \models^\omega G$,

(vi) F is $\forall \overline{V} G$ with $\overline{V}$ a set of predicate variables and for all bounded assignments $\boldsymbol{b}$ to $\overline{V}$, $\mathcal{M}$, $(\boldsymbol{a} - \boldsymbol{a} \urcorner \overline{V}) \cup \boldsymbol{b} \models^\omega G$,

(vii) F is $v_{i_1} = v_{i_2}$ and $\boldsymbol{a}(v_{i_1})$ is $\boldsymbol{a}(v_{i_2})$.

A formula F is *ω-satisfiable* if there are $\mathcal{M}$, $\boldsymbol{a}$ (bounded assignment) such that $\mathcal{M}, \boldsymbol{a} \models^\omega F$.

A set S of formulas is ω-satisfiable if there are $\mathcal{M}$, $\boldsymbol{a}$ (bounded assignment) such that for all $F \in S$, $\mathcal{M}, \boldsymbol{a} \models^\omega F$.

A formula is ω-valid, $\models^\omega F$, iff $\mathcal{M}, \boldsymbol{a} \models^\omega F$ for all $\mathcal{M}$ and for all bounded assignments to the free variables of F.

REMARK I. If F is $-\&\bar{G}$ and $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} F$, then there is $G \in \bar{G}$ such that $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} -G$.

REMARK II. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are bounded assignments in $\mathcal{M}$ and S is a set of variables, then $(\boldsymbol{a} - \boldsymbol{a} \urcorner S) \cup (\boldsymbol{b} \urcorner S)$ is also a bounded assignment.

REMARK III. If F is $-\forall \bar{v} G$ and $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} F$, then there is a bounded assignment $\boldsymbol{b}$ to $\bar{v}$ such that $\mathcal{M}$, $(\boldsymbol{a} - \boldsymbol{a} \urcorner \bar{v}) \cup \boldsymbol{b} \models^{\omega} - G$.

REMARK IV. Similarly if F is $-\forall \bar{V} G$ and $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} F$, then there is a bounded assignment $\boldsymbol{b}$ to $\bar{V}$ such that $\mathcal{M}$, $(\boldsymbol{a} - \boldsymbol{a} \urcorner \bar{V}) \cup \boldsymbol{b} \models^{\omega} - G$.

REMARK V (Karp [7]). A formula in $L_{k,k}$ in which all first order quantifiers are followed by finite sets of variables isω -satisfied in a ω-chain of structures $\mathcal{M}$ by a bounded assignment in $\mathcal{M}$ if and only if it is satisfied (in the usual sense) in $\cup\{M_n : n \in \omega\}$.

REMARK VI. If the language $L^2_{k,k}$ ($L^{2+}_{k,k}$, $L^{2-}_{k,k}$) has individual and predicate constants, the notion of ω-chain of structures should be changed as follows:

An *ω-chain of structures for a language with constants* $\mathfrak{A}$ is a pair $\langle \mathcal{M}, \boldsymbol{\alpha} \rangle$ where $\mathcal{M}$ is an ω-chain of structures for the language without constants and $\boldsymbol{\alpha}$ is a bounded assignment to the constants, i.e. a function that maps each p-placed predicate constant into a set of p-tuples of elements of $\cup\{M_n : n \in \omega\}$ for $p \neq 0$, and for some fixed $\boldsymbol{n} \in \omega$ each individual constant into some $M_{\boldsymbol{n}}$.

The notion of ω-satisfaction for languages with constants, $\langle \mathcal{M}, \boldsymbol{\alpha} \rangle$, $\boldsymbol{a} \models^{\omega} F$, under the bounded assignment $\boldsymbol{a}$ to a set of variables including the free variables of F, would then be obtained from the notion of ω-satisfaction for languages without constants, $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} F$, by changing case (ii) and (vii) to read:

(ii′) F is $X^p(x_1, \ldots, x_p)$ and $\langle \boldsymbol{\delta}(x_1), \ldots, \boldsymbol{\delta}(x_p) \rangle \in \boldsymbol{\delta}(X^p)$, where X^p is either a p-placed predicate variable or a p-placed predicate constant, x_i for $i = 1, \ldots, p$ is either an individual variable or an individual constant, and $\boldsymbol{\delta}$ is $\boldsymbol{a} \cup \boldsymbol{\alpha}$,

(vii′) F is $x_1 = x_2$ and $\boldsymbol{\delta}(x_1)$ is $\boldsymbol{\delta}(x_2)$, where x_1, x_2, and $\boldsymbol{\delta}$ have the same meaning as in (ii′).

Hence we see that a formula with constants is ω-satisfied in an ω-chain of structures for languages with constants iff the same formula is ω-satisfied in an ω-chain of structures for languages without constants when the constants in the formula are considered as free variables.

REMARK VII (Karp [7]). If $M_n = M$ for all $n \in \omega$, then, for any formula F, $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} F$ iff M, $\boldsymbol{a} \models F$. Thus standard structures may be considered as particular ω-chains of structures, and ω-satisfiability a generalization of the notion of satisfiability. Hence $\models^{\omega} F$ implies $\models F$, but the converse is not true in general.

