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**A Variational Principle, in General Relativity,  
for the Equilibrium  
of an Elastic Body also Capable of Couple Stresses.**

LUCIANO BATTALIA (\*)

**SUMMARY** - We prove a variational theorem for the equilibrium of an elastic body, also capable of couple-stresses but not of heat conduction, in general relativity. It can be considered as the relativization of the classical principle of stationary potential energy.

**1. Introduction.**

In classical physics equilibrium problems are often proved to be equivalent to variational principles. One of these principles is the principle of stationary potential energy, firstly formulated within the linear theory for infinitesimal deformations and then extended to the non linear theory for finite deformations but, in both cases, for materials not capable of couple-stresses (non-polar materials)-see for example [12], [14].

In this work we present an extension of this equivalence theorem to general relativity and we take also polar materials (and finite deformation) into account. The work is based on the Eulerian and Lagrangian theories of continuous media in general relativity as formulated by Bressan (§ 2), and on a certain variational principle, involving the

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variation of world lines. This principle was first formulated by Taub in [13] and improved by Fock cf. [19], § 47; then it was extended by Schöpf and Bressan to the non-polar and polar cases respectively.

As a preliminary, in § 4 we state the dynamical equations in mixed form in general relativity for polar materials, thus extending the corresponding result of Bressan for non-polar materials, cf. [2].

In § 5 we present and prove the equivalence of the aforementioned variational principle to the corresponding conservation equations, which include the dynamical equations of the equilibrium.

In § 6 we compare our principle with the classical principle of stationary potential energy.

## 2. Preliminaries.

We first recall some fundamental concepts of the Eulerian and Lagrangian theories of continuous media in general relativity presented by Bressan in [1]-[7].

Let  $S_4$  be the space time of general relativity with the metric <sup>(1)</sup>

$$(1) \quad ds^2 = -g_{\alpha\beta} dx^\alpha dx^\beta$$

whose signature is determined by the condition that it can everywhere be reduced locally to the pseudo-Euclidean form:

$$(2) \quad g_{rs} = \delta_{rs} ; \quad g_{\alpha 0} = g_{0\alpha} = -\delta_{\alpha 0} ,$$

where  $\delta_{\alpha\beta}$  is the Kronecker symbol.

The co-ordinate system  $(x)$  is assumed to be admissible: the hypersurfaces  $x_0 = \text{const}$  are space like and  $x^0$  increases towards future.

Let  $\mathcal{C}$  be a body. We may consider the process  $\mathcal{F}$  of the universe in  $S_4$  as consisting only of the motion  $\mathcal{M}$  of  $\mathcal{C}$ , the temperature distribution in the world-tube  $W_{\mathcal{C}}$  of  $\mathcal{C}$ , and the metric tensor field over  $S_4$ .

We consider only regular motions of the body  $\mathcal{C}$  for which, among other things,  $\mathcal{C}$  can be regarded as a collection of material points. We denote the typical one of them by  $P^*$ , and the four velocity (intrinsic acceleration) of  $\mathcal{C}$  at its material point  $P^*$  by  $u^\alpha(A^\alpha)$ .

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<sup>(1)</sup> Greek and Latin indices run over 0, 1, 2, 3, and 1, 2, 3 respectively.

The frame  $(x)$  is called locally natural and proper at the event point  $x^e$  if there (2) and

$$(3) \quad g_{\alpha\beta,\gamma} = 0 ; \quad u^\alpha = \delta^\alpha_0 \quad (f, \alpha = \partial f / \partial x^\alpha)$$

hold.

We use the notation  $T^{\dots}_{\dots|\gamma}$  for the covariant derivative:

$$(4) \quad T^\alpha_{\beta|\gamma} = T^\alpha_{\beta,\gamma} - \left\{ \begin{matrix} \sigma \\ \beta\gamma \end{matrix} \right\} T^\alpha_\sigma + \left\{ \begin{matrix} \alpha \\ \sigma\gamma \end{matrix} \right\} T^\sigma_\beta.$$

We also consider the spatial projector

$$(5) \quad g_{\alpha\beta}^\perp = g_{\alpha\beta} + u_\alpha u_\beta$$

and the following notations

$$(6) \quad T^{\dots\perp}_{\dots\alpha} = g^\perp_{\alpha\gamma} T^{\dots\gamma}; \quad T^{\perp\gamma\dots}_{\alpha\dots} = T^{\perp\gamma\dots}_{\alpha\dots},$$

recalling that the index  $\alpha$  of the tensor  $T^{\dots\alpha}$  is said to be spatial if  $T^{\dots\alpha} u^\alpha = 0$  or, equivalently, if  $T^{\dots\alpha} = T^{\dots\perp\alpha}$ .

