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Limits to some interpolation theorems

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Limits to Some Interpolation Theorems.

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Summary - In this paper it is shown that the interpolation theorems of Chang and of Maehara and Takeuti hold only for languages in which the identity is a logical symbol.

Sommario - Nel presente articolo si mostra che i teoremi di interpolazione di Chang e di Maehara e Takeuti valgono solo per linguaggi in cui l'uguaglianza è un simbolo logico.

Introduction.

Chang's first theorem in [1] improves Craig's interpolation theorem. The latter can be stated as follows.

Let F and G be two formulas of a first order language. Suppose that $F \to G$ is a valid formula. Then there is a formula Z the predicates of which occur both in F and in G such that $F \to Z$ and $Z \to G$ are valid formulas.

If we denote by $R_1, ..., R_m$ the predicates in F that do not occur in G, and by $S_1, ..., S_n$ the predicates in G that do not occur in F, and by $x_1, ..., x_p$ the other predicate and individual free variables that occur either in F or in G (constants may be considered as free variabres), then the validity of the formula $F \to G$ corresponds to the validity of the formula $\forall x_1, ..., x_p (\exists R_1, ..., R_m F \to \forall S_1, ..., S_n G)$ and the validity of the formulas $F \to Z$ and $Z \to G$ corresponds to the validity of formula $\forall x_1, ..., x_p ((\exists R_1, ..., R_m F \to Z) \& (Z \to \forall S_1, ..., S_n G))$.

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Chang does not consider only the case that the variables $x_1, ..., x_p$ are all universally quantified, but he assumes that some of the individual variables among $x_1, ..., x_p$ are quantified existentially.

For $1 \leqslant i \leqslant p$ let Q_i be either $\forall$ or $\exists$ if x_i is an individual variable, otherwise let Q_i be $\forall$.

Chang's theorem can be stated as follows.

Let F, G, $R_1, ..., R_m$, $S_1, ..., S_n$, $x_1, ..., x_p$ be as before.

Suppose that $Q_1 x_1, ..., Q_p x_p (\exists R_1, ..., R_m F \rightarrow \forall S_1, ..., S_n G)$ is a valid formula. Then there is a first order formula Z whose predicate variables occur in both F and G, such that the formula

$$Q_1 x_1, ..., Q_p x_p \big((\exists R_1, ..., R_m F \rightarrow Z) \& (Z \rightarrow \forall S_1, ..., S_n G) \big)$$

is valid.

Indeed Chang shows that the formulas

$$Q_1 x_1, ..., Q_p x_p (\exists R_1, ..., R_m F \rightarrow Z) \quad \text{and} \quad \forall x_1, ..., x_p (Z \rightarrow \forall S_1, ..., S_n G)$$

are valid.

It should be remarked that Chang's theorem is stated in a language where the identity is a logical symbol.

The proof of Chang's theorem reguires non elementary notions such as special ω_1-saturated models.

In some remarks in [1] Chang asks whether there is a simpler proof of his theorem and whether it can be extended to infinitary languages.

In [2] Maehara and Takeuti give a positive answer to Chang's questions proving a theorem that implies Chang's theorem and using only regular techniques for cut free proof theory. Furthermore they remark that their proof can easily be extended to an infinitary language $L_{\omega_1, \omega}$ where countable conjunctions are allowed.

In our languages we can assume that the only connectives are — and &, and the only quantifier is $\forall$, while the other connectives and quantifier may be considered as metalinguistic abbreviations. An occurrence of $\forall$ is universal (existential) if it is in the scope of an even (odd) number of negation symbols.

A second order positive language $L_{\omega, \omega}^{2+}$ is a second order language where the predicate variables are quantified only universally.

A first order formula in $L_{\omega, \omega}^{2+}$ is a formula without bound predicate variables.

Let $G(\overline{A})$ be a formula whose predicate and individual variables which are free are in the set $\overline{A}$.

By a variant of $G(\overline{A})$ we mean a formula that can be obtained from $G(\overline{A})$ substituting for the variables in $\overline{A}$ variables which are free for the corresponding variables in $\overline{A}$ and which correspond to the viables in $\overline{A}$ through a function f that preserves the type of the varibles. We will use the symbol $G(\overline{A}/f)$ do denote the variant obtained from $G(\overline{A})$ through the function f.