REMARK VIII (Karp [7]). The formula

$$\&\Big\{\forall v_1 - P(v_1, v_1),\ \forall v_1 \forall v_2 - \&\{-P(v_1, v_2),\ -P(v_2, v_1),\ v_1 = v_2\},$$
$$\forall v_1 \forall v_2 \forall v_3 - \&\big\{\&\{P(v_1, v_2),\ P(v_2, v_3)\},\ -P(v_1, v_3)\big\},$$
$$\forall v_1 - \forall v_2 - P(v_1, v_2),\ \forall\{v_n : n \in \omega\} - \&\{P(v_n, v_{n+1}) : n \in \omega\}\Big\}$$

is not satisfiable in any standard model, but it is ω-satisfiable in the ω-chain of structures $\mathcal{M}$ where M_n is the set $\{0, \dots, n\}$, under the bounded assignment $\boldsymbol{a} = \big\{(P$, the strict ordering of the natural numbers$)\big\}$.

REMARK IX. Suppose that the formula F' is obtained from the formula F through a change of bound variables, then F is ω-satisfiable iff F' is. Therefore the assumptions that each variable occurring free in a formula does not occur bound in the same formula and that no bound variable occur in more than one set following a quantifier, do not cause any loss of generality from the point of view of ω-satisfaction, and therefore from now on we will assume that either the formulas satisfy these assumptions or we immediately perform a change of bound variables that makes the formula satisfy the assumptions and we keep the same symbol for the formula.

LEMMA. Let $\{\forall \bar{v}_i F_i : i \in I\} \subseteq s$. If s, $\big\{F_i(\bar{v}_i/g_i) : g_i \in {}^{\bar{v}_i}\{x : x$ is an individual variable$\}$, $i \in I\big\}$ is not ω-satisfiable, then also s is not ω-satisfiable.

PROOF. Indeed if there are $\mathcal{M}$, $\boldsymbol{a}$ such that $\mathcal{M}$, $\boldsymbol{a} \models^{\omega} s$, $\{\forall \bar{v}_i F_i :$ $i \in I\}$, then for all bounded assignment $\boldsymbol{b}$ to $\cup\{\bar{v}_i : i \in I\}$, $\mathcal{M}$, $\boldsymbol{a} \cup \boldsymbol{b} \models^{\omega} s$, F_i for all $i \in I$. Hence, in particular, $\mathcal{M}$, $\boldsymbol{a} \cup \boldsymbol{b}_g \models^{\omega} s$, F_i for all $i \in I$, where $\boldsymbol{b}_g$ is a bounded assignment to $\cup\{\bar{v}_i : i \in I\}$ such that if $v' \in \bar{v}_i$ and $v'' \in \bar{v}_j$ and $g_i(v') = g_j(v'')$ then $\boldsymbol{b}_g(v') = \boldsymbol{b}_g(v'')$, and if $v \in \bar{v}_i$ for some $i \in I$ and $g_i(v) \in \operatorname{dom} \boldsymbol{a}$ then $\boldsymbol{b}_g(v) = \boldsymbol{a}(g_i(v))$, while if $g_i(v) \notin \operatorname{dom} \boldsymbol{a}$ then $\boldsymbol{b}_g(v)$ is any fixed element α, and $g = \cup \{g_i \colon i \in I\}$.

Let $\boldsymbol{b}'_g$ be $\cup \{g_i^{-1} \colon i \in I\} \cdot \boldsymbol{b}_g$ which is a function due to the definition of $\boldsymbol{b}_g$. Then it is clear that $\mathcal{M}$, $\boldsymbol{a} \cup \boldsymbol{b}'_g \models^{\omega} s$, $F_i(\bar{v}_i/g_i)$ for all $i \in I$, and for all g and for all $i \in I$ $\mathcal{M}$, $\boldsymbol{a} \cup \{(v, \alpha) \colon v \in \cup \{(\operatorname{rng} g - \operatorname{dom} \boldsymbol{a}) \colon g = \cup \{g_i \colon i \in I\} \text{ for all } g_i\}\} \models^{\omega} s$, $F_i(\bar{v}_i/g_i)$, and hence from the same ω-chain of structures under the same bounded assignment $\models^{\omega} s$, $\{F_i(\bar{v}_i/g_i) \colon i \in I, g_i \in \bar{v}_i\{x \colon x \text{ is an individual variable}\}\}$; a contradiction.

Chapter II

CONSISTENCY PROPERTY AND MODEL EXISTENCE THEOREM

II.1. – Consistency property for $L^{2-}_{k,k}$.

Since k is a strong limit cardinal of cofinality ω, we may assume that $k = \cup \{k_n \colon n \in \omega\}$, where $2^{k_n} \leqslant k_{n+1}$.

Let us now define the notion of consistency property for $L^{2-}_{k,k}$.

Let $C = \cup \{C_n \colon n \in \omega\}$ be a set such that $|C_n| = k_n$ and for all $m, n \in \omega$ if $m \neq n$ then $C_m \cap C_n = \emptyset$.