We also use the spatial derivative and divergence of any tensor:

$$(7) \quad T^{\dots\beta\perp}_{\dots\alpha\perp\gamma} = g^\perp_{\gamma\delta} T^{\dots\beta\perp}_{\dots\alpha\perp\delta}; \quad T^{\dots\beta\perp}_{\dots\alpha\perp\beta} = g^\perp_{\beta\delta} T^{\dots\beta\perp}_{\dots\alpha\perp\delta}$$

and the natural decomposition of a tensor with respect to the index  $\alpha$ . This decomposition for a vector  $v_\alpha$  is:

$$(8) \quad v_\alpha = v_\alpha^\perp - v_\beta u^\beta u_\alpha$$

and  $v_\alpha^\perp (-v_\beta u^\beta u_\alpha)$  is called the spatial (temporal) part of  $v_\alpha$  <sup>(2)</sup>.

The proper density  $\varrho$  of total internal energy is defined by:

$$(9) \quad dm = e^{-2} \varrho dC$$

where  $dm$  is the proper gravitational mass of the element  $dC$  of  $C$  and  $dC$  the actual proper volume of  $dC$ .

(2) For more details on this subject see Cattaneo [8].

Now we fix a process  $\mathcal{F}^*$  (reference process) physically possible for the universe (containing  $\mathcal{C}$ ) and we define  $S_4^*$ ,  $W_{\mathcal{C}}^*$ ,  $\mathcal{M}^*$ ,  $g_{\lambda\mu}^*$ ,  $dC^*$  to be corresponding instances of  $S_4$ ,  $W_{\mathcal{C}}$ ,  $\mathcal{M}$ ,  $g_{\alpha\beta}$  and  $dC$ . We consider an admissible system of co-ordinates ( $y$ ) and the intersection  $S_3^*$  of  $W_{\mathcal{C}}^*$  with the hypersurface  $y^0=0$ . For every material point  $P^*$  of  $\mathcal{C}$  we shall use to co-ordinate  $y^L$  of the intersection of  $S_3^*$  with the world line of  $P^*$  as the  $L$ -th material coordinate. The spatial metric in  $S_3^*$  will be called  $ds^{*2}$ :

$$(10) \quad ds^{*2} = a_{LM}^* dy^L dy^M \quad \text{where} \quad a_{rs}^* = \hat{a}_{rs}(y^L) = g_{rs}^*(0, y^1, y^2, y^3).$$

The co-ordinates  $y^L$  and their increments  $dy^L$  then characterize the material points  $P^*$  and the linear infinitesimal material elements at  $P^*$ .

Let  $\Sigma^*$  be the intrinsic state of  $\mathcal{C}$  in  $\mathcal{F}^*$ ; we call  $k^*dC^*$  the proper gravitational mass of  $dC$  in  $\Sigma^*$ . It is related to the actual volume  $dC$  by the following definition of the density  $k$ :

$$(11) \quad k^*dC^* = k dC; \quad \text{hence} \quad k^* = k\mathcal{D}, \quad \mathcal{D} = dC/dC^*.$$

As a consequence of this definition  $k$  satisfies the continuity equation

$$(12) \quad (ku^\alpha)_{;\alpha} \equiv \frac{Dk}{Ds} + ku^\alpha_{;\alpha} = 0.$$

Let us set  $c^{-2}wk dC = c^{-2}\varrho dC - k dC$ . Then by the equivalence principle of mass and energy  $wk dC$  can be regarded as the internal energy of  $dC$ . Furthermore:

$$(13) \quad \varrho = k(c^2 + w).$$

We also set

$$(14) \quad \varrho^* = \varrho\mathcal{D} = k^*(c^2 + w).$$

An arbitrary motion of  $\mathcal{C}$  in the system of co-ordinates ( $x$ ) is represented by the equations <sup>(3)</sup>

$$(15) \quad x^\alpha = \hat{x}^\alpha(t, y^1, y^2, y^3) [= \hat{x}^\alpha(t, y^L) = \hat{x}^\alpha(y^{\mathcal{L}}), t \equiv y^0].$$

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<sup>(3)</sup> Capital and lower case letters represent material and space time indices respectively.

For the regularity conditions on the functions (15) and their indetermination in the choice of the time parameter  $t = t(x)$  see Bressan [1]; here we only remember that  $t = t(x)$  can be chosen in such a way that

$$(16) \quad u_L^\dagger \equiv u^e x_L^e = 0, \quad (x_L^e = \partial x^e / \partial y^L)$$

holds at an arbitrarily pre-assigned event  $\xi$ .