An occurrence of a formula is negative (positive) if it is within the scope of an even (odd) number of negation symbols.

The theorem of Maehara and Takeuti, theorem 4 in [2], can be stated as follows.

Let S be a valid sentence in which all the occurrences of variants of $G(\overline{A})$ are positive. Then there is a first order formula $C(\overline{A})$ whose only free variables are in $\overline{A}$, such that $C(\overline{A}) \to G(\overline{A})$ is valid and the sentence S', obtained from S substituting for each variant $G(\overline{A}/f)$ of $G(\overline{A})$ the corresponding variant $C(\overline{A}/f)$ of $C(\overline{A})$, is valid.

To obtain this result, Maehara and Takeuti show and use another new interpolation theorem, their first interpolation theorem, theorem 3 in [2], that can be stated as follows.

Let $(F' \to G')(\overline{A})$ be a formula of $L^{2+}_{\omega,\omega}$ whose only free variables are in $\overline{A}$, and let f be a function from $\overline{A}$ into a set $\overline{B}$ of variables such that f preserves the type of the variables and the variables in $\overline{B}$ are free for the variables of their counterimages in $\overline{A}$. Let $F \to G$ be $(F' \to G')(\overline{A}/f)$.

Suppose that $F \to G$ is a valid formula. Then there is a first order formula $C(\overline{A})$ such that: 1) every variable occurring in $C(\overline{A})$ which is not in $\overline{A}$ occurs in both F and G, 2) $F \to C(\overline{A}/f)$ is a valid formula, and 3) $C(\overline{A}) \to G(\overline{A})$ is a valid formula.

Also this theorem is proved in [2] using only regular techniques for cut free proof theory.

It should be remarked that in [2] the language used is without a logical symbol for the identity.

Of course the theorems of Maehara and Takeuti stated above are stated differently in [2], using sequents instead of formulas, in order to apply more easily a cut free natural deduction.

Unfortunatly there is a mistake in the first interpolation theorem of Maehara and Takeuti.

In this paper we want to show that the interpolation theorems of Chang and of Maehara and Takeuti hold only for languages where the identity is a logical symbol.

To do this we will produce counterexamples to show where the

first interpolation theorem of Maehara and Takeuti fails and to show that what it is proved in Chang's interpolation theorem does not hold in a language without identity. Furthermore we will extend the rules of inference to adapt them to a language with identity, and obtain there a proof of the (slightly changed) interpolation theorems of Maehara and Takeuti for the languages $L_{\omega,\omega}^{2+}$ and $L_{\omega_1,\omega}^{2+}$, thus saving all that can be saved of the work of Maehara and Takeuti.

1. Preliminaries.

In our language there will be the symbol t for truth.

A *sequent* in a language is an ordered pair of sets, Δ and Γ, of formulas such that $-\,\&\,\{-\,G,\,\&\,\Delta\colon G\in\Gamma\}$ is a formula. Our notation for a sequent will be $\Delta\to\Gamma$.

An *initial sequent* is a sequent either of the type $\to t$ or of the type $F\to F$ where F is an atomic formula.

A *proof* for $L_{\omega,\omega}^{2+}$ without identity is a finite sequence of sequents in $L_{\omega,\omega}^{2+}$ without identity such that each sequent in the sequence is either an initial sequent or is obtained from previous sequents in the sequence through one of the rules described by the following schemas where Δ, Δ', Γ, Γ' are finite sets of formulas of $L_{\omega,\omega}^{2+}$ without identity and $\bar{X}$ is a fixed set of variables:

$$(,-)\quad \frac{\Delta, F\to\Gamma}{\Delta\to\Gamma, -F};\qquad (-,)\quad \frac{\Delta\to\Gamma, F}{\Delta, -F\to\Gamma};$$

$$(++)\quad \frac{\Delta\to\Gamma}{\Delta'\cup\Delta\to\Gamma\cup\Gamma'},\ \text{where either}\ \Delta\neq\emptyset\ \text{or}\ \Gamma\neq\emptyset;$$

$$(\&,)\quad \frac{\Delta, F_1, F_2\to\Gamma}{\Delta, F_1\,\&\,F_2\to\Gamma};\qquad (,\&)\quad \frac{\Delta\to\Gamma, F_1|\Delta\to\Gamma, F_2}{\Delta\to\Gamma, F_1\,\&\,F_2};$$