Let $\underline{C}^p = \cup \{\underline{C}^p_n \colon n \in \omega\}$ be a set such that $|\underline{C}^p_n| = k_n$ and for all $m, n, p, p' \in \omega$ if $\langle m, p \rangle \neq \langle n, p' \rangle$ then $\underline{C}^p_m \cap \underline{C}^p_n = \emptyset$.

Let L_n be the language obtained from $L^{2-}_{k,k}$ by adding $\cup \{C_i \colon i \in n\}$ as individual variables and $\cup \{\underline{C}^p_i \colon i \in n\}$ as p-placed predicate variables, for all $p \in \omega$, $p \neq 0$.

Σ is a *consistency property* for $L^{2-}_{k,k}$ with respect to $\{C_n \colon n \in \omega\}$, $\{\cup \{\underline{C}^p_n \colon p \in \omega\} \colon n \in \omega\}$ if Σ is a set of sets s of formulas whose free variables in $L^{2-}_{k,k}$ are all in a set V^* of cardinality $< k$ (we can take $k_0 \geqslant |V^*|$) such that all of the following conditions hold:

(C0) For all $s \in \Sigma$, $|s| < k$ and there is an n (depending on s) such that all formulas in s are in L_n.

(C1) If Z is an atomic formula then either Z is either t or $x = x$ and $-Z \notin s$, or Z is neither t nor $x = x$ and either $Z \notin s$ or $-Z \notin s$.

(C2) If $\{--F_i\colon i\in I\}\subseteq s$ and $|I|<k$, then $s\cup\{F_i\colon i\in I\}\in\Sigma$.

(C3) If $\{\&\overline{F}_i\colon i\in I\}\subseteq s$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ we have $0<|\overline{F}_i|<k_m$, then $s\cup(\cup\{\overline{F}_i\colon i\in I\})\in\Sigma$.

(C4) If $\{-\&\overline{F}_i\colon i\in I\}\subseteq s$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ we have $0<|\overline{F}_i|<k_m$ then there is a functions $f\in\times\{\overline{F}_i\colon i\in I\}$ such that $s\cup\{-f(i)\colon i\in I\}\in\Sigma$.

(C5) If $\{\forall\overline{v}_iF_i\colon i\in I\}\subseteq s$ and $|I|<k$ and for all $i\in I$ the variables in $\overline{v}_i$ are individual and there is $m\in\omega$ such that for all $i\in I$ we have that $|\overline{v}_i|<k_m$, then for the first natural number n such that the formulas in s are all in L_{n-1} we have that $s\cup\{F_i(\overline{v}_i/f_i)\colon$ for all functions f_i from $\overline{v}_i$ into $\cup\{C_j\colon j\leqslant n\}\cup V^*$, and for all $i\in I\}\in\Sigma$, where $F_i(\overline{v}_i/f_i)$, as every where else in this paper, has the meaning already specified in the definition of immediate subformula in section I.2.

(C6) If $\{-\forall\overline{V}_iF_i\colon i\in I\}\subseteq s$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ $|\overline{V}_i|<k_m$, then for the first natural number n such that no element of $\cup\{C_n^p\colon p\in\omega\}\cup C_n$ is in s and for all $p\in\omega$ $|C_n^p|\geqslant\geqslant k_m+|I|$ and $|C_n|\geqslant k_m+|I|$, for all 1—1 place preserving functions f from $\cup\{\overline{V}_i\colon i\in I\}$ into $\cup\{\underline{C}_n^p\colon p\in\omega\}\cup C_n$ we have that $s\cup\{-F_i(\overline{V}_i/f\urcorner\forall_i)\colon i\in I\}\in\Sigma$.

[(C6′) If $\{-\forall\overline{V}_iF_i\colon i\in I\}\subseteq s$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ $|\overline{V}_i|\leqslant k_m$, then there are a natural number n and a place preserving function f from $\cup\{\overline{V}_i\colon i\in I\}$ into $\cup\{\underline{C}_n^p\colon p\in\omega\}\cup C_n$ such that $s\cup\{-F_i(\overline{V}_i/f\urcorner\overline{V}_i)\colon i\in I\}\in\Sigma$.

Clearly (C6) implies (C6′)].

(C7) (i) If $\{v_{i_1}=v_{i_2}\colon i\in I\}\subseteq s$ and $|I|<k$, then $s\cup\{v_{i_2}=v_{i_1}\colon i\in I\}\in\Sigma$.

(ii) If $\{Z_i(v_{i_1}), v_{i_1}=v_{i_2}\colon i\in I\}\subseteq s$ and $|I|<k$ and for all $i\in I$ Z_i is an atomic or negated atomic formula, then $s\cup\{Z_i(v_{i_2})\colon i\in I\}\in\Sigma$ where $Z_i(v_{i_2})$ is the formula obtained from $Z_i(v_{i_1})$ by substituting v_{i_2} for one occurrence of v_{i_1}.

Remark. We actually defined two different notions of consistency property, say CP and CP', according to whether we include (C6) or (C6′). Clearly if Σ is CP then it is also CP'. So once we have proved that if s belongs to a CP' then it is ω-satisfiable (model existence theorem) it will follow that the same will be true for the sets belonging to a CP.