Let now  $T_{\beta \dots M \dots}^{\alpha \dots L \dots}$  be a double tensor field—cf. Ericksen [9]—associated to the event point  $x^e$  and the material point  $y^L$ . We shall think of it as depending from  $x^e$ ,  $y^L$  and the time parameter  $t$ , with  $x^e$  and  $y^L$  connected by means of the equations of motions (15). Then the total covariant derivative based on the map (15) is defined by

$$(17) \quad T_{\beta \dots M \dots; P}^{\alpha \dots L \dots} = T_{\dots; l e} x_e^e + T_{\dots; P}$$

It depends on the particular representation of the motion through the time parameter  $t$ . In case (16) holds it is called Lagrangian spatial derivative and denoted by  $T_{\dots; P}$ .

Then we introduce the first and second deformation gradient

$$(18) \quad \alpha_L^e = \overset{\perp}{g}_\sigma^e x_L^\sigma; \quad \alpha_{LM}^e = \alpha_{L|M}^e,$$

the first and second Cauchy-Green tensor

$$(19) \quad C_{ML} = \alpha_{eM} \alpha_L^e; \quad C_{MLB} = \alpha_M^e \alpha_{eLB},$$

and lastly the deformation tensor  $\varepsilon_{LM}$

$$(20) \quad a_{LM}^* + 2\varepsilon_{LM} = C_{LM}.$$

We have (\*)

$$(21) \quad \alpha_{[AB]}^e = 0; \quad C_{[AB]} = \varepsilon_{[AB]} = 0; \quad C_{L[AB]} = 0.$$

The spatial inverse  $\mathcal{D}y_e^L$  of  $\alpha_L^e$  is defined by

$$(22) \quad \alpha_L^e \gamma_\sigma^L = \overset{\perp}{\mathcal{D}}g_\sigma^e \quad \text{hence} \quad u^e \gamma_L^e = 0.$$

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(\*) We use the notations  $2T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$ ;  $2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$ .

If  $X^{e\sigma}$  is an arbitrary Eulerian double tensor, its mixed and Lagrangian counterparts are defined by

$$(23) \quad Ke^M = X^{e\sigma} \gamma_\sigma^M; \quad Y^{LM} = \frac{1}{\mathfrak{D}} \gamma_e^L K^{eM}.$$

In the sequel  $X^{e\sigma}$  will denote the Eulerian stress tensor so that its mixed and Lagrangian counterparts defined by (23) are called first and second Piola stress tensor, see [15].

We are interested in an elastic body  $\mathcal{C}$  capable of couple-stress but not of heat conduction and we assume, also for the sequel the absence of electromagnetic phenomena. We consider an event  $\mathfrak{E}$  in the world tube  $W_{\mathcal{C}}$  of  $\mathcal{C}$  and we suppose that the vector  $d\sigma_{\alpha\alpha}$  represents the infinitesimal oriented spatial surface  $d\sigma$  at  $\mathfrak{E}$ . We admit that the forces exerted by the material elements contiguous to the negative face of  $d\sigma$  on those contiguous to the positive one are characterizable by means of the resultant  $dR^\alpha$  and the intrinsic resultant moment  $d\mathcal{M}^\alpha \cdot dR^\alpha$  is expressed by

$$(24) \quad dR^\alpha = X^{\alpha\beta} d\sigma_\beta \quad \text{with} \quad X^{\alpha\beta} u_\beta = 0 = u_\alpha X^{\alpha\beta}$$

while  $d\mathcal{M}^\alpha$  is expressed by

$$(25) \quad d\mathcal{M}^\alpha = \overset{\perp}{\varepsilon}_{\alpha\varrho\sigma} m^{e\sigma\beta} d\sigma_\beta \quad \text{with} \quad m^{(e\sigma)\beta} = 0 = u_\varrho m^{e\sigma\beta} = m^{e\sigma\beta} u_\beta,$$

where  $\overset{\perp}{\varepsilon}_{\alpha\varrho\sigma}$  is the spatial Ricci tensor  $u^e \varepsilon_{\varrho\alpha\beta\gamma}$  <sup>(5)</sup>.

As well as the stress  $X^{\alpha\beta}$ , the tensor  $m^{\alpha\beta\gamma}$  of couple stress depends on  $\mathfrak{E}$  but not on  $d\sigma_\beta$ .

We define

$$(26) \quad \mathcal{M}^{\alpha\beta} = m^{\alpha\lambda\beta}{}_{|\lambda} + m^{\beta\lambda\alpha}{}_{|\lambda} + \nu^\alpha u^\beta + u^\alpha \nu^\beta \quad (\nu^\alpha = 2m^{:(\alpha\varrho\sigma)} u_{\varrho|\sigma}).$$

Then the total energy tensor is:

$$(27) \quad \mathfrak{U}_{\alpha\beta} = \varrho u_\alpha u_\beta + X_{(\alpha\beta)} + \mathfrak{U}_{\alpha\beta}.$$

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<sup>(5)</sup>  $\mathfrak{E}_{\alpha\beta\gamma\delta}$  is the usual permutation symbol with  $\mathfrak{E}_{0123} = 1$ . The Ricci tensor is defined by  $\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g} \mathfrak{E}_{\alpha\beta\gamma\delta}$   $\{g = \det \|g_{\alpha\beta}\| < 0\}$ .