$$(\forall,)\quad \frac{\Delta, F(v)\to\Gamma}{\Delta, \forall x F(x)\to\Gamma}\ \text{where}\ v\ \text{is an individual variable;}$$

$$(,\forall v)\quad \frac{\Delta\to\Gamma, F(v)}{\Delta\to\Gamma, \forall v F(v)}\ \begin{array}{l}\text{where } v \text{ is an individual variable that does}\\ \text{not occur free in the lower sequent and } v\notin\bar{X};\end{array}$$

$$(,\forall V)\quad \frac{\Delta\to\Gamma, F(V)}{\Delta\to\Gamma, \forall V F(V)}\ \begin{array}{l}\text{where } V \text{ is a predicate variable that does not}\\ \text{occur free in the lower sequent and } V\notin\bar{X}.\end{array}$$

A sequent is said to be *provable* if it as a sequent of a proof.

A sequent $\Delta \to \Gamma$ is said to be valid if the formula $- \& \{- G, \& \Delta : G \in \Gamma\}$ is valid.

The notions of validity and provability are related by the following:

COMPLETENESS THEOREM. A sequent is valid if and only if it is provable.

2. The status of the first interpolation theorem of Maehara and Takeuti.

To explain more easily the exact point where the first interpolation theorem of Maehara and Takeuti fails, let us state it in its original form:

Let $\Delta \to \Gamma$ be a provable sequent and let it be $\Pi(\overline{A}/f) \to \Lambda(\overline{A}/f)$, $\overline{A}$ free variables in $\Pi \to \Lambda$. Let (Π_1, Π_2) be a partition of Π, and let (Λ_1, Λ_2) be a partition of Λ. There is a first order formula C, the interpolant, satisfying the following conditions:

1) each free variable in C which is not in $\overline{A}$ occurs both in $\Pi_1 \to \Lambda_1$ and in $\Pi_2 \to \Lambda_2$;

2) $\Pi_1(\overline{A}/f) \to \Lambda_1(\overline{A}/f)$, $C(\overline{A}/f)$ is provable;

3) $C, \Pi_2 \to \Lambda_2$ is provable.

The proof of this theorem in [13] is by induction on the length of a proof of $\Delta \to \Gamma$.

First of all we have to consider the cases in which $\Delta \to \Gamma$ is an initial sequent either of the type $\to t$ or of the type $F \to F$. In the first case the theorem is obvious, while in the second case we have to consider the following four subcases:

1) Π_1 is Π and Λ_1 is Λ,

2) Π_1 is Π and Λ_1 is empty,

3) Π_1 is empty and Λ_1 is Λ,

4) Π_1 is empty and Λ_1 is empty.

It is easy to see that $- t, \Lambda, - \Pi$ are the interpolants required by the theorem for the subcases 1), 2), 3) respectively.

It is not true that t is the interpolant required by the theorem for the subcase 4). On the contrary we will provide initial sequents of

this type that have no interpolant once they are partitioned according to subcase 4), and thus we will obtain counterexamples to the first interpolation theorem of Maehara Takeuti.

Let $\Delta \rightarrow \Gamma$ be $P(b) \rightarrow P(b)$, where P is a unary predicate variable and b an individual variable. Let $\overline{A}$ be $\{R\}$, and $f(R)$ be P, let Π be $P(b)$ and Λ be $R(b)$. Let Π_1 be ϕ and Λ_1 be ϕ. $\Delta \rightarrow \Gamma$ is provable and $\Pi_2 \rightarrow \Lambda_2$ is $P(b) \rightarrow R(b)$. Let us show that there is no interpolant in this case.

Indeed if C was an interpolant its only atomic subformulas would be of the type t and $R(x)$ where x is an individual variable quantified in C. In order to satisfy condition (2), $C(\overline{A}/f)$ has to be equivalent to t, and therefore also C has be equivalent to t since C is $C(\overline{A}/f)(\{P\}/f^{-1})$ for f is a 1-1 map. Then condition (3) becomes equivalent to: t, $P(b) \rightarrow \rightarrow R(b)$ is provable, which is not true. So there cannot be any interpolant.

Even if we weaken the initial theorem requiring that condition (1) holds only for predicate variables, allowing C to contain any individual variable, we will not obtain a positive result. Indeed the same counterexample would still hold.