But there is more to it.

THEOREM. The two notions are equivalent in the sense that if $\bar{s}$ belons to Σ' which is a CP' then there is also Σ which is a CP such that $\bar{s}$ belongs to it.

PROOF. Let $\Sigma'_0 = \{\bar{s}\}$. Let Σ'_{n+1} be the set of all the sets $s^* \in \Sigma'$ such that there is $s^{**} \in \Sigma'_n$ and $s^* \supseteq s^{**}$ and s^* is obtained from s^{**} applying one of the steps (C2), (C3), (C4), (C5), (C6′), (C7). Clearly $\bigcup \{\Sigma'_n : n \in \omega\} = \Sigma'$.

Let us define by induction on n sets Σ_n of sets of formulas, functions g_n from Σ_n in Σ'_n that extend g_{n-1} for $n > 0$, functions $h_{n,s}$ from the set $s \in \Sigma$ in the set $g_n(s)$ and functions $f_{n,s}$ from the free variables occurring in the formulas of $s \in \Sigma$ onto the free variables occurrins in the formulas of $\{h_{n,s}(F) : F \in s\}$, in such a way that for all $F \in s$ $h_{n,s}(F(\bar{v})) = F(\bar{v}/f_{n,s} \urcorner \bar{v})$ where $\bar{v}$ is the set of all free variables in F, as follows.

$\Sigma_0 = \{\bar{s}\} = \Sigma'_0$; g_0, $h_{0,\bar{s}}$, $f_{0,\bar{s}}$ are the identity.

Suppose that Σ_n, g_n, $h_{n,s}$, $f_{n,s}$ were already defined for all $s \in \Sigma_n$.

Then proceed according to the following cases, where S is any subset of s satisfying the conditions stated below at each step.

1) $S = \{--F_i : i \in I\}$ and $|I| < k$. Then let $s \cup \{F_i : i \in I\} = s' \in \Sigma_{n+1}$. Let

$$h_{n+1,s'} = h_{n,s} \cup \{(F_i, F'_i) : --F'_i = h_{n,s}(--F_i),\ i \in I\}.$$

Let

$$f_{n+1,s'} = f_{n,s} \quad \text{and} \quad g_{n+1}(s') = g_n(s) \cup \{F'_i : --F'_i = h_{n,s}(--F_i),\ i \in I\}.$$

2) $S = \{\&\bar{F}_i : i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\bar{F}| < k_m$. Then let $s \cup (\bigcup \{\bar{F}_i : i \in I\}) = s' \in \Sigma_{n+1}$. Let

$h_{n+1,s'} = h_{n,s} \cup \{(F_i, F'_i) : F'_i = F_i(\bar{v}_i/f_{n,s} \urcorner \bar{v}_i)$ where $F_i \in \bar{F}_i$ and $\bar{v}_i$ is the set of free variables in F_i and $i \in I\}$. Let $f_{n+1,s'} = f_{n,s}$ and

$g_{n+1}(s') = g_n(s) \cup \{F'_i : F'_i = F_i(\bar{v}_i/f_{n,s} \urcorner \bar{v}_i)$ where $F_i \in \bar{F}_i$, $\bar{v}_i$ is the set of free variables in F_i and $i \in I\}$.

3) $S = \{-\&\bar{F}_i : i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\bar{F}_i| < k_m$. Then let $f' \in \times \{\bar{G}_i : \bar{G}_i = \{F_i(\bar{v}_i/f_{n,s} \urcorner \bar{v}_i) : F_i \in \bar{F}_i$ and $\bar{v}_i$ are the free variables in $F_i\}, i \in I\}$ be such that $g_n(s) \cup \{-f'(i) : i \in I\} \in \Sigma'_{n+1}$. Let $f \in \times \{\bar{F}_i : i \in I\}$ be such that $f(i)(\bar{v}_i/f_{n,s} \urcorner \bar{v}_i) = f'(i)$ where again $\bar{v}_i$ are the free variables in $f(i)$.

Let

$$s \cup \{-f(i) : i \in I\} = s' \in \Sigma_{n+1}\,.$$

Let

$$h_{n+1,s'} = h_{n,s} \cup \{(-f(i), -f'(i)) : i \in I\}\,.$$

Let $f_{n+1,s'} = f_{n,s}$ and let $g_{n+1}(s') = g_n(s) \cup \{-f'(i) : i \in I\}$.