The temporal part of the conservation equations

$$(28) \quad \mathfrak{U}^{\alpha\beta}{}_{;\beta} = 0,$$

can satisfactorily be taken as the relativization of the first principle of thermodynamics and can be put into the form

$$(29) \quad K \frac{Dw}{Ds} + \frac{dl^{(i)}}{Ds} = 0$$

where  $dl^{(i)}$  is the work of internal contact forces

$$(30) \quad \frac{dl^{(i)}}{Ds} = -u_{\alpha}(X^{(\alpha\beta)} + \mathcal{M}^{\alpha\beta})_{;\beta}.$$

The spatial part of (28)

$$(31) \quad \overset{\perp}{g}_{\alpha\gamma}(\rho u^{\gamma} u^{\beta} + X^{(\gamma\beta)} + \mathcal{M}^{\gamma\beta})_{;\beta} = 0$$

can be taken as the relativization of the first Cauchy equation of continuous media <sup>(6)</sup>.

### 3. Variation of world lines in the elastic adiabatic case.

Introducing the Lagrangian counterpart.

$$(32) \quad m_{*}^{B LM} = \mathfrak{D}^{-2} \gamma_{\beta}^B \gamma_{\lambda}^L \gamma_{\mu}^M m^{\beta\lambda\mu}$$

of  $m^{\beta\lambda\mu}$ , the constitutive equations of the elastic body  $\mathcal{C}$  capable of

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<sup>(6)</sup> For more detail on the definition of  $\mathcal{M}^{\alpha\beta}$  and on the acceptability in general relativity of the equations (29), (31) and the expression (30) of the work of internal contact forces see the works of Bressan. When thermodynamic and electromagnetic phenomena are taken into account, Eckart's tensor  $Q_{\alpha\beta}$  and the electromagnetic tensor  $E^{\alpha\beta}$  should also be added to the expression (27) of  $U^{\alpha\beta}$ . However no equation involving  $Q_{\alpha\beta}$  or  $E_{\alpha\beta}$  will be considered in the sequel.



couple stress but not of heat conduction are

$$(33) \quad \begin{aligned} w &= \hat{w}(\varepsilon_{LM}, C_{BLM}, y^L); & m_*^{BLM} &= K^* \frac{\partial w}{\partial C_{[BL]M}} + \lambda \varepsilon^{*BLM} \\ Y^{(LM)} &= -K^* \frac{\partial w}{\partial \varepsilon_{LM}} - 2m_*^{(LAB} C^{M)S} C_{SAB}; & \frac{\partial w}{\partial C_{(BL)M}} &= 0 \end{aligned}$$

where  $\lambda = \lambda(\varepsilon_{LM}, C_{BLM})$  and  $\varepsilon^{*BLM}$  is the spatial Ricci tensor is  $S_3^*$ .

Now we suppose the field  $g_{\alpha\beta}$  and the motion  $\mathcal{M}$  of  $\mathbb{C}$  as given in  $S_4$ . We consider a regular motion  $\mathcal{M}_\lambda$  depending on the real parameter  $\lambda$

$$(34) \quad z^\alpha = *z^\alpha(\lambda, y^\Sigma) \quad \text{with } *z^\alpha(0, y^\Sigma) = \hat{x}^\alpha(y^\Sigma),$$

and the functions  $z^\alpha(\lambda, x^e)$  defined by the condition

$$(35) \quad z^\alpha = z^\alpha[\lambda, \hat{x}^e(y^\Sigma)] = *z^\alpha(\lambda, y^\Sigma);$$

furthermore we set

$$(36) \quad \xi^e = \xi^e(x) = \left[ \frac{\partial z^e(\lambda, x)}{\partial \lambda} \right]_{\lambda=0}.$$

It follows that  $\xi^e d\lambda$  is the displacement of the material point  $y^\mu$  (at the instant  $t = y^0$ ) in the correspondence  $\mathcal{M}_0 \rightarrow \mathcal{M}_{d\lambda}$ .