In order to save the first interpolation theorem of Maehara Takeuti from this counterexample we have to add the hypothesis that each predicate variable in $\overline{A}$ is mapped into itself by f.

But this is not enough. Actually, just because the added hypothesis involves only predicate variables, we can produce another counterexample stemming out of the same idea as the previous one, this time based on the individual variables. And if we add a further hypothesis similar to the previous one but on the individual variables, still we could not obtain an interpolant because it may happen that no predicate variable is allowed in C.

Let $\Delta \rightarrow \Gamma$ be $P(b) \rightarrow P(b)$, where P and b are as before. Let $\overline{A}$ be $\{a\}$, and $f(a)$ be b, let Π be $P(b)$ and Λ be $P(a)$. Again let Π_1 be ϕ and Λ_1 be ϕ, so that $\Pi_2 \rightarrow \Lambda_2$ is $P(b) \rightarrow P(a)$. Let us show that even in this case there is no interpolant C. Indeed the only atomic subformula of C should be t, and, in order to satisfy condition (2), we should have $C(\overline{A}/f)$ (which is C) equivalent to t, and therefore we can take t for C. Condition (3) becomes equivalent to: t, $P(b) \rightarrow P(a)$ is provable, which is not true. So also in this case there is no interpolant. It should be remarked that even if $\overline{A}$ was $\{a, b\}$, still there would be no interpolant.

In order to overcome these difficulties, we have to allow more

freedom to the interpolant C, and therefore we have to allow some other predicate variable in C. To preserve the spirit of the theorem, the most natural solution is to move to a language where the identity is a logical symbols, and therefore the identity can occur freely in the interpolant.

A notion of proof adequate to this new language can be obtained allowing all the sequents in $L^{2+}_{\omega,\omega}$ even those with the identity and adding to the initial sequents the sequents of the type $\rightarrow x = x$ where x is a individual variable, and adding the four schemas of rules of inference:

$$(=,) \ \frac{\varDelta, a = b \rightarrow \varGamma}{\varDelta, b = a \rightarrow \varGamma}; \qquad (,=) \ \frac{\varDelta \rightarrow \varGamma, a = b}{\varDelta \rightarrow \varGamma, b = a};$$

$$(s,) \ \frac{\varDelta, a = b, Z(a) \rightarrow \varGamma}{\varDelta, a = b, Z(b) \rightarrow \varGamma}; \qquad (,s) \ \frac{\varDelta, a = b \rightarrow \varGamma, Z(a)}{\varDelta, a = b \rightarrow \varGamma, Z(b)}$$

where $Z(b)$ is the formula obtained from the atomic formula $Z(a)$ substituting b for an occurrence of a in $Z(a)$.

With the usual techniques, it is easy to show the completeness of this notion of proof, and hence of provable, with respect to the notion of validity for models of languages where the identity is a logical symbol.

To make the role of $\Pi \rightarrow \varLambda$ with respect to $\varDelta \rightarrow \varGamma$ more clear, we can introduce metavariables, that is symbols that have types and can substitute a variable of the same type to give rise to metaformulas and metasequents. Clearly a metaformula becomes a formula and a metasequent becomes a sequent once variables of the same type are substituted for the metavariables.

Thus a metaformula (a metasequent) is but a shema for a formula (a sequent). In our case we can think of $\overline{A}$ as a set of metavariables in the metasequent $\Pi \rightarrow \varLambda$ without free variables which is a schema for the sequent $\varDelta \rightarrow \varGamma$. f becomes a function (not necessarily 1-1) from the metavariables in $\overline{A}$ onto the variables in a set $\overline{B}$.

The correct first Maehara Takeuti's interpolation theorem can now be stated as follows:

THEOREM 1. Let $\varDelta \rightarrow \varGamma$ be a provable sequent, and let it be $\Pi(\overline{A}/f) \rightarrow \varLambda(\overline{A}/f)$ where $\overline{A}$ is a set of metavariables and f is such that it is 1-1 on the predicate metavariables, its range is $\overline{B}$, a set of variables free in $\varDelta \rightarrow \varGamma$, and no predicate variable in $\overline{B}$ occurs in $\Pi \rightarrow \varLambda$. Let (Π_1, Π_2) be a partition of Π, and $(\varLambda_1, \varLambda_2)$ be a partition of $\varLambda$. There

exists a first order metaformula C, the interpolant, satisfying the following conditions:

