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**Artinian and noetherian factorized groups**

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# Artinian and Noetherian Factorized Groups.

Bernhard Amberg (*)

## Introduction.

If the group $G = AB$ is factorized by two subgroups $A$ and $B$, the question arises what can be said about the structure of the factorized group $G$ if the structure of its subgroups $A$ and $B$ is known. If $A$ and $B$ are artinian (noetherian),it is easy to see that $G = AB$ satisfies the minimum (maximum) condition for normal subgroups (Corollary 3.3). Sections 4 and 5 below show that for soluble groups much more can be said. Thus, a soluble group which is factorized by two artinian subgroups is artinian (Theorem 5.5), and a soluble group which is factorized by two nilpotent noetherian subgroups is noetherian (Theorem 4.3). Additional information on the structure of soluble artinian (noetherian) groups $G = AB$ which are factorized by two locally nilpotent subgroups $A$ and $B$ is collected in Theorems 4.4 and 5.7. For instance, the Hirsch-Plotkin radical of such a group $G$ is always « factorized » as a product of a subgroup of $A$ and a subgroup of $B$.

Some of the results in the first three sections are perhaps of independent interest. In section 1 it is shown that the « factorized » normal subgroups of a factorized group form a complete lattice (Corollary 1.4), and in section 2 a generalization of the well-known Theorem of Kegel and Wielandt stating that a finite group factorized by two nilpotent subgroups is soluble, is given (Theorem 2.3). Theorem 1.7 contains

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a criterion for a normal subgroup of a factorized group to be « factorized » which is fundamental at various places of the paper.

The author wishes to thank Professor O. H. Kegel for the kind permission to include some unpublished results of his in section 5.

**Notations.**

$AB$ = set of all elements $ab$ where $a \in A$ and $b \in B$

$X \subseteq Y$ = $X$ is a subgroup of $Y$

$\mathfrak{n}X$ = normalizer of $X$

$\mathfrak{R}G$ = Hirsch-Plotkin-radical of the group $G$

$\mathfrak{D}G$ = semi-radicable radical of $G$

a group is artinian (noetherian) if its subgroups satisfy the minimum (maximum) condition

## 1. – Elementary properties of factorized groups.

A subset $S$ of a factorized group $G = AB$ is called *factorized* (with respect to the factorization $G = AB$) if the following condition holds:

$$(*) \qquad \text{If } ab \in S \text{ where } a \in A,\ b \in B, \text{ then } a \in S\,.$$

(See Wielandt [11]).

The following lemma, which is easy to prove, shows that the asymmetry in the definition $(*)$ vanishes if $S$ is a subgroup of $G$.

Lemma 1.1 (Wielandt [11]). *If the group $G = AB$ is factorized by two subgroups $A$ and $B$, then the following conditions for the subgroup $S$ of $G$ are equivalent*:

(*a*) *$S$ is factorized,*

(*b*) *$S = (A \cap S)(B \cap S)$ and $A \cap B \subseteq S$.*

Remark 1.2. Let $S$ be a subgroup of the group $G = AB$ which is factorized by two subgroups $A$ and $B$ such that $A \cap B \subseteq S$, and let $Y = Y(S) = (A \cap S)(B \cap S)$. *If $Y$ is a subgroup, then it is the* *largest factorized subgroup of* $G = AB$ *which is contained in* $S$. For if $Z$ is a factorized subgroup of $G = AB$ such that $Y \subseteq Z \subseteq S$, then by Lemma 1.1 $Z = (A \cap Z)(B \cap Z)$, so that $Z \subseteq (A \cap S)(B \cap S) = Y$. Hence $Z = Y$, and the assertion follows.

The next lemma exhibits some closure properties of the set of all factorized subgroups of a factorized group $G = AB$.

LEMMA 1.3. *Let the group* $G = AB$ *be factorized by the subgroups* $A$ *and* $B$.

(*a*) *The intersection and the union of* (*arbitrary many*) *factorized subsets of* $G$ *are factorized subsets of* $G$.

(*b*) *If* $S$ *is a subset of* $G$ *and* $N$ *is a normal subgroup of* $G$, *then* $S/N$ *is factorized in* $G/N = (AN/N)(BN/N)$ *if and only if* $S$ *is factorized in* $G = AB$.

(*c*) *The product of two factorized subgroups* $N$ *and* $M$ *with* $MN = MN$ *and* $M(B \cap N) = (B \cap N)M$ *is a factorized subgroup of* $G$.

PROOF. (*a*) Let $S_i$, $i \in I$, be a collection of factorized subsets of $G$. If $ab$ is in $S = \bigcap_{i \in I} S_i$, where $a \in A$ and $b \in B$, then $ab \in S_i$ for each $i \in I$. Hence $a \in S_i$ for each $i \in I$, so that $a \in S$. It follows that $S$ is a factorized subset of $G$.

If $ab$ is in $V = \bigcup_{i \in I} S_i$, where $a \in A$ and $b \in B$, then $ab \in S_i$ for at least one $i \in I$. Since $S_i$ is factorized, $a \in S_i$. In particular $a$ is in $V$, so that $V$ is a factorized subset of $G$.

(*b*) Let $N$ be a normal subgroup of $G$ and $S/N$ be a factorized subset of $G/N = (AN/N)(BN/N)$. If $ab \in S$ where $a \in A$ and $b \in B$, then $abN/N = aNbN \in S/N$. Since $S/N$ is factorized, $aN \in S/N$, so that $a \in S$. Hence $S$ is a factorized subset of $G = AB$.