Consider the functional

$$(37) \quad I = \int_{C_4} \varrho \sqrt{-g} dx \quad (dx = dx_0 dx^1 dx^2 dx^3)$$

and the variation  $\xi^e d\lambda$  of the motion  $\mathcal{M}$  of  $\mathbb{C}$  that is of class  $C^{(e)}$  and satisfies

$$(38) \quad \xi^e = 0 = \xi^e_{,\sigma}$$

on the boundary  $\mathcal{F}C_4$  of  $C_4$ .

Then in [6] it is proved that for these variations we have

$$(39) \quad \delta \int_{C_4} \varrho \sqrt{-g} dx = d\lambda \int_{C_4} (\varrho u^e u^\sigma + X^{(e\sigma)} + \mathcal{M}^{e\sigma})_{,\sigma} \xi_e \sqrt{-g} dx.$$

Hence the validity of the variational condition

$$(40) \quad \delta I = 0$$

is equivalent to the conservation equations (28).

#### 4. Mixed form of the dynamical equations.

In the absence of electromagnetic and heat conduction phenomena the spatial part of  $\mathfrak{U}^{\alpha\beta}{}_{|\beta} = 0$  is:

$$(41) \quad \frac{1}{2} \mathfrak{g}_{\alpha\beta} (\varrho u^\gamma u^\beta + X^{(\beta\gamma)} + \mathcal{M}^{\gamma\beta})_{|\beta} \equiv (\varrho \mathfrak{g}_{\alpha\gamma} + X_{(\gamma\beta)}) A^\beta + \frac{1}{2} \mathfrak{g}_{\alpha\gamma} X^{(\gamma\beta)}{}_{|\beta} + \frac{1}{2} \mathfrak{g}_{\alpha\gamma} \mathcal{M}^{\gamma\beta}{}_{|\beta} = 0 .$$

We call  $M^\alpha = -\frac{1}{2} \mathfrak{g}_{\alpha}^{\sigma} \mathcal{M}^{\sigma\beta}{}_{|\beta}$  the resultant of the forces due to couple stresses and, in analogy with Bressan [2], we introduce the resultant

$$(42) \quad M^{*\alpha} = \mathfrak{D} M^\alpha$$

of the same forces per unit volume of reference configuration.

By a useful formula proved in [2] we have

$$(43) \quad \mathfrak{D} X^{(\beta\gamma)}{}_{|\beta} = K^{\gamma M}{}_{|M} .$$

Let us multiply (41) by  $\mathfrak{D} (\neq 0)$  then by (14), (23), (41), (42) and (43) we have

$$(44) \quad \mathfrak{D} \frac{1}{2} \mathfrak{g}_{\alpha\gamma} (\varrho u^\gamma u^\beta + X^{(\gamma\beta)} + \mathcal{M}^{\gamma\beta})_{|\beta} = (\varrho^* \mathfrak{g}_{\alpha\beta} + K_{\alpha}^M \alpha_{\beta M}) A^\beta + \frac{1}{2} \mathfrak{g}_{\alpha\gamma} K^{\gamma M}{}_{|M} - M_{\alpha}^* ,$$

from which the following mixed form of the dynamical equations follows

$$(45) \quad (\varrho \mathfrak{g}_{\alpha\beta} + K_{\alpha}^M \alpha_{\beta M}) A^\beta + \frac{1}{2} \mathfrak{g}_{\alpha\gamma} K^{\gamma M}{}_{|M} - M_{\alpha}^* = 0 \quad (?).$$

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(?) Recalling that  $K_{\alpha}^M = \alpha_{\alpha L} Y^{LM}$ , from (45) we can also deduce the completely Lagrangian form of the same equations.

### 5. Variational principle for equilibrium problems.

We now suppose that the space-time  $S_4$  is stationary and introduce a stationary system of reference  $(x)$ . Let  $C_3$  be the intersection of the world tube  $W_C$  of  $C$  with the hypersurface  $x^0 = 0$ .

We want to consider the equilibrium of  $C$  with respect to the stationary frame of reference  $(x)$ . We identify the arbitrary parameter  $t$  in the equations (15) of the motion of  $C$  with  $x^0$  and denote the configuration of  $C$  in  $C_3$  by  $x^r = \chi^r(y^L)$ . Then the equations of the motion of  $C$  can be put into the form

$$(46) \quad \begin{aligned} x_0 &= t \\ x^r &= x^r(t, y^2) = \chi^r(y^2). \end{aligned}$$

The functions  $\chi^r(y^L)$  are such that (46) satisfy the regularity conditions requested for the equations of motion.