(A) all the metavariables in C are free and belong to $\overline{A}$, each free predicate variable in C occurs in both $\Pi_1 \rightarrow \Lambda_1$ and $\Pi_2 \rightarrow \Lambda_2$, each free individual variable in C occurs in $\Pi_2 \rightarrow \Lambda_2$;

(B) $\Pi_1(\overline{A}/f) \rightarrow \Lambda_1(\overline{A}/f)$, $C(\overline{A}/f)$ is provable;

(C) $C(\overline{A}/f')$, $\Pi_2(\overline{A}/f') \rightarrow \Lambda_2(\overline{A}/f')$ is provable, where f' is a 1-1 function and $f'(\overline{A})$ is a set of variables that do not occur in C, $\Pi_2 \rightarrow \Lambda_2$.

Note that the introduction of $=$ as a logical symbol forced us to weaken condition (A) with respect to condition (1). Remark further that condition (C) is but another way of stating the third condition in the first interpolation theorem of Maehara and Takeuti using the metavariables.

The requirement on f that no predicate variable in $\overline{B}$ occurs in $\Pi \rightarrow \Lambda$ can be met since in $\Delta' \rightarrow \Gamma'$ is the result of the application of one of the rules $(-,)$, $(,-)$, $(++)$, $(\&,)$, $(,\&)$, $(\forall,)$, $(,\forall v)$, $(,\forall)$-$(=,)$, $(,=)$, $(s,)$, $(,s)$ to $\Delta \rightarrow \Gamma$ and $\Delta' \rightarrow \Gamma'$ is $\Pi'(\overline{A}/f) \rightarrow \Lambda'(\overline{A}/f)$ where f satisfies the requirement, then $\Delta \rightarrow \Gamma$ is $\Pi(\overline{A}/f) \rightarrow \Lambda(\overline{A}/f)$ and again f satisfies the requirement, and hence this requirement can go through any proof by induction on the length of a proof, and also through the correct steps of the proof of Maehara and Takeuti.

PROOF. Let us prove this theorem by induction on the length of a proof of $\Delta \rightarrow \Gamma$.

Suppose that $\Delta \rightarrow \Gamma$ is an initial sequent. We have only to check the subcase where Maehara and Takeuti failed and the cases newly introduced by the use of the identity.

Let $\Delta \rightarrow \Gamma$ be $F \rightarrow F$, where F is an atomic formula, and let it be $G_1(\overline{A}/f) \rightarrow G_2(\overline{A}/f)$ with P the predicate variable or metavariable in G_1 and G_2, and let G_1 be $P(\alpha_1, ..., \alpha_m)$ and let G_2 be $P(\alpha'_1, ..., \alpha'_m)$ and let F be $P'(v_1, ..., v_m)$. Let C be $\& \{t, \& \{v_i = \alpha_i : 1 \leqslant i \leqslant m$ and v_i is not $\alpha_i\}$, $\& \{v_i = \alpha'_i : 1 \leqslant i \leqslant m$ and v_i is not $\alpha'_i\}\}$. Indeed it is trivial to check condition (A), and also condition (B) and (C) can be checked through simple computations.

Let us now assume that $\Delta \rightarrow \Gamma$ is $\rightarrow b = b$ and that $\Pi \rightarrow \Lambda$ is $\rightarrow a_1 = a_2$. Since Π is empty, there are only two cases to be considered: either Λ_1 is $a_1 = a_2$ or Λ_1 is empty. It is clear that in the first case $-t$ is an interpolant, while in the second case $a_1 = a_2$ is an interpolant.

For the steps of the induction that concern the schemas of rules $(=,)$ and $(, =)$, it is easy to see that the interpolant for the sequent below with respect to any partition is the interpolant for the sequent above with respect to the naturally corresponding partition.

For the steps that concern the schemas of rules $(s,)$ and $(, s)$, we have to consider four cases for each rule.

Let $\Delta \to \Gamma$ be the sequent above in the schema, $\Delta' \to \Gamma'$ the sequent below in the schema; let $\Pi \to \Lambda$ be the sequent that becomes $\Delta \to \Gamma$ through the substitution, and $\Pi' \to \Lambda'$ the sequent that becomes $\Delta' \to \Gamma'$ through the substitution. Let $c = d$ be the metaformula of $\Pi' \to \Lambda'$ that becomes $a = b$ in $\Delta' \to \Gamma'$, and Z' the metaformula in $\Pi' \to \Lambda'$ that becomes $Z(b)$ in $\Delta' \to \Gamma'$.