Conversely, let $S$ be a factorized subset of $G = AB$. If $abN \in S/N$, then $ab \in S$. Since $S$ is factorized, $a \in S$. Hence $aN \in S/N$, so that $S/N$ is factorized in $G/N = (AN/N)(BN/N)$.

(*c*) If $N$ and $M$ are factorized subgroups of $G$, by Lemma 1.1 $N = (A \cap N)(B \cap N)$, $A \cap B \subseteq N$, and $M = (A \cap M)(B \cap M)$, $A \cap B \subseteq M$. Clearly $A \cap B \subseteq NM$. Furthermore

$$NM = (A \cap N)(B \cap N)M = (A \cap N)M(B \cap N) =$$
$$= (A \cap N)(A \cap M)(B \cap M)(B \cap N) \subseteq (A \cap NM)(B \cap NM),$$

so that $NM = (A \cap NM)(B \cap NM)$. By Lemma 1.1 the product $NM$ is factorized.

Corollary 1.4. *Let the group $G = AB$ be factorized by two subgroups $A$ and $B$.*

(*a*) *The set of all factorized subsets of $G$ forms a complete sublattice of the lattice of all subsets of $G$.*

(*b*) *The set of all factorized normal subgroups of $G$ forms a complete sublattice of the lattice of all normal subgroups of $G$.*

For any subgroup $S$ of the factorized group $G = AB$ we consider the subset $X(S) = AS \cap BS$ of $G$.

Lemma 1.5. *If $S$ is a factorized subgroup of the group $G = AB$ which is factorized by two subgroups $A$ and $B$, then $S = X(S) = AS \cap BS$.*

Proof. If $x$ is an element of $X(S)$, then $x = as = bt$ where $a \in A$, $b \in B$ and $s, t \in S$. Hence $b^{-1}a = ts^{-1}$ is in $S$. Since $S$ is a subgroup, also $a^{-1}b = (b^{-1}a)^{-1}$ is in $S$. Since $S$ is factorized, $a^{-1}$ and $a$ are in $S$. Hence $x = as$ is in $S$. Thus $X(S) \subseteq S$, so that $S = X(S)$.

Corollary 1.6. *If $N$ is a factorized normal subgroup of the group $G = AB$ which is factorized by two subgroups $A$ and $B$, then $G/N = A^*B^*$ where $A^* \simeq A/(A \cap N)$, $B^* \simeq B/(B \cap N)$ and $A^* \cap B^* = 1$.*

Proof. By Lemma 1.5 $N = AN \cap BN$, since $N$ is factorized. Hence $G/N = AB/N = (AN/N)(BN/N) = A^*B^*$ where $A^* = AN/N \simeq A/(A \cap N)$ and $B^* = BN/N \simeq B/(B \cap N)$. Furthermore $A^* \cap B^* = (AN/N) \cap (BN/N) = (AN \cap BN)/N = 1$.

The following basic theorem contains a criterion for a normal subgroup of a factorized group to be factorized. Part (*b*) of it is essentially Lemma 1 of Sesekin [9].

Theorem 1.7. *Let $S$ be a subgroup of the group $G = AB$ which is factorized by two subgroups $A$ and $B$ such that $AS = SA$ and $BS = SB$.*

(*a*) *$X = X(S) = AS \cap BS$ is the smallest factorized subgroup of $G$ which contains $S$,*

(*b*) $X = S(A \cap BS) = S(B \cap AS) = (A \cap BS)(B \cap AS)$.

Proof. (*a*) $AS$ and $BS$ are factorized subgroups of $G$, since they contain $A$ or $B$. By Lemma 1.3 (*a*) $X(S) = AS \cap BS$ is also a factorized subgroup of $G$. Clearly $X(S)$ contains $S$. Let $T$ be any factorized subgroup of $G$ with $S \subseteq T \subseteq X = X(S)$. By Lemma 1.5 $T = AT \cap BT$. Hence $X = AS \cap BS \subseteq AT \cap BT = T$, so that $X = T$.

(*b*) By Dedekind's modular law $S(A \cap BS) = AS \cap BS = S \cdot (B \cap AS)$. Trivially $(A \cap BS)(B \cap AS) \subseteq AS \cap BS$. To show the other inclusion consider first an element $x$ in $S$. Then $x = ab$ where $a \in A$, $b \in B$. Hence $a = xb^{-1} \in A \cap BS$ and $b = a^{-1}x \in B \cap AS$, so that $x = ab \in (A \cap BS)(B \cap AS)$. It follows that

$$AS \cap BS = S(B \cap AS) \subseteq (A \cap BS)(B \cap AS)(B \cap AS) = (A \cap BS)(B \cap AS) .$$

This proves (*b*).

Theorem 1.7 (*a*) shows that a normal subgroup $N$ of the factorized group $G = AB$ is factorized if and only if it equals $X(N) = AN \cap BN$. This subgroup can be determined by means of Theorem 1.7 (*b*).

REMARK 1.8. The symmetric group of degree 4, $G = S_4$, has a factorization $G = AB$ where $A$ is a symmetric group of degree 3 and $B$ is a cyclic group of order 4. Clearly $A \cap B = 1$. The only proper normal subgroups of $G$ are the commutator subgroup $G^{(1)}$ which is isomorphic to the alternating group of degree 4 and the Fitting subgroup $F = \mathfrak{F}(G)$ which is a four group. One computes that $G = FA$ and that $FB$ has order 8. Hence also $X(F) = AF \cap BF$ has order 8 and is not subnormal in $G$. Thus in general $X(N)$ is only a subgroup though $N$ is a normal subgroup of the factorized group $G = AB$.