4) $S = \{\forall \bar{v}_i F_i : i \in I\}$ and $|I| < k$ and for $i \in I$ the variables in $\bar{v}_i$ are individual and there is $m \in \omega$ such that for all $i \in I$ $|\bar{v}_i| < k_m$. Let p be the first natural number such that all the formulas in s are in L_{p-1}. Let p' be the first natural number such that all the formulas in $g_n(s)$ are in $L_{p'-1}$. Let φ_i be the set of all functions from $\bar{v}_i$ into $\cup\{C_j : j \leqslant p\} \cup V^*$. Let

$$s' = s \cup \{F_i(\bar{v}_i/f_i) : f_i \in \varphi_i,\ i \in I\} \in \Sigma_{n+1}\,.$$

Let $h_{n+1,s'} = h_{n,s} \cup \Big\{\big(F_i(\bar{v}_i/f_i),\ F(\overline{w}_i/(f_{n,s} \cup g_i) \urcorner \overline{w}_i)\big) : i \in I,\ f_i \in \varphi_i,\ \overline{w}_i$ are the free variables in F_i, and g_i is the function from $\bar{v}_i$ into $\cup\{C_j : j \leqslant p'\} \cup V^*$ such that for all $v \in \bar{v}_i$ if $f_i(v) \in$ Domain of $f_{n,s}$ then $g_i(v) = f_i \cdot f_{n,s}(v)$, if $f_i(v) \in (\cup\{C_j : j \leqslant p'\} \cup V^*)$ — Domain of $f_{n,s}$ then $g_i(v) = f_i(v)$, while $g_i(v) = v_0$, a fixed variable in V^*, otherwise$\Big\}$. Let

$$f_{n+1,s'} = f_{n,s} \cup \{(f_i(v), g_i(v)) : v \in \bar{v}_i,\ f_i \text{ and } g_i \text{ are defined as before},\ i \in I\}\,.$$

Let

$$g_{n+1}(s') = g_n(s) \cup \{h_{p+1,s'}(F_i(\bar{v}_i/f_i)) : i \in I,\ f_i \in \varphi_i\}\,.$$

5) $S = \{-\forall \overline{V}_i F_i : i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $|\overline{V}_i| < k_m$. Let n' be the first natural number such that no element of $\cup\{\underline{C}^p_{n'} : p \in \omega\} \cup C_{n'}$ is in s and for all $p \in \omega$ $|\underline{C}^p_{n'}| \geqslant k_m + |I|$ and $|C_{n'}| \geqslant k_m + |I|$. For all $1-1$ place preserving functions from $\cup\{\overline{V}_i : i \in I\}$ into $\cup\{\underline{C}^p_{n'} : p \in \omega\} \cup C_{n'}$ let $s \cup \{-F_i(\overline{V}_i/f \urcorner \overline{V}_i) : i \in I\} = s'_f \in \Sigma_{n+1}$. Let $s^* = g_n(s) \in \Sigma'_n$; $S^* = h_{n,s}(S)$. Clearly $S^* \subseteq s^*$ and $S^* = \{-\forall \overline{V}_i F_i(\overline{w}_i/f_{n,s} \urcorner \overline{w}_i) : \overline{w}_i$ is the set of the free variables in $\forall \overline{V}_i F_i$, $i \in I\}$. Therefore there is a natural number n'' and a place preserving function g from $\cup\{\overline{V}_i : i \in I\}$ into $\cup\{\underline{C}^p_{n''} : p \in \omega\} \cup C_{n''}$ such that

$$s \cup \{-F_i(\overline{V}_i/g \urcorner \overline{V}_i) : i \in I\} \in \Sigma'_{n+1}\,.$$

Let

$$h_{n+1,s'_f} = h_{n,s} \cup \Big\{ \big(-F_i(\overline{V}_i/f \urcorner \overline{V}_i),\ -F_i\big(\overline{w}_i \cup \overline{V}_i/(f_{n,s} \cup g) \urcorner (\overline{w}_i \cup \overline{V}_i)\big)\big):$$

$$f,\ g,\ \overline{w}_i \text{ are defined as above, } i \in I \Big\}.$$

Let

$$f_{n+1,s'_f} = f_{n,s} \cup \big\{ (f(v), g(v)) : v \in \cup \{ \overline{V}_i : i \in I \} \big\} ;$$

note that this definition is correct since f is a $1-1$ function. Let

$$g_{n+1}(s'_f) = g_n(s) \cup \big\{ h_{n+1,s'_f}(-F_i(\overline{V}_i/f \urcorner \overline{V}_i)) : i \in I \big\} .$$

6) Let $S = \{v_{i_1} = v_{i_2} : i \in I\}$ and $|I| < k$. Let $s \cup \{v_{i_2} = v_{i_1} : i \in I\} = s' \in \Sigma_{n+1}$. Let

$$h_{n+1,s'} = h_{n,s} \cup \big\{ \big(v_{i_2} = v_{i_1},\ f_{n,s}(v_{i_2}) = f_{n,s}(v_{i_1})\big) : i \in I \big\} .$$

Let $f_{n+1,s'} = f_{n,s}$. Let

$$g_{n+1}(s') = g_n(s) \cup \big\{ h_{n+1,s'}(v_{i_2} = v_{i_1}) : i \in I \big\} .$$