The hypothesis of stationarity of  $(x)$  implies

$$(47) \quad g_{\alpha\beta,0} = 0.$$

Consider the functional

$$(48) \quad J^* = J^*(\chi^r) = \int_{S_3^*} \varrho^* \sqrt{a^*} dy \quad (dy = dy^2, dy^2 \cdot dy^2)$$

where  $\varrho^* = \varrho^*(\varepsilon_{LM}, C_{BLM,y})$  and  $J^*$  depends of  $\chi^r$  through  $\varepsilon_{LM}$  and  $C_{[BL]M}$ , and an arbitrary variation  $\delta\chi^r$  of  $\chi^r$ , that is of class  $C^{(e)}$  and vanishes on the boundary  $\mathcal{F}S_3^*$  of  $S_3^*$  together with  $\delta\chi^r_{,L}$ .

We shall prove the following:

a) the first principle of thermodynamics holds for the body  $C$  (not capable of heat conduction);

b) for the aforementioned variations  $\delta\chi^r$  we have

$$(49) \quad \delta J^* = \int_{S_3^*} \{ (\varrho^* g_{r\varrho}^\perp + K_r^L \alpha_{\varrho L}) A^\varrho + g_{r\varrho} K^{\varrho M}_{LM} - M_r^* \} \delta\chi^r \sqrt{a^*} dy,$$

Hence the validity of the variational condition

$$(50) \quad \delta J^* = 0$$

is equivalent to the dynamical equations (45) of the equilibrium <sup>(8)</sup>.

Let  $\mathcal{C}$  undergo the motion (46). We first prove (a). Since heat conduction is absent, the first principle of thermodynamics reads

$$(51) \quad k \frac{Dw}{Ds} + \frac{dl^{(i)}}{Ds} = 0 .$$

By (46)  $D\varepsilon_{LM}/Ds = 0 = DC_{[BL]M}/Ds$ . Hence on the one hand from  $w = \tilde{w}(\varepsilon_{LM}, C_{[BL]M})$  we deduce  $Dw/Ds = 0$ . With regard to  $dl^{(i)}/Ds$  we know from Bressan [6] that <sup>(9)</sup>

$$(52) \quad \frac{dl^{(i)}}{Ds} = \frac{1}{\mathfrak{D}} \frac{d^*l^{(i)}}{Ds} = \frac{1}{\mathfrak{D}} [Y^{(BR)} + 2m^{(BLM} C^{RS)} C_{SLM}] \cdot \frac{D\varepsilon_{BR}}{Ds} - \frac{m^{BLM}}{\mathfrak{D}} \frac{DC^{[BL]M}}{Ds} ,$$

so that, on the other hand, we have  $dl_{(i)}/Ds = 0$ .

Now we consider the set  $C_4$  of the points  $x^0$  of  $W_{\mathcal{C}}$  with  $|x^0| \leq a + 1$  (a real positive). Let  $C_4^a$  be the subset of  $C_4$  where  $|x^0| \leq a$ , let  $C_4^+$  be the subset where  $a \leq x^0 \leq a + 1$ , and let  $C_4^-$  be the subset where  $-(a + 1) \leq x^0 \leq -a$ .

Then we consider any function, of class  $C^{(3)}$ ,  $\varphi(\xi)$ , of the real variable  $\xi$ , for which

$$(53) \quad \begin{cases} \varphi'(0) = \varphi''(0) = \varphi'''(0) = 0 \\ \varphi(0) = 1; \quad \varphi(1) = \varphi'(1) = 0 \\ \int_0^1 \varphi(\xi) d\xi = 0 . \end{cases}$$

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<sup>(8)</sup> It must also be remembered that the vector on the left hand side of (45) is purely spatial, hence it vanishes if and only if its components with  $\alpha = 1, 2, 3$  vanish.

<sup>(9)</sup> If  $dl^{(i)}/Ds$  and  $d^*l_{(i)}/Ds$  represent the power of internal contact forces for unity of actual and reference configuration respectively, we have

$$d^*l^{(i)} = \mathfrak{D} dl^{(i)} .$$

Next we introduce the arbitrary variations  $\delta\chi^r$ , of class  $C^{(3)}$ , that vanish on  $\mathcal{F}S_3^*$  together with  $\delta\chi^{r,L}$ ; we also consider the corresponding  $\delta x^r$  of the  $x^r$  as given by (46) and we set  $\delta x^0 = 0$ . We regard these  $\delta x^r$  as functions  $\delta x^r(x^s)$  of the space time co-ordinates  $x^s$  by means of the inverses of the functions  $x^r = \chi^r(y^L)$ . Then  $\delta x^r = 0 = \delta x^{r,s}$ , on  $\mathcal{F}C_3$ . We also suppose that, for  $x^r$  in  $C_3$ , the functions that appear in (39) are given by

$$(54) \quad \begin{cases} \xi_0 d\lambda = 0 & \text{in } C_4 \\ \xi^r(x^0, x^r) d\lambda = \begin{cases} \delta x^r(x^s) & |x^0| < a \\ \varphi(x^0 - a) \delta x^r(x^s) & a \leq x^0 \leq a + 1 \\ \varphi(-x^0 - a) \delta x^r(x^s) & -(a + 1) \leq x^0 \leq -a. \end{cases} \end{cases}$$

By (53) these functions are of class  $C^{(3)}$  in  $C_4$  and vanish on  $\mathcal{F}C_4$  together with  $\xi_{e,\sigma}$ .