## 2. – A generalization of the theorem of Kegel and Wielandt.

In this section Theorem 1.7 is used to obtain a generalization of the theorem of Kegel and Wielandt stating that a finite group which is factorized by two nilpotent subgroups must be soluble; see for instance Huppert [1], Hauptsatz 4.3, p. 674.

How do properties of the normal subgroup $N$ of the factorized group $G = AB$ transfer to properties of the subgroup $X(N) = AN \cap BN$? Trivial examples show that $X(N)$ may be infinite while $N$ is finite. However, the following holds.

LEMMA 2.1. *If $N$ is a finite normal subgroups of the group $G = AB$ which is factorized by two subgroups $A$ and $B$, then $|X:D|$ is finite, where $X = X(N) = AN \cap BN$ and $D = A \cap B$.*

PROOF. By Theorem 1.7 (*b*) $X = X(N) = AN \cap BN = A^*B^*$ where $A^* = A \cap BN$ and $B^* = B \cap AN$. By Dedekind's modular law

$$AN = AN \cap AB = A(AN \cap B) = AB^* .$$

Hence

$$|B^*:D| = |B^*:(B^* \cap D)| = |B^*:(B^* \cap A)| = |AB^*:A| = \\ = |AN:A| = |N:(A \cap N)|$$

is finite. Similarly one obtains that $|A^*:D|$ is finite. Thus also

$$|X:D| = |A^*B^*:D| = |A^*B^*:B^*|\,|B^*:D| = \\ = |A^*:(A^* \cap B^*)|\,|B^*:D| = |A^*:D|\,|B^*:D|$$

is finite. This proves the lemma.

A finite normal subgroup of a group which is factorized by two locally nilpotent subgroups must be soluble. This is a consequence of the following result.

PROPOSITION 2.2. *If $N$ is a finite normal subgroup of the group $G = AB$ which is factorized by two locally nilpotent subgroups $A$ and $B$ then $X = X(N) = AN \cap BN$ is (locally nilpotent)-by-(finite and soluble); if $A$ and $B$ are nilpotent, then $X$ is soluble.*

PROOF. By Theorem 1.7 (*b*) $X = A^*B^*$ where $A^* = A \cap BN$ and $B^* = B \cap AN$. Let $D = A \cap B$. Since $N$ is finite, $|X:D|$ is finite by Lemma 2.1. By the theorem of Poincaré $X/D_X$ is finite. Hence $X/D_X = (A^*/D_X)(B^*/D_X)$ is a finite group factorized by two nilpotent subgroups $A^*/D_X$ and $B^*/D_X$. By the theorem of Kegel and Wielandt $X/D_X$ is soluble. As a subgroup of $A$ and $B$, the group $D_X$ is locally nilpotent. Hence $X$ is (locally nilpotent)-by-(finite and soluble). If $A$ and $B$ are nilpotent, $D_X$ is nilpotent, so that $X$ is soluble. This proves the proposition.

Proposition 2.2 is a generalization of the theorem of Kegel and Wielandt. It can be used to obtain further generalizations of this theorem. A group $G$ is (*almost*) *radical* if every epimorphic image $H \neq 1$ of $G$ contains a locally nilpotent (or finite) normal subgroup $N \neq 1$. $G$ is (*almost*) *hyperabelian* if every epimorphic image $H \neq 1$ of $G$ contains an abelian (or finite) normal subgroup $N \neq 1$.

THEOREM 2.3. *If the almost hyperabelian* [*almost radical*] *group* $G = AB$ *is factorized by two locally nilpotent subgroups* $A$ *and* $B$, *then* $G$ *is hyperabelian* [*radical*].

PROOF. Assume that 1 is the only abelian [locally nilpotent] normal subgroup of the epimorphic image $H \neq 1$ of $G$. Then $H$ is likewise an almost hyperabelian [almost radical] group which is factorized by two locally nilpotent subgroups. Hence there exists a finite normal subgroup $N \neq 1$ of $H$. By Proposition 2.2 $N$ is soluble. Thus there exists an abelian characteristic subgroup $M \neq 1$ of $N$ which is a non-trivial abelian normal subgroup of $H$. This contradiction proves the theorem.

## 3. – Chain conditions.

Is every group which is factorized by two artinian (noetherian) subgroups likewise artinian (noetherian)? At least we always have the following elementary

LEMMA 3.1. *If the group* $G = AB$ *is factorized by two artinian* (*noetherian*) *subgroups* $A$ *and* $B$, *then* $G$ *satisfies the minimum* (*maximum*) *condition for subgroups* $X$ *of* $G$ *with* $AX = XA$.

PROOF. Let $S_i$ be a set of subgroups of $G$ with $AS_i = S_iA$ such that $S_{i+1} \subseteq S_i$ ($S_i \subseteq S_{i+1}$). Since $A$ is artinian (noetherian),

$$A \cap S_i = A \cap S_{i+1} \text{ for almost all } i\,.$$

Since $B$ is artinian (noetherian) and by Dedekind's modular law

$$AS_i = AS_i \cap AB = A(B \cap AS_i) = A(B \cap AS_{i+1}) = AS_{i+1}$$

for almost all $i$. Thus the following holds:

$$A \cap S_i = A \cap S_{i+1} \text{ and } AS_i = AS_{i+1} \text{ for almost all } i\,.$$

From this and by Dedekind's modular law it follows that

$$S_i = S_i(A \cap S_i) = S_i(A \cap S_{i+1}) = S_{i+1} \cap AS_i = AS_{i+1} = S_{i+1}$$

for almost all $i$. Thus $G$ satisfies the minimum (maximum) condition for subgroups $S_i$ of $G$ such that $AS_i = S_iA$.