7) Let $S = \big\{ Z_i(v_{i_1}),\ v_{i_1} = v_{i_2} : i \in I \big\}$ and $|I| < k$ and for all $i \in I$ $Z_i(v_{i_1})$ is an atomic or negated atomic formula. Let $Z_i(v_{i_2})$ be the formula obtained from $Z_i(v_{i_1})$ by substituting v_{i_2} for one occurrence of v_{i_1}. Let $s \cup \big\{ Z_i(v_{i_2}) : i \in I \big\} = s' \in \Sigma^2_{n+1}$. Let $h_{n+1,s'} = h_{n,s} \cup \big\{ Z_i(v_{i_2}),\ Z'_i(v'_{i_2})\big) : i \in I$ and Z'_i is $Z_i(\overline{w}_i/f_{n,s} \urcorner \overline{w}_i)$ where $\overline{w}_i$ are the free variables in Z_i, $v'_{i_2} = f_{n,s}(v_{i_2})$ and $Z'_i(v'_{i_2})$ is obtained from Z'_i substituting v'_{i_2} for the occurrence of $f_{n,s}(v_{i_1})$ that corresponds to the occurrence of v_{i_1} in $Z_i(v_{i_1})$ that is changed in $v_{i_2}\big\}$. Let $f_{n+1,s'} = f_{n,s}$. Let

$$g_{n+1}(s') = g_n(s) \cup \big\{ h_{n+1,s'}\big(Z_i(v_{i_2})\big) : i \in I \big\} .$$

Let Σ_{n+1} be the least set that satisfies the previously stated conditions. Hence $g_n : \Sigma_n \to \Sigma'_n$ is completely defined for all $n \in \omega$.

Let $\Sigma = \cup \{ \Sigma_n : n \in \omega \}$.

At this point it is clear that to show that Σ is a *CP* it is enough to show that Σ satisfies (C1).

Indeed suppose that there is $s \in \Sigma$, say $\underline{s}$, such that either $-t \in \bar{s}$

or $x \neq x \in \underline{s}$ or $Z \in \underline{s}$ and $-Z \in \underline{s}$ where Z is an atomic formula. Since $\underline{s} \in \Sigma$ there is $n \in \omega$ such that $\underline{s} \in \Sigma_n$. Then $g_n(\underline{s})$ belongs to $\Sigma_n' \subseteq \Sigma'$.

If $-t \in \underline{s}$ then $h_{n,\underline{s}}(-t) = -t \in g_n(\underline{s})$, a contradiction since Σ' is a CP'. If $x \neq x \in \underline{s}$ then $h_{n,\underline{s}}(x \neq x)$ is $f_{n,\underline{s}}(x) \neq f_{n,\underline{s}}(x) \in g_n(\underline{s})$, a contradiction since $f_{n,\underline{s}}$ is a function and Σ' is a CP'. If $Z \in \underline{s}$ and $-Z \in \underline{s}$ where Z is an atomic formula and $\bar{v}$ is the set of free variables occurring in Z, then $h_{n,\underline{s}}(Z)$ is $Z(\bar{v}/f_{n,\underline{s}} \urcorner \bar{v}) \in g_n(\underline{s})$ and $h_{n,\underline{s}}(-Z)$ is $-Z(\bar{v}/f_{n,\underline{s}} \urcorner \bar{v}) \in g_n(\underline{s})$, a contradiction since again $f_{n,\underline{s}}$ is a function and Σ' is a CP'.

Therefore also Σ satisfies (C1) and it is a CP.

II.2. – Model existence theorem.

Model existence Theorem. If s is a set of formulas of $L^{2-}_{k,k}$, $|s| = k_0 < k$, and s belongs to S, a consistency property CP' with respect to $\{C_n: n \in \omega\}$, $\{\cup\{\underline{C}_n^p: p \in \omega\}: n \in \omega\}$, then s is ω-satisfied in an ω-chain of structures by a bounded assignment.

Moreover the n-th set in the chain has cardinality less or equal to k_m.

Proof. By a good split of a set of at most k formulas we shall mean a sequence $\langle s_m: m \in \omega\rangle$ such that $|s_m| \leqslant k_m$, every formula of the type either $\&\bar{F}$ or $-\&\bar{F}$ in s_m has $|\bar{F}| \leqslant k_m$, every formula of the type $-\forall \bar{V} F$ in s_m has $|\bar{V}| < k_m$, every formula of the type $\forall \bar{v} F$ in s_m has $|\bar{v}| \leqslant k_m$ and the variables in $\bar{v}$ are all individual.

Let us define, by induction on n, sets $s_n \in S$, and good splits $\langle s_{n,m}: m \in \omega\rangle$ of each s_n as follows.

$$s_0 = s\,; \qquad \langle s_{0,m}: m \in \omega\rangle \text{ is any good split of } s_0\,.$$

Suppose that s_h, $s_{h,m}: m \in \omega\rangle$ have been defined for all $k \leqslant n$.

Let

$$s_n' = s_{n,n} \cup \big\{\forall \bar{v} F: \forall \bar{v} F \in \cup\{s_{n,i}: i < n\}\big\} \cup \big\{c = d: c = d \in \cup\{s_{n,i}: i < n\}\big\}\,.$$

Clearly $s_n' \subseteq s_n$, $|s_n'| \leqslant k_n$ and all conjunction sets and quantification sets in s_n' have cardinality $\leqslant k_n$.