For the schema $(s,)$ the cases are:

a) $c = d \in \Pi_1'$, $Z' \in \Pi_1'$;   b) $c = d \in \Pi_2'$, $Z' \in \Pi_1'$;

c) $c = d \in \Pi_1'$, $Z' \in \Pi_2'$;   d) $c = d \in \Pi_2'$, $Z' \in \Pi_2'$;

while for the schema $(, s)$ the cases are:

a) $c = \mathrm{d} \in \Pi_1'$, $Z' \in \Lambda_1'$;   b) $c = d \in \Pi_2'$, $Z' \in \Lambda_1'$;

c) $c = d \in \Pi_1'$; $Z' \in \Lambda_2'$;   d) $c = d \in \Pi_2'$, $Z' \in \Lambda_2'$.

In case $a)$, $b)$ it is easy to see that if C is an interpolant for the sequent above in either $(s,)$ or $(, s)$ with respect to a partition naturally corresponding to a partition for the sequent below, then C in case $a)$ and $- \& \{- C, c = d\}$ in case $b)$ are interpolants for the sequent below with respect to the partition stated.

Cases $c)$ and $d)$ are a little bit more complicated. Again let $c = d$ be the metaformula of $\Pi' \to \Lambda'$ that becomes $a = b$ in $\Delta' \to \Gamma'$. Let c' be the individual variable or metavariable in $\Pi \to \Lambda$ that becomes b through the rule, and let d' be the individual variables or metavariable in $\Pi' \to \Lambda'$ that becomes the occurrence of b in $Z(b)$ in $\Delta' \to \Gamma'$ that comes from a through the rule. We have the following subcases:

c.1), d.1) $c \in \overline{A}$ or c occurs in $\Pi_2' \to \Lambda_2'$, and $c' \in \overline{A}$ or c' occurs in $\Pi_2' \to \Lambda_2'$;

c.2), d.2) $c \in \overline{A}$ or c occurs in $\Pi_2' \to \Lambda_2'$, and $c' \notin \overline{A}$ and c' does not occur in $\Pi_2' \to \Lambda_2'$;

c.3) $c \notin \overline{A}$ and c does not occur in $\Pi_2' \to \Lambda_2'$, and either $c' \in \overline{A}$ or c' occurs in $\Pi_2' \to \Lambda_2'$;

c.4) $c \notin \overline{A}$ and c does not occur in $\Pi_2' \to \Lambda_2'$, and $c' \notin \overline{A}$ and c'
does not occur in $\Pi_2' \to \Lambda_2'$.

If C is an interpolant for $\Delta \to \Gamma$ with respect to the partition
naturally corresponding to the stated partition, let C' be be $\&\{C, c = d,$
$c = c', d = d'\}$ and let C' be $\&\{C, c = c', d = d'\}$. Then it is not dif-
ficult to check that C' in case c.1), C' in case d.1), $- \forall c' - C'$ in case c.2),
$- \forall c' - C'$ in case d.2), $- \forall c - C'$ in case c.3), $- \forall c \forall c' - C'$ in
case c.4) will be interpolants, for $\Delta' \to \Gamma'$ with respect to the stated
partition.

The steps of the induction that concern the logical connectives and
the quantification of the predicate variables where already considered
by Maehara and Takeuti in [13] and they do not need to be repeated.
So we are left with the cases of the individual quantification.

Let $\Delta \to \Gamma$ be the sequent above in the schema $(\forall,)$, $\Delta' \to \Gamma'$ be
the sequent below in the schema; let $\Pi \to \Lambda$ be the sequent that be-
comes $\Delta \to \Gamma$ through the substitution, and $\Pi' \to \Lambda'$ the sequent that
becomes $\Delta' \to \Gamma'$ through the substitution. Let $F(v)$ be the formula
of Δ that becomes $\forall x F(x)$ of Δ' through the schema. Let $F^*(v^*)$ be
the metaformula of Π that becomes $F(v)$ through the substitution,
and let $\forall x F^*(x)$ be the metaformula of Π' that becomes $\forall x F(x)$ through
the substition. Let (Π_1', Π_2') and (Λ_1', Λ_2') be partitions of Π' and Λ'
respectively. There are two cases to be considered: $a)$ $\forall x F^*(x) \in \Pi_1'$,
$b)$ $\forall x F^*(x) \in \Pi_2'$.