A subgroups $S$ of a group $G$ is called *permutable* (or quasinormal) if for every subgroup $X$ of $G$ the sets $SX$ and $XS$ are equal, so that $SX = XS$ is a subgroup of $G$. Obviously every normal subgroup is permutable.

COROLLARY 3.2. *If the group $G = AB$ is factorized by two artinian (noetherian) subgroups $A$ and $B$, one of which is permutable, then $G$ is artinian (noetherian).*

COROLLARY 3.3. *If the group $G = AB$ is factorized by two artinian (noetherian) subgroups $A$ and $B$, then $G$ satisfies the minimum (maximum) condition for permutable subgroups, in particular for normal subgroups of $G$.*

## 4. – Polycyclic groups.

A noetherian soluble group is called polycyclic. The structure of polycyclic groups is well-known; see for instance Robinson [7], chapter 3. It would be interesting to know whether every (soluble) group factorized by two polycyclic subgroups is polycyclic. In this section a positive solution is given for soluble groups factorized by two nilpotent subgroups. These results are based on the following lemma on groups which have a factorization with three factors. The author is indebted to Professor J. Roseblade for pointing out the applicability of this method in this situation.

Let $\mathfrak{Z}$ be an epimorphism inherited class of generalized nilpotent groups such that every finite $\mathfrak{Z}$-group is nilpotent.

LEMMA 4.1. *If the finitely generated hyper-(abelian-by-finite) group $G = AB = AC = BC$ is factorized by three $\mathfrak{Z}$-subgroups $A$, $B$ and $C$, then $G$ is nilpotent.*

PROOF. If $H$ is an epimorphic image of $G$, then there exists a subgroup $N$ of $G$ such that $H = G/N$. Then

$$G/N = (AN/N)(BN/N) = (AN/N)(CN/N) = (BN/N)(CN/N)$$

is factorized by three $\mathfrak{Z}$-subgroups $AN/N \simeq A/(A \cap N)$, $BN/N \simeq B/$ $/(B \cap N)$ and $CN/N \simeq C/N \cap C$. If $H$ is finite, $AN/N$, $BN/N$ and $CN/N$ are nilpotent. Application of Kegel [4], Folgerung 2, p. 44, yields that $H$ is nilpotent. Thus every finite epimorphic image of $G$ is nilpotent. By a theorem of Robinson $G$ is nilpotent; see [8], Theorem 10.51, p. 194.

If the class $\mathfrak{Z}$ of generalized nilpotent groups is also inherited by subgroups, we obtain

COROLLARY 4.2. *If $N$ is a normal $\mathfrak{Z}$-subgroup of the group $G = AB$ which is factorized by two $\mathfrak{Z}$-subgroups $A$ and $B$ and if $X(N) = AN \cap$ $\cap BN$ is finitely generated and hyper-(abelian-by-finite), then $X(N)$ is nilpotent.*

PROOF. By Theorem 1.6 (*b*)

$$X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$$

is factorized by three $\mathfrak{Z}$-subgroups $N$, $A \cap BN$ and $B \cap AN$. Hence $X(N)$ is nilpotent by Lemma 4.1.

This leads to the following result.

THEOREM 4.3. *If every factorized subgroup of the hyper-(abelian-by-finite) group $G = AB$ which is factorized by two locally nilpotent subgroups $A$ and $B$ is finitely generated, then $G$ is polycyclic.*

PROOF. By Theorem 2.3 (*a*) $G$ is hyperabelian. If $G/N = (AN/N)\cdot$ $\cdot(BN/N)$ is an epimorphic image of $G$, then $G/N$ is likewise hyperabelian and factorized by two locally nilpotent subgroups. If $S/N$ is a factorized subgroup of $G/N$, then $S$ is a factorized subgroup of $G$; see Lemma 1.3 (*a*). Since $S$ is finitely generated, so is $S/N$. Hence $G/N$ has the same properties as $G$.

Assume that $G$ is not polycyclic. Since $G$ is finitely generated, there exists an epimorphic image $H \neq 1$ of $G$ which is not polycyclic, but all its proper epimorphic images are; see for instance Robinson [8], Lemma 6.17, p. 11. Since $H$ is hyperabelian, there exists an abelian normal subgroup $M \neq 1$ of $H$. Since $H/M$ is polycyclic, $H$ is soluble. Without loss of generality let $G = H$. Let $K = G^{(k)}$ be the last non-trivial term of the derived series of $G$. Then $G/K$ is polycyclic, so that $K$ is not polycyclic. By Theorem 1.7 (*b*)

$$X = X(K) = AK \cap BK = KA^* = KB^* = A^*B^*$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. Hence $X$ is a hyperabelian group factorized by three locally nilpotent subgroups $K$, $A^*$ and $B^*$. By Corollary 4.2 $X$ is nilpotent. Hence $X$ is polycyclic as a finitely generated nilpotent group. Then also $K$ is polycyclic. This contradiction proves the theorem.

The *ascending radical series* of the group $G$ is defined by

$$\mathfrak{R}_0 G = 1\,, \qquad \mathfrak{R}_{\alpha+1} G/\mathfrak{R}_\alpha G = \mathfrak{R}(G/\mathfrak{R}_\alpha G)\,,$$

$$\mathfrak{R}_\gamma G = \bigcup_{\alpha<\gamma} R_\alpha G \text{ for limit ordinals } \gamma\,,$$

where $\mathfrak{R}X$ is the Hirsch-Plotkin radical of the group $X$. The *ascending Fitting series* of the group $G$ is defined similarly where the Hirsch-Plotkin radical is replaced by the Fitting subgroup. Note that for polycyclic groups the two notations coincide.