Define

$$s_n^{(1)} = s_n \cup \{F: --F \in s_n'\}\,,$$

$$s_n^{(2)} = s_n^{(1)} \cup \big(\cup\{\bar{F}: \&\bar{F} \in s_n'\}\big)\,,$$

$s_n^{(3)} = s_n^{(2)} \cup \{-f(\overline{F}): -\&\overline{F} \in s_n'\}$ where $f \in \times\{\overline{F}: -\&\overline{F} \in s_n'\}$ and f is such that if $s_n^{(2)} \in S$ so does $s_n^{(3)}$ (such an f exists by (C6′)),

$s_n^{(4)} = s_n^{(3)} \cup \{F(\bar{v}_F/f_F): \forall \bar{v}_F F \in s_n'$, for all $f_F: \bar{v}_F \to (\cup\{C_i: i \leqslant m\} \cup V^*)\}$, where the variables in $\bar{v}_F$ are all individual and m is the least natural number such that the formulas of $s_n^{(3)}$ are in L_{m-1},

$s_n^{(5)} = s_n^{(4)} \cup \{-F(\overline{V}_F/f): -\forall \overline{V}_F F \in s_n'\}$ where f is a place preserving function from $\cup\{\overline{V}_F: -\forall \overline{V}_F F \in s_n'\}$ to $\cup\{\underline{C}_n^p: p \in \omega\} \cup C_n$ and f and m are such that if $s_n^{(4)} \in S$ so does $s_n^{(5)}$ (such function f and natural number m exist by (C6′)),

$s_n^{(6)} = s_n^{(5)} \cup \{d = c: c = d \in s_n'\}$,

$s_n^{(7)} = s_n^{(6)} \cup \{Z(d): Z(c),\ c = d \in s_n'$ and Z is an atomic or negated atomic formula$\}$.

Notice that for all natural numbers n and for all $i = 1, \dots, 7$, $s_n^{(i)} \in S$ due to the conditions (C2), (C3), (C4), (C5), (C6′), (C7).

Define $s_{n+1} = s_n^{(7)}$.

Define $\langle s_{n+1,m}: m \in \omega \rangle$ as a good split of s_{n+1} such that $s_{n+1,m} = s_{n,m}$ for all $m \leqslant n$ and $s_{n+1,m} \supseteq s_{n,m}$ for all $m > n$.

Let $s_\omega = \cup\{s_n: n \in \omega\}$. The set s_ω can be used to define an ω-chain of structures using $\{C_n: n \in \omega\}$. The closure conditions $s_n^{(6)}$ and $s_n^{(7)}$ can be used to show that the relation $\sim$ defined as $c \sim d$ if either $c = d \in s_\omega$ or c is d is an equivalence relation on $\cup\{C_i: i \in \omega\} \cup v^*$ where v^* is the set of the individual variables in V^*, and

$$\sim_m = \sim \cap \Big((\cup\{C_i: i \in m\} \cup v^*) \times (\cup\{C_i: i \in m\} \cup v^*) \Big)$$

is an equivalence relation on $\cup\{C_i: i \in m\} \cup v^*$ such that

$$\sim_m \cap \Big((\cup\{C_i: i \in m'\} \cup v^*) \times (\cup\{C_i: i \in m'\} \cup v^*) \Big) = \sim_{m'}$$

for all $m' < m < \omega$.

Let $\{c/\sim: c \in \cup\{C_i: i \leqslant n\} \cup v^*\}$ be M_n. Consider the ω-chain of structures $\langle M_n: n \in \omega \rangle$. Consider the following bounded assignment $\boldsymbol{a}_n$: for all $c \in \cup\{C_i: i \leqslant n\} \cup v^*$, $\boldsymbol{a}_n(c) = c/\sim$, and for all predicate variables $V^p \in \cup\{\underline{C}_i^p: i \in \omega\} \cup (V^* - v^*)$,

$$\boldsymbol{a}_n(V^p) = \{(c_1/\sim, \dots, c_p/\sim): V^p(c_1, \dots, c_p) \in s_\omega,\ c_1, \dots, c_p \in \cup\{C_i: i \in \omega\} \cup v^*\}.$$

This is well defined since if $c_1 = d_1, \ldots, c_p = d_p \in s_\omega$ and $V^p(c_1, \ldots \ldots, c_p) \in s_\omega$ and $c_1, d_1, \ldots, c_p, d_p \in \operatorname{dom} \boldsymbol{a}_n$ then also $V^p(d_1, \ldots, d_p) \in s_\omega$.

Then an induction on the rank of any formula in s_ω shows that it is ω-satisfied in the ω-chain of structures $\langle M_n : n \in \omega \rangle$ under the assignment $\boldsymbol{a}_n$ described above, where n is such that the formula is in L_n, once the following properties of s_ω are established:

not both an atomic formula and its negation occur in s_ω,

if $--F \in s_\omega$ then $F \in s_\omega$,

if $\&\overline{F} \in s_\omega$ and $F \in \overline{F}$ then $F \in s_\omega$,

if $-\&\overline{F} \in s_\omega$ then there is $F \in \overline{F}$ such that $-F \in s_\omega$,

if $\forall \bar{v} F \in s_\omega$ then the variables in $\bar{v}$ are all individual and for all functions f from $\bar{v}$ into $\cup\{C_i : i \leqslant n\} \cup v^*$ we have that $F(\bar{v}/f) \in s_\omega$,

if $-\forall \overline{V} F \in s_\omega$ then there is a place preserving function f from $\overline{V}$ into

$$\cup\{\cup\{\underline{C}_i^p : i \in \omega\} : p \in \omega\} \cup (\cup\{C_i : i \leqslant n\}) \cup V^*$$

such that $-F(\overline{V}/f) \in s_\omega$,

if $c = d \in s_\omega$ then also $d = c \in s_\omega$,

if $Z(c)$, $c = d \in s_\omega$ where Z is an atomic or negated atomic formula, then also $Z(d) \in s_\omega$.