For these variations we have, remembering that  $\varrho \sqrt{-g}$  does not depend on  $x^0$ ,

$$(55) \quad \begin{aligned} \delta \int_{\sigma_4^+} \varrho \sqrt{-g} dx &= \int_{\sigma_4^+} \frac{\partial \varrho}{\partial x^r} \delta x^r \varphi(x^0 - a) \sqrt{-g} dx = \\ &= \int_{C_3} \frac{\partial \varrho}{\partial x^r} \delta x^r \sqrt{-g} d_3 x \int_a^{a+1} \varphi(x^0 - a) dx^0 = 0, \end{aligned}$$

and likewise

$$(56) \quad \delta \int_{\sigma_4^-} \varrho \sqrt{-g} dx = 0.$$

Then, as  $\varrho u_\alpha u_\beta + X_{\alpha\beta} + \mathcal{M}_{\alpha\beta}$  does not depend on  $x^0$ ,

$$(57) \quad \begin{aligned} d\lambda \int_{\sigma_4^+} U_{\varrho\sigma}{}^{\mu\sigma} \xi^e \sqrt{-g} &= \int_{\sigma_4^+} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{\mu\sigma} \varphi(dx^0 - a) \delta x^r \sqrt{-g} dx \\ &= \int_{C_3} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{\mu\sigma} \delta x_r \sqrt{-g} d_3 x \int_a^{a+1} \varphi(x^0 - a) dx^0 = 0, \end{aligned}$$

and analogously

$$(58) \quad d\lambda \int_{\sigma_4^-} (\varrho u_\rho u_\sigma + X_{(\rho\sigma)} + \mathcal{M}_{\rho\sigma})^{\mu\sigma} \xi^e \sqrt{-g} dx = 0.$$

We then have

$$(59) \quad \int_{C_4^+} \varrho \sqrt{-g} dx = \int_{-a}^a dx^0 \int_{C_3} \varrho \sqrt{-g} d_3x = 2a \int_{C_3} \varrho \sqrt{-g} d_3x = \\ = 2a \int_{S_3^*} \varrho \sqrt{a^*} \mathcal{D} dy = 2a \int_{S_3^*} \varrho^* \sqrt{a^*} dy ,$$

hence, see also (55) and (56),

$$(60) \quad \delta \int_{C_4} \varrho \sqrt{-g} dx = 2a \delta \int_{S_3^*} \varrho^* \sqrt{a^*} dy$$

for the variations  $\xi^e d\lambda$  and  $\delta\chi^r$  considered above. Furthermore, remembering (57), (58), and that the temporal part of  $\mathfrak{U}^{\alpha\beta}{}_{;\beta}$  vanishes identically (first principle of thermodynamics), we have

$$(61) \quad d\lambda \int_{C_4} (\varrho u_e u_\sigma + X_{(e\sigma)} + \mathcal{M}_{\varrho\sigma})^{;\sigma} \xi^e \sqrt{-g} dx = \\ = \int_{C_4^+} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{;\sigma} \delta x^r \sqrt{-g} dx = \\ = \int_{C_4^+}^{\perp} g_{re} (\varrho u^e u^\sigma + X^{(e\sigma)} + \mathcal{M}^{e\sigma})_{;\sigma} \delta x^r \sqrt{g} dx = \\ = \int_{-a}^a dx^0 \int_{C_3}^{\perp} g_{re} (\varrho u^e u^\sigma + X^{(e\sigma)} + \mathcal{M}^{e\sigma})_{;\sigma} \delta x^r \sqrt{-g} g d_3x = \\ = 2a \int_{S_3^*}^{\perp} g_{re} (\varrho u^e u^\sigma + X^{(e\sigma)} + \mathcal{M}^{e\sigma})_{;\sigma} \delta \chi^r(y) \mathcal{M} \sqrt{a^*} dy = \\ = 2a \int_{S_3^*}^{\perp} \{(\varrho^* g_{re}^{\perp} + K_r^L \alpha_{eL}) A^e + g_{re} K^{eM}{}_{;M} - M_r^{\cdot\cdot}\} \delta \chi^r \sqrt{a^*} dy .$$

From (60), (61) and (39) we obtain the validity of

$$(62) \quad \delta \int_{S_3^*} \varrho^* \sqrt{a^*} dy = \int_{S_3^*}^{\perp} \{(\varrho^* g_{re}^{\perp} + K_r^L \alpha_{eL}) A^e + g_{re} K^{eM}{}_{;M} - M_r^{\cdot\cdot}\} \delta \chi^r \sqrt{a^*} dy$$

for the variations  $\delta\chi^r$  of  $\chi^r$  that are of class  $C^{(3)}$  and vanish on  $\mathcal{F}S_3^*$  together with their first partial derivatives. Q.E.D.