The following theorem contains some useful properties of polycyclic groups factorized by two nilpotent subgroups.

THEOREM 4.4. *Let the polycyclic group $G = AB$ be factorized by two nilpotent subgroups $A$ and $B$.*

(*a*) *Each term of the ascending Fitting series of $G$ is factorized,*

(*b*) *If $N$ is a normal subgroups of $G$, then $X(N) = AN \cap BN$ is subnormal in $G$; in particular $A \cap B$ is subnormal in $G$,*

(*c*) *If $G \neq 1$, then there exists a subnormal subgroup $S \neq 1$ of $G$ which is contained in $A$ or $B$,*

(*d*) *If $N$ is a nilpotent normal subgroup of $G$, then $X(N) = AN \cap BN$ is contained in the Fitting subgroup of $G$ and hence is a nilpotent subnormal subgroup of $G$,*

(*e*) *If $G$ is not of prime order and $A \neq G$ or $B \neq G$, then there exists a factorized normal subgroup $N$ of $G$ with $1 \neq N \neq G$,*

(*f*) *If $A \neq G$ or $B \neq G$, then every maximal factorized normal subgroup of $G$ contains $A$ or $B$,*

(*g*) *If $A \cap B$ is finite, then the maximal torsion normal subgroup $T = \mathfrak{T}G$ of $G$ is factorized.*

PROOF. (*a*) Assume there is a polycyclic group $G = AB$ factorized by two nilpotent subgroups $A$ and $B$ of minimal derived length in which the Fitting subgroup $R = \mathfrak{R}G$ of $G$ is not factorized. By

Theorem 1.7

$$X = X(R) = AR \cap BR =$$
$$= R(A \cap BR) = R(B \cap AR) = (A \cap BR)(B \cap AR)\,.$$

Hence $X$ is factorized by three nilpotent subgroups $R$, $A \cap BR$ and $B \cap AR$. By Corollary 4.2 $X$ is nilpotent. If $H/R$ is the Fitting subgroup of $G/R$, then by induction $H/R$ is factorized. By Lemma 1.5 $H = AH \cap BH$. Hence $R \subseteq X \subseteq H$. Since $H/R$ is nilpotent, $X/R$ is subnormal in $H/R$, so that $X$ is subnormal in $G$. It follows that $X \subseteq R$, so that $R = X$ is factorized by Theorem 1.7 (*a*). This contradiction proves (*a*).

(*b*) By (*a*) the Fitting subgroup $\mathfrak{F}(G/N)$ of $G/N = (AN/N)\cdot(BN/N)$ is factorized. Hence by Lemma 1.1 it contains $(AN/N) \cap (BN/N)$. Since $\mathfrak{F}(G/N)$ is nilpotent, $(AN/N) \cap (BN/N)$ is subnormal in $\mathfrak{F}(G/N)$ and $G/N$.

By Theorem 1.7 (*b*) $X = X(N) = N(A \cap BN) = N(B \cap AN)$. Hence $X/N$ is contained in $(AN/N) \cap (BN/N)$ which is subnormal in $G/N$. Since $(AN/N) \cap (BN/N)$ is nilpotent, $X/N$ is subnormal in $G/N$. Hence $X$ is subnormal in $G$. This proves (*b*).

(*c*) Since $G \neq 1$ is soluble, $F = \mathfrak{F}G \neq 1$. By (*a*) $F$ is factorized, so that by Lemma 1.1 $F = (A \cap F)(B \cap F)$. Then $A \cap F \neq 1$ or $B \cap F \neq 1$ is a nontrivial subnormal subgroup of $G$ contained in $A$ or $B$, since $F$ is nilpotent. This proves (*c*).

(*d*) By Theorem 1.7 (*b*)

$$X = X(N) = AN \cap BN =$$
$$= N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)\,.$$

Since $N$, $A \cap BN$ and $B \cap AN$ are nilpotent, $X$ is nilpotent by Corollary 4.2. Since $X$ is subnormal in $G$ by (*b*), $X$ is contained in $\mathfrak{F}G$.

(*e*) If $G \neq 1$ is nilpotent, then $A \neq G$ or $B \neq G$ is subnormal in $G$. Hence this subgroup is contained in a proper normal subgroup of $G$ which is factorized. If $G$ is not nilpotent, $F = \mathfrak{F}G$ is a factorized normal subgroup of $G$ with $1 \neq F \neq G$ by (*a*).

(*f*) Let $M$ be a maximal factorized normal subgroup of $G$. By Lemma 1.5 or Theorem 1.7 (*a*) $X(M) = AM \cap BM = M$, since $M$ is factorized. Then $G/M = (AM/M)(BM/M)$, $AM/M \neq G/M$ or $BM/M \neq \neq G/M$, since otherwise $G = AM = BM = M$. By (*e*) $G/M$ is cyclic of prime order and $AM = M$ or $BM = M$ (or both), so that $A \subseteq M$ or $B \subseteq M$.

(*g*) Since $G$ is polycyclic, $T = \mathfrak{T}G$ is finite. By (*b*) $X = X(T) = = AT \cap BT$ is subnormal in $G$. By Lemma 2.1 $|X:D| = |X:(A \cap B)|$ is finite. Since $D = A \cap B$ is finite, $X$ is finite. As a finite subnormal subgroup of $G$, $X$ is contained in $T$. Hence $T = X$ is factorized by Theorem 1.7 (*a*). This proves the theorem.