A set for which these properties hold is called a Hintikka set, and therefore s_ω is a Hintikka set.

Indeed if F is an atomic formula or a negated atomic formula in s_ω it will be in some L_n and then $\mathcal{M}, \boldsymbol{a}_n \models^\omega F$, due to the construction of $\mathcal{M}$ and of $\boldsymbol{a}_n$.

Suppose that the claim has been verified for all formulas in s_ω of rank less then the cardinal o. Let F be a formula in s_ω of rank o.

If F is in L_n and is $--F'$ then the rank of F' is $< o$ and F' is in s_ω and F' is in L_n, hence $\mathcal{M}, \boldsymbol{a}_n \models^\omega F'$, whence $\mathcal{M}, \boldsymbol{a}_n \models^\omega F$.

If F is in L_n and is $\&\overline{F}$ then the rank of each $F' \in \overline{F}$ is $< o$ and each $F' \in s_\omega$ and is in L_n, hence $\mathcal{M}, \boldsymbol{a}_n \models^\omega F'$ for all $F' \in \overline{F}$, whence $\mathcal{M}, \boldsymbol{a}_n \models^\omega F$.

If F is in L_n and is $-\&\overline{F}$ then there is $F' \in \overline{F}$ which is in s_ω and in L_n and rank $-F' < o$, hence $\mathcal{M}, \boldsymbol{a}_n \models^\omega -F'$, whence $\mathcal{M}, \boldsymbol{a}_n \models^\omega G$.

If F is in L_n and is $\forall\bar{v}F'$ then for all m and for all functions $f\colon \bar{v}\to\cup\{C_i\colon i\leqslant m\}\cup v^*$, $F'(\bar{v}/f)\in s_\omega$ and is in $L_{\mathrm{Max}(n,m)}$ and the rank of $F'(\bar{v}/f)$ is the same of the rank of F' which is smaller than o, hence for all m and f we have $\mathcal{M}$, $\boldsymbol{a}_{\mathrm{Max}(n,m)}\models^\omega F'(\bar{v}/f)$, and for all bounded assignments $\boldsymbol{b}$ to the variables in $\bar{v}$, $\mathcal{M}$, $(\boldsymbol{a}_{\mathrm{Max}(n,m)}-\boldsymbol{a}_{\mathrm{Max}(n,m)}\urcorner\bar{v})\cup\boldsymbol{b}\models^\omega F'$ whence $\mathcal{M}$, $\boldsymbol{a}_{\mathrm{Max}(n,m)}\models^\omega F$, but the free variables in F are in L_n and therefore $\mathcal{M}$, $\boldsymbol{a}_n\models^\omega F$.

If F is in L_n and is $-\forall\bar{V}F'$ then there is $m\in\omega$ and there is a place preserving function f from $\bar{V}$ into $\cup\{\cup\{\underline{C}_i^p\colon i\in\omega\}\colon p\in\omega\}\cup\cup(\{C_i\colon i\leqslant m\})\cup V^*$ such that $-F'(\bar{V}/f)\in s_\omega$ and is in $L_{\mathrm{Max}(n,m)}$ and the rank of $-F'(\bar{V}/f)$ is the same as the rank of $-F'$ which is smaller than o, hence $\mathcal{M}$, $\boldsymbol{a}_{\mathrm{Max}(n,m)}\models^\omega -F'(\bar{V}/f)$ and there is a bounded assignment $\boldsymbol{b}$ to the variables in $\bar{V}$ such that $\mathcal{M}$, $(\boldsymbol{a}_{\mathrm{Max}(n,m)}-\boldsymbol{a}_{\mathrm{Max}(n,m)}\urcorner\bar{V})\cup\cup\boldsymbol{b}\models^\omega -F'$, whence $\mathcal{M}$, $\boldsymbol{a}_{\mathrm{Max}(n,m)}\models^\omega F$, but the free variables in F are in L_n and therefore $\mathcal{M}$, $\boldsymbol{a}_n\models^\omega F$.

Since there are no other type of formulas in s_ω we can conclude that a formula of s_ω which is in L_n is ω-satisfied in the ω-chain of structures $\mathcal{M}$ under the bounded assignment $\boldsymbol{a}_n$. Hence all the formulas in $s\in S$ which are all in the same L_n are ω-satisfied in the ω-chain of structures $\mathcal{M}$ under the bounded assignment $\boldsymbol{a}_n$.

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