In case $a)$ let C be an interpolant for $\Delta \to \Gamma$ with respect to the
partitions $\big((\Pi_1' - \{\forall x F^*(x)\}) \cup \{F^*(v^*)\}, \Pi_2'\big)$ of Π and (Λ_1', Λ_2') of Λ.
It is easy to check that C is also an interpolant for $\Delta' \to \Gamma'$ with
respect to the partitions (Π_1', Π_2') of Π' and (Λ_1', Λ_2') of Λ'.

In case $b)$ let C be an interpolant for $\Delta \to \Gamma$ with respect to the
partitions $\big(\Pi_1'(\Pi_2' - \{\forall x F^*(x)\}) \cup \{F^*(v^*)\}\big)$ of Π and (Λ_1', Λ_2') of Λ.
There are two subcases to be considered: b.1) v^* is v and v does not
occur in $\Pi_2' \to \Lambda_2'$, b.2) either v^* is not v (and hence $v^* \in \overline{A}$) or v oc-
curs in $\Pi_2' \to \Lambda_2'$. It is again easy to see that in subcase b.2) C is
also an interpolant for $\Delta' \to \Gamma'$ with respect to the partitions (Π_1', Π_2')
of Π' and (Λ_1', Λ_2') of Λ'. In subcase b.1) through simple computations
it can be shown that $- \forall v - C$ is an interpolant for $\Delta' \to \Gamma'$ with respect
to the partitions (Π_1', Π_2') of Π' and (Λ_1', Λ_2') of Λ'.

Let us now consider the schema $(, \forall v)$. Let $\Delta \to \Gamma$ be the se-
quent above in the schema, $\Delta' \to \Gamma'$ the sequent below in the schema.
Let $\Pi \to \Lambda$ be the sequent that becomes $\Delta \to \Gamma$ through the substi-
tution, and $\Pi' \to \Lambda'$ the sequent that becomes $\Delta' \to \Gamma'$ through the

substitution. Let $F(v)$ be the formula of Γ that becomes $\forall x F(x)$ through the schema. We are assuming that v does not occur in $\varDelta' \rightarrow \Gamma'$ and $f(\overline{A})$. Let $F^*(v)$ be the metaformula of $\varDelta$ that becomes $F(v)$ through the substitution, and let $\forall x F^*(x)$ be the metaformula of $\varDelta'$ that becomes $\forall x F(x)$ through the substitution. Let (Π_1', Π_2') and $(\varLambda_1', \varLambda_2')$ be partitions of Π' and $\varLambda'$ respectively. There are two cases to be considered: a) $\forall x F^*(x) \in \varLambda_1'$, b) $\forall x F^*(x) \in \varLambda_2'$.

In case a) let C be an interpolant for $\varDelta \rightarrow \Gamma$ with respect to the partitions (Π_1', Π_2') of Π and $\big((\varLambda_1' - \{\forall x\, F^*(x)\}) \cup \{F^*(v)\}, \varLambda_2' \big)$ of $\varLambda$. It is easy to check that C is also an interpolant for $\varDelta' \rightarrow \Gamma'$ with respect to the partitions (Π_1', Π_2') of Π' and $(\varLambda_1', \varLambda_2')$ of $\varLambda'$. In case b) let C be an interpolant for $\varDelta \rightarrow \Gamma$ with respect to the partitions (Π_1', Π_2') of Π and $\big(\varLambda_1' (\varLambda_2' - \{\forall x F^*(x)\}) \cup \{F^*(v)\} \big)$ of $\varLambda$. Through simple computations it can be shown that $\forall v C$ is an interpolant for $\varDelta' \rightarrow \Gamma'$ with respect to the partitions (Π_1', Π_2') of Π' and $(\varLambda_1', \varLambda_2')$ of $\varLambda'$.

Thus the proof of theorem 1 is compete.

3. The status of the second interpolation theorem of Maehara and Takeuti.

In [2], Maehara and Takeuti use their first interpolation theorem to show the second one, the theorem we refer to in page 2. Therefore it is important to see whether the theorem proved above can be used to show the second theorem of Maehara Takeuti, or a corresponding theorem, and whether the further requirement introduced is essential to the second result.