Remarks 4.5. (*a*) The Fitting subgroup of a (finite soluble) group $G = AB$ factorized by two subgroups $A$ and $B$ need not be factorized if only one of the two subgroups $A$ and $B$ is nilpotent; see Remark 1.8. (*b*) The special case of Theorem 4.4 (*a*) that the Fitting subgroup of a finite soluble group factorized by two nilpotent subgroups is factorized, was also shown by different arguments in Pennington [6]. (*c*) Theorem 4.4 (*f*) extends results of Itô in [2] for finite groups factorized by two abelian subgroups, and of Kegel in [3] for finite groups factorized by two nilpotent subgroups. (*d*) Trivial examples show that Theorem 4.4 (*g*) becomes false if $A \cap B$ is infinite. (*h*) The special case of Theorem 4.3 that a group factorized by two noetherian abelian subgroups is polycyclic, was also proved by Sesekin in [10].

## 5. – Černikov groups.

A group $G$ is a *Černikov group* if it possesses a normal subgroup $D$ of finite index which is the direct product of finitely many quasicyclic subgroups of type $p^\infty$ for finitely many primes $p$. $D = \mathfrak{D}G$ is called the *semi-radicable radical* of $G$. Every Černikov group is artinian, and every soluble artinian group is a Černikov group; see for instance Robinson [7], Theorem 3.12, p. 68.

Is every group $G$ which is factorized by two Černikov subgroups a Černikov group? Kegel has shown that this is the case whenever $G$ is soluble. His proof is based on the following considerations.

The first lemma is well-known.

Lemma 5.1. *If the group $G = AB$ is factorized by two subgroups $A$ and $B$, and if $A^*$ and $B^*$ are subgroups of finite index in $A$ resp. $B$, then the subgroup $S = \langle A^*, B^*\rangle$ of $G$ has finite index in $G$.*

PROOF. The group $G$ can be expressed in the following way:

$$G = AB = \bigcup_{i,j} a_i A^* B^* b_j = \bigcup_{i,j} a_i S b_j = \bigcup_{i,j} (a_i S a_i^{-1}) k_j$$

where $k_{ij} = a_i b_j$. Thus $G$ is covered by finitely many right cosets with respect to conjugates of $S$. Application of a theorem of B. H. Neumann yields that one of the conjugates of $S$ has finite index in $G$; see for instance Robinson [7], Lemma 4.17, p. 105. Hence also $S$ has finite index in $G$.

LEMMA 5.2. *If the group $G = AB$ is factorized by two subgroups $A$ and $B$ if $K$ is a subgroup of $G$ such that the normal subgroups $A^*$ of $A$ and $B^*$ of $B$ are contained in a conjugate subgroup of $K$, then there is a subgroup of $G$ conjugate to $K$ which contains $A^*$ and $B^*$.*

PROOF. Assume for instance that $A^* \subseteq K$ and $B^* \subseteq K^g$ where $g = ab$ with $a \in A$ and $b \in B$. Then $B^{*a^{-1}} = B^{*g^{-1}} \subseteq K$, and hence $A^*, B^* \subseteq K^a$.

If $X$ is a group, $\pi X$ denotes the set of all primes $p$ for which there exists an element of order $p$ in $X$.

PROPOSITION 5.3. *Let the group $G = AB$ be factorized by two subgroups $A$ and $B$ and let $A^*$ be a normal subgroup of $A$ and $B^*$ be a normal subgroup of $B$. If $A^*$ and $B^*$ are finite and $S = \langle A^*, B^* \rangle$ is soluble and $\mathfrak{n}S$ is factorized, then $S$ is finite and $\pi S = \pi A^* \cup \pi B^*$.*

PROOF. Let $\pi = \pi S$. If $S$ is a $\pi$-group, then $S$ is finite as a finitely generated soluble torsion group. Thus it may be assumed that $S$ is not a $\pi$-group. There exists a smallest positive integer $n$ such that the quotient group $S^{(n)}/S^{(n+1)}$ is not a $\pi$-group. Since $S$ is finitely generated and $S/S^{(n+1)}$ is finite, $S^{(n)}$ is finitely generated, so that $S^{(n)}/S^{(n+1)}$ is a finitely generated abelian group. If $S^{(n)}/S^{(n+1)}$ is finite, choose a prime divisor $q$ in $\pi(S^{(n)}/S^{(n+1)})$ which is not in $\pi$, and let $K/S^{(n+1)}$ be the set of elements in $S^{(n)}/S^{(n+1)}$ whose orders are relatively prime to $q$. If $S^{(n)}/S^{(n+1)}$ is infinite, choose any prime number $q$ not in $\pi$, and let $K/S^{(n+1)}$ be the subgroup of $S^{(n)}/S^{(n+1)}$ which is generated by the $q$-th powers of the elements of $S^{(n)}/S^{(n+1)}$. In both cases $K$ is a characteristic subgroup of $S$ with finite index in $S$. By hypothesis the subgroup $\mathfrak{n}S$ of $G$ is factorized. The group $S/K$ is a finite soluble subgroup of $\mathfrak{n}S/K$; and the subgroups $A^*K/K$ and $B^*K/K$ are contained in a $q$-complement of the finite soluble group $S/K = \langle A^*K/K, B^*K/K \rangle$. Since the

$q$-complements of a finite soluble group are conjugate, application of Lemma 5.2 yields that $A^*K/K$ and $B^*K/K$ are contained in a $q$-complement of $S/K$. This contradiction shows that $S$ must be a $\pi$-group.