## 6. Comparison with a classical principle.

We now consider an elastic body  $\mathcal{C}$  not capable of couple stresses, in classical physics. If  $W$  is the elastic potential energy, the constitutive equations of  $\mathcal{C}$  are <sup>(10)</sup>

$$(63) \quad Y^{LM} = -\mu^* \frac{\partial W}{\partial \varepsilon_{LM}}$$

where  $\mu^*$  is the mass density in the reference configuration.

We suppose that the body forces are conservative, that is there exists a potential  $U = U(x)$  such that

$$(64) \quad f^r = U,{}^r.$$

Then the equations of the equilibrium of  $\mathcal{C}$  are

$$(65) \quad \left( \mu^* \frac{\partial W}{\partial \alpha_{rL}} \right)_{;L} + \mu^* U,{}^r = 0.$$

Consider the functional

$$(66) \quad J(x^r) = \int_{\mathcal{C}} \mu^* W dC^* - \int_{\mathcal{C}} \mu^* U dC^*$$

and any variation  $\delta x^r$  of  $x^r$  that vanishes on  $\mathcal{F}\mathcal{C}$ . Then it has been proved that the variational condition

$$(67) \quad \delta J(x^r) = 0,$$

is equivalent to the equilibrium equations.

If we remember that in general relativity body forces are taken into account by means of the metric, we see that our theorem in §5 generalizes this classical equivalence theorem involving the statio-

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<sup>(10)</sup> The quantities  $Y^{LM}$ ,  $\varepsilon_{LM}$  etc., occurring in classical physics, are defined in substantially the same way as we did in general relativity. See for example [14], [15].

narity of potential energy. Furthermore in our principle we also take into account possibly vanishing couple stresses and the energy due to the mass of  $C$ , as requested by the principle of equivalence of mass and energy.

## REFERENCES

- [1] A. BRESSAN, *Cinematica dei sistemi continui in relatività generale*, Ann. Mat. Pura Appl., **62** (1963), p. 99.
- [2] A. BRESSAN, *Termodinamica e magneto-visco-elasticità con deformazioni finite in relatività generale*, Rend. Sem. Mat. Univ. Padova, **34** (1964), p. 1.
- [3] A. BRESSAN, *Elasticità relativistica con coppie di contatto*, Ricerche di mat., **15** (1966), p. 169.
- [4] A. BRESSAN, *Coppie di contatto in relatività*, Ann. Scuola Normale Sup. Pisa, **20** (1966), p. 63.
- [5] A. BRESSAN, *Sui conduttori ideali in relatività*, Ann. Mat. Pura Appl., **94** (1972), p. 177.
- [6] A. BRESSAN, *Principi variazionali relativistici in presenza di coppie di contatto*, Ann. Mat. Pura Appl., **94** (1972), p. 201.
- [7] A. BRESSAN, *Relativistic theories of materials*, being printed by Springer, Berlin.
- [8] C. CATTANEO, *Proiezioni naturali e derivazione trasversa in una varietà Riemanniana e metrica iperbolica normale*, Ann. Mat. Pura Appl., **48** (1969), p. 361.
- [9] J. L. ERICKSEN, *Tensor fields*, Appendix in Handbuch der Physik III/I, Springer (1960).
- [10] V. A. FOCK, *The theory of space, time, and gravitation*, Pergamon Press, London (1964).
- [11] H. G. SCHÖPF, *Allgemein relativistische Prinzipien der Kontinuumsmechanik*, Ann. Physik, **12** (1964), p. 377.
- [12] A. SIGNORINI, *Lezioni di Fisica Matematica*, Veschi, Roma (1963).
- [13] A. H. TAUB, *A general variational principle for perfect fluid*, Phys. Rev. **94** (1954), p. 1468.
- [14] C. TRUESDELL - W. NOLL, *The non linear field theories of mechanics*, Handbuch der Physik III/III, Springer, Berlin (1965).
- [15] C. TRUESDELL - R. A. TOUPIN, *The classical field theories*, Handbuch, der Physik III/I, Springer, Berlin (1960).
- [16] L. M. ZUBOV, *Variational principles in the nonlinear field theories of mechanics*, Prikl. Math. Meckh., **35** (1971), p. 406.

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