Our Theorem 2 will be the second interpolation theorem of Maehara and Takeuti but for a language with identity.

PROOF. The same reasons as in the proof of the corresponding result in [2] are valid, once we have remarked that there are no free variables in $G(\overline{A})$ and therefore there will be no consequences due to the difference between (A) and (1) of the definition of interpolant in theorem 1 and in the first interpolation theorem of Maehara Takeuti; and that a variant of $G(\overline{A})$ with a second order quantifier cannot appear through one of the inference rules of the type either $(=,)$ or $(,=)$ or $(s,)$ or $(,s)$, and that the cases concerning these rules are then trivial.

Now let us show that we cannot hope for a better result.

Indeed we shall provide a counterexample to the second interpolation theorem of Maehara Takeuti that will point out the need to introduce the identity as a logical symbol.

Let S be $\to \forall x \exists y\, G(x, y)$ where $G(x, y)$ is $\forall P\big(P(x) \to P(y)\big)$ and $\overline{A}$ is $\{x, y\}$. First of all, it is obvious that S is provable. If there was an interpolant it could have only t as atomic subformula in order to satisfy condition (i) in the statement of the second interpolation theorem of Maehara and Takeuti as it was given in the introduction, and we could take either t or $-t$ for the interpolant. To satisfy condition (ii) of the same theorem, either $\to \forall x \exists y\, t$ or $\to \forall x \exists y - t$ should be provable, and therefore we have to take t as interpolant. But condition (iii) would become $\to \forall x\, \forall y\big(t \to \forall P\big(P(x) \to P(y)\big)\big)$, which is false. It is easy to see that the interpolant for this S is $x = y$ in our theorem 2.

4. The status of Chang's interpolation theorem.

One may ask whether even for Chang's theorem which follows easily from theorem 2 with the method in [2], there is the need that the identity is a logical symbol.

The following counterexample shows that if the identity is not a logical symbol, then Chang's theorem fails in its version deriving from the proof, i.e. replacing the last formula in the statement of the theorem in the introduction by the formulas $Q_1 x_1, ..., Q_p x_p (\exists R_1, ..., R_m F \to Z)$ and $\forall x_1, ..., x_p, S_1, ..., S_n (Z \to G)$ whose validity implies the validity of that formula.

Let $\forall x \exists y\big(\exists R\big(R(x) \to R(x)\big) \to \forall S\big(S(x) \to S(y)\big)\big)$ be the formula for which we want to find an interpolant according to Chang. It is obvious that the formula is valid and that $x = y$ is an interpolant. But if we do not allow the identity as a logical symbol, the interpolant should be either t or $-t$, since it should not contain predicate variables. To have $\models \forall x \exists y\big(\exists R\big(R(x) \to R(x)\big) \to C\big)$ the interpolant should be equivalent to t; and hence we should have $\models \forall x\, \forall y\big(t \to \forall S\big(S(x) \to S(y)\big)\big)$ which is not true; and we have shown what we claimed.

5. The extensions of the theorems of Maehara and Takeuti to $L^{2+}_{\omega_1, \omega}$.

As it was done by Maehara and Takeuti in [2], also our theorem 1 and the second interpolation theorem of Maehara and Takeuti for a language with identity can be extended to the infinitary language $L^{+2}_{\omega_1, \omega}$.

In this language we allow as formulas countable conjunctions $\&\{F_i: i \in I\}$ of formulas F_i with $|I| < \omega_1$. Sequents will become pairs of countable sets of formulas.

We will have to change the schemas of the inference rules $(\&,)$ and $(, \&)$ to

$$(\&,)' \quad \frac{\Delta \cup \{F_i: i \in I\} \to \Gamma}{\Delta, \& \{F_i: i \in I\} \to \Gamma}$$

and

$$(,\&)' \quad \frac{\Delta \to \Gamma, \ F_i \ \text{for all} \ i \in I}{\Delta \to \Gamma, \& \{F_i: i \in I\}} \quad \text{respectively} .$$

Thus we obtain a cut-free proof theory for $L_{\omega_1,\omega}^{+2}$ in which at each step only a formula in the second sequent of an inference is affected, and in which a completeness theorem can be proved.

Therefore the proofs of theorem 1 and of the second interpolation theorem of Maehara and Takeuti for a language with identity can be repeated almost verbally and they will yeild the wanted results.

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