The following two theorems are essentially due to Kegel.

THEOREM 5.4. *If the locally-soluble-by-finite group $G = AB$ is factorized by two locally-normal-by-finite subgroups $A$ and $B$, then $G$ is locally finite and $\pi G = \pi A \cup \pi B$.*

PROOF. There exists a locally normal subgroup $A^*$ of $A$ with finite index $|A:A^*|$, and there exists a locally normal subgroup $B^*$ of $B$ with finite index $|B:B^*|$. It follows that $A^*$ is generated by finite normal subgroups of $A$ and $B^*$ is generated by finite normal subgroups of $B$. — Furthermore, there exists a locally soluble normal subgroup $L$ of $G$ with finite $G/L$. If $A_1 = A^* \cap L$ and $B_1 = B^* \cap L$, then the indices $|A:A_1|$ and $|B:B_1|$ are likewise finite. It follows from Proposition 5.3 that $S = \langle A_1, B_1 \rangle$ is a locally finite $\pi$-group for $\pi = \pi A \cup \cup \pi B$. — By Lemma 5.1 the index $|G:S|$ is finite. If $S_G$ is the largest normal subgroup of $G$ which is contained in $S$, then by the theorem of Poincaré $G/S_G$ is finite. As a subgroup of $S$, the group $S_G$ is a locally finite $\pi$-group. This proves that $G$ is a locally finite $\pi$-group.

THEOREM 5.5. *If the almost hyperabelian group $G = AB$ is factorized by two Černikov subgroups $A$ and $B$, then $G$ is a Černikov group.*

PROOF. Assume that the theorem is false, and let the almost hyperabelian group $G = AB$ with Černikov subgroups $A$ and $B$ be a counterexample where the sum of the primary ranks of $\mathfrak{D}A$ and $\mathfrak{D}B$ and then also the sum of the finite indices $A/\mathfrak{D}A$ and $B/\mathfrak{D}B$ are minimal. By Theorem 5.4 $G$ is locally finite. Since $G$ is factorized by two artinian subgroups, it satisfies the minimum condition for normal subgroups by Corollary 3.3.

By a theorem of Baer every hyperfinite group with minimum condition for normal subgroups is a Černikov group; see for instance Robinson [7], Corollary 2 on p. 148. Since $G$ is not a Černikov group, there exists an epimorphic image $H$ of $G$ which contains no nontrivial finite normal subgroups. In particular $H$ is not a Černikov group. $H$ is also factorized by two Černikov subgroups such that the corresponding induction quantities are minimal. Without loss of generality let $G = H$.

If $M$ is a minimal normal subgroup of $G$, then $M$ is abelian and therefore an infinite elementary abelian $p$-group. By Theorem 1.7 ($b$)

$$X = X(M) = AM \cap BM = AM^* = MB^* = A^* B^*$$

where $A^* = A \cap BM$ and $B^* = B \cap AM$. If $X$ is a Černikov group, then $M$ is finite. This contradiction shows that $X$ is not a Černikov group. As a subgroup of $G$, $X$ is almost hyperabelian. Then the induction quantities of $X = A^* B^*$ and $G = AB$ are equal. Hence $A = A^* = A \cap BM$ and $B = B^* = B \cap AM$, so that $G = X = AM = BM$.

Assume that the radicable subgroup $\mathfrak{D}A$ of finite index in $A$ has a nontrivial $p$-part $(\mathfrak{D}A)_p$. Then $(\mathfrak{D}A)_p$ is a normal subgroup of $A$ which centralizes $M$. In particular $G$ contains nontrivial finite normal subgroups. This contradiction shows that $(\mathfrak{D}A)_p = 1$. Similarly $(\mathfrak{D}B)_p = 1$.

By Lemma 5.1 the subgroup $S = \langle \mathfrak{D}A, \mathfrak{D}B \rangle$ has finite index in $G$. By Proposition 5.3 $S$ is a $p'$-group. Hence $G$ is an extension of a $p'$-group by a finite group. Then the normal $p$-subgroup $M$ of $G$ must be finite. This contradiction proves the theorem.

The following lemma slightly generalizes Proposition 1.6 of O. H. Kegel [5], p. 538.

LEMMA 5.6. *The following properties of the group $G = AB$ which is factorized by two subgroups $A$ and $B$ are equivalent*:

($a$) *$G$ is a Černikov group,*

($b$) *$A$ and $B$ are Černikov groups and $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B) = (\mathfrak{D}B) \cdot (\mathfrak{D}A)$.*

PROOF. If ($a$) holds, then $G$ is an extension of the radicable artinian abelian normal subgroups $\mathfrak{D}G$ of $G$ by a finite group $G/\mathfrak{D}G$. As subgroups of $G$ also $A$ and $B$ are Černikov groups. Hence the semi-radicable radicals $\mathfrak{D}A$ and $\mathfrak{D}B$ of $A$ resp. $B$ are abelian with finite factor groups $A/\mathfrak{D}A$ and $B/\mathfrak{D}B$. By Lemma 5.1 the subgroup $S = \langle \mathfrak{D}A, \mathfrak{D}B \rangle$ of $G$ has finite index in $G$. Hence $\mathfrak{D}G \subseteq S$, since $\mathfrak{D}G$ is the intersection of all subgroups of finite index in $G$. Since $\mathfrak{D}A$ and $\mathfrak{D}B$ are radicable, they are contained in $\mathfrak{D}G$, so that $S = \mathfrak{D}G$. Since $\mathfrak{D}G$ is abelian, $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B)$. This proves ($b$).

If ($b$) holds, then $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B)$ is factorized by two radicable artinian abelian subgroups $\mathfrak{D}A$ amd $\mathfrak{D}B$. Since $\mathfrak{D}G$ is factorized by two abelian subgroups, it is metabelian by a theorem of Itô; see for instance Huppert [1], Satz 4.4, p. 674. As a soluble group factorized by two artinian subgroups $\mathfrak{D}G$ is a Černikov group by Theorem 5.5. Since $A$ and $B$ are Černikov groups, the groups $A/\mathfrak{D}A$ and $B/\mathfrak{D}B$ are finite. By Lemma 5.1 also $G/\mathfrak{D}G$ is finite. Hence $G$ is a Černikov group, and (*b*) follows.

The following theorem which corresponds to Theorem 4.4 for polycyclic groups, contains some useful properties for soluble Černikov groups which are factorized by two locally nilpotent subgroups.

THEOREM 5.7. *Let the soluble Černikov group $G = AB$ be factorized by two locally nilpotent subgroups $A$ and $B$.*

(*a*) *Each term of the ascending radical series of $G$ is factorized,*

(*b*) *If $N$ is a normal subgroup of $G$, then $X(N) = AN \cap BN$ is ascendant in $G$; in particular $A \cap B$ is ascendant in $G$,*

(*c*) *If $G \neq 1$, then there exists a subnormal subgroup $S \neq 1$ of $G$ which is contained in $A$ or $B$,*

(*d*) *If $N$ is a locally nilpotent normal subgroups of $G$, then $X(N) = AN \cap BN$ is contained in the Hirsch-Plotkin radical of $G$ and is therefore a locally nilpotent ascendant subgroup of $G$.*

PROOF. (*a*) Assume that the theorem is false, and let $G = AB$ be a counterexample with minimal derived length. Let $R = \mathfrak{R}G$ be the Hirsch-Plotkin radical of $G$. By Theorem 1.7 (*b*)

$$X = X(R) = AR \cap BR = RA^* = RB^* = A^*B^*$$

where $A^* = A \cap BR$ and $B^* = B \cap AR$. Since $G \neq 1$ is soluble, $R \neq 1$. If $p$ is a prime, the maximal $p$-subgroups of $A^*$, $B^*$ and $R$ are normal, since these groups are locally nilpotent. Since $X$ is a Černikov group, it follows from Kegel [5], Theorem 1.9, p. 540, that $X$ possesses normal Sylow-$p$-subgroups for every prime $p$. Since $X$ is locally finite, this implies that $X$ is locally nilpotent; see for instance Robinson [8], chapter 6. By induction the Hirsch-Plotkin radical $H/R$ of $G/R$ is factorized, so that by Lemma 1.5 $H = AH \cap BH$. It follows that $R \subseteq X \subseteq H$. Since every subgroup of a locally nilpotent Černikov group is ascendant, $X$ is ascendant in ($H$ and) $G$. Hence $X$ is an ascendant locally nilpotent subgroup of $G$, so that $X \subseteq R$ and therefore $X = R$; see for instance Robinson [7], Theorem 2.31, p. 57. Thus $R$ is factorized by Theorem 1.7 (*a*). This proves (*a*).

(*b*) By (*a*) the Hirsch-Plotkin radical $\mathfrak{R}(G/N)$ of $G/N = (AN/N)\cdot$ $\cdot(BN/N)$ is factorized. Hence by Lemma 1.1 it contains $(AN/N) \cap$ $\cap (BN/N)$. Since $\mathfrak{R}(G/N)$ is hypercentral, $(AN/N) \cap (BN/N)$ is ascendant in $\mathfrak{R}(G/N)$ and $G/N$.

By Theorem 1.7 (*b*) $X = X(N) = N(A \cap BN) = N(B \cap AN)$. Hence $X/N$ is contained in $(AN/N) \cap (BN/N)$ which is ascendant in $G/N$. Since $(AN/N) \cap (BN/N)$ is hypercentral, $X/N$ is ascendant in $G/N$. Hence $X$ is ascendant in $G$.

(*c*) If $G$ is finite, statement (*c*) is contained in Theorem 4.4 (*c*). If $G$ is infinite, by Lemma 5.6 $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B)$, and $\mathfrak{D}G$ is abelian. Since $\mathfrak{D}G$ is infinite, $\mathfrak{D}A \neq 1$ or $\mathfrak{D}B \neq 1$. Thus there exists a nontrivial element in $\mathfrak{D}A$ or $\mathfrak{D}B$ which generates a subnormal subgroup of defect 2 in $G$.

(*d*) By Theorem 1.7 (*b*)

$$X = X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)\,.$$

Since $N$, $A \cap BN$ and $B \cap AN$ are locally nilpotent, their Sylow-$p$-subgroups are normal for each prime $p$. Since $X$ is a Černikov group, application of Kegel [5], Theorem 1.9, p. 540, yields that $X$ contains normal Sylow-p-subgroups for each prime $p$. Since $X$ is locally finite, this implies that $X$ is locally nilpotent; see for instance Robinson [8], section 6. By (*b*) $X(N)$ is ascendant in $G$. Hence $X(N) \subseteq R = \mathfrak{R}G$; see for instance Robinson [7], Theorem 2.31, p. 57.

Remark 5.8. The special case of Theorem 5.5 that a group factorized by two artinian abelian subgroups is artinian was also proved by Sesekin in [9].

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