RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*, tome 54 (1975), p. 1-29

<http://www.numdam.org/item?id=RSMUP_1975__54__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 54 (1975)

# On Centralizers of Involutions Having a Component of Type $A_6$ and $A_7$.

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Recently, M. Aschbacher [1] has shown under certain assunptions, that every finite simple group $G$ containing an involution $t$ such that $C_G(t)$ is not 2-constrained contains a subgroup of standard type. So it is of fundamental interest for the theory of finite simple groups to classify finite groups by standard subgroups.

A standard subgroup $A$ of a finite group $G$ is a quasisimple group such that $C_G(A)$ is a group of even order and satisfies certain further properties.

Aschbacher [2] has classified all simple groups with a standard subgroup $A$ such that $A/Z(A) = A_n$ and that $C(A)$ has a 2-rank of at least 2.

On the other hand, the case of 2-rank 1 is of considerable interest as well. The Mathieu group $M_{12}$ contains an involution $t_2$ such that $C(t_2) = \langle t_2 \rangle \times S$, where $S$ is isomorphic to $\Sigma_5$. The Higman Sims simple group contains an involution with centralizer isomorphic to $Z_2 \times \mathrm{Aut}(A_6)$. Both groups have been classified by these centralizers (cf. [4] and [5]).

In this paper, we consider centralizers of the form $Z_2 \times \Sigma_6$ and $Z_2 \times \Sigma_7$. We shall prove the following theorems:

THEOREM A. Let $G$ be a finite group of even order containing an involution $t$ such that $C_G(t)$ is isomorphic to the direct product of a group of order 2 and the symmetric group on 6 letters. Then $G$ has a subgroup of index 2.

THEOREM B. Let $G$ be a finite group of even order containing an involution $t$ such that $C_G(t)$ is isomorphic to the direct product of a

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group of order 2 and the symmetric group on 7 letters. Then $G$ has a subgroup of index 2.

The methods used in the proof are elementary. Throughout the paper we assume that $G$ has no subgroup of index 2; we use the Thompson transfer lemma (cf. [3]) to derive a contradiction. The crucial fact seems to be that in the Sylow-2-subgroup of our centralizer we have two elementary groups of order 16, say $E_1$ and $E_2$, which « should » be conjugate in the centralizer, but are not. (This is contrary to the situation in [4] leading to the Higman Sims group).

Theorem B will be a corollary of the proof of theorem A. We will only have to redo parts of §§ 1-4. Then we will see that the two elementary groups which are the basis for the whole proof, are not conjugate in $G$, so §§ 5-8 can be applied.

Now we fix some notation. $G$ is a finite group having no subgroup of index 2, $t \in G$ is an involution such that $H := C_G(t) = \langle t \rangle \times \Sigma$, $\Sigma \cong \Sigma_6$. We choose a fixed Sylow-2-subgroup of $H$, say $T_0$, where $T_0 = \langle t \rangle \times S_0$ such that $S_0$ is a Sylow-2-subgroup of $\Sigma$.

$\Sigma_u$ and $A_n$ always denote the symmetric resp. alternating group on $n$ letters, $E_k$ denotes an elementary abelian group of order $k$, $D_n$ denotes a dihedral group of order $n$. If a group $X$ operates on a group $B$, then put $A_X(B) := N_X(B)/C_X(B)$.

For this paper, it is useful to define the Thompson subgroup $J(T)$ of a 2-group $T$ as follows: $J(T) = \langle E/E \leqslant T, E \cong E_{16} \rangle$. If $X$ is a subgroup of $G$, $N(X)$ and $C(X)$ always stand for $N_G(X)$ and $C_G(X)$.

When we regard a permutation representation of a group $X$ on a set $S = \{s_1, s_2, \ldots, s_n\}$, then we describe the action of an element $x \in X$ on $S$ as follows:

$$x \triangleq (s_{j11}, s_{j12}, \ldots, s_{j1r_1}), \ldots, (s_{in1}, \ldots, s_{jnr_n}) .$$

For sake of convenience, we do not always assume that the representation of $X$ on $S$ is faithful.

The remainder of the notation follows [3] and is fairly standard; for example we use the « bar convention » for homomorphic images, $V(\mathrm{ccl}_G(g); G_0)$ denotes the weak closure of $g$ in $G_0$ with respect to $G$.

## 1. The structure of $H$.

As $S_0$ is a Sylow-2-subgroup of the symmetric group on 6 letters, we may assume that $S_0$ is generated by the elements $i_2 = (5, 6)$,

$i_4 = (1, 2)(3, 4)$, $i_6 = i_2 i_4$, $a_1 = (1, 4)(2, 3)$ and $a_2 = (1, 2)(5, 6)$. We have $(a_1 a_2)^2 = i_4$.

We see that $S_0' = T_0' = \langle i_4 \rangle$ and that $Z(T_0) = \langle t, i_2, i_4 \rangle$. $T_0$ contains precisely 2 elementary subgroups of order 16, namely $E_1 = Z(T_0)\langle a_1 \rangle$ and $E_2 = Z(T_0)\langle a_2 \rangle$. It is clear that $Z(T_0) = E_1 \cap E_2$.

The involutions $i_2$, $i_4$, and $i_6$ represent the three conjugacy classes in $\Sigma$, $i_2$ having 15 conjugates, $i_4$ having 45 conjugates and $i_6$ having 15 conjugates in $H$.

Altogether, $H$ contains 7 classes of involutions, all the involutions of $Z(T_0)$ being representatives of the different classes. This shows that if $t$ is conjugate to any other class of involutions of $H$ in $G$, then $t$ is conjugate to this class in $N_G(T_0)$.

Finally, we have that $N_H(E_1) = E_1\langle d_1, a_2 \rangle$, where $d_1 \in \Sigma$ can be taken as $(1, 2, 3)$, and that $N_H(E_2) = E_2\langle d_2, a_1 \rangle$ with $d_2 = (1, 6, 4)(2, 5, 3)$ Then $a_2$ inverts $d_1$ and $a_1$ inverts $d_2$.

## 2. The first centralizer case.

In this paragraph, we want to show that $2^6$ divides the order of $G$. So assume the contrary.

LEMMA 2.1. *$N_G(T_0)$ has order $2^5 \cdot 3$. $G$ has precisely 3 classes of involutions. $E_1$ and $E_2$ are not conjugate in $G$. $N_G(E_i)$ controls the fusion on $E_i$ for $i = 1, 2$.*

PROOF. By the Thompson transfer lemma, all involutions of $T_0$ must be conjugate to some involution of $S_0$, but $S_0$ contains involutions of at most 3 different classes. Hence, fusion must take place in $Z(T_0)$ under the action of $N_G(T_0)$. The group $T_0' = \langle i_4 \rangle$ is characteristic in $T_0$, so there must be precisely 3 different classes in $Z(T_0)$ and $N(T_0)$ must have order $2^5 \cdot 3$.

If $E_1$ and $E_2$ are conjugate in $G$, they are conjugate in $N(T_0)$ by Burnside's lemma; but this is not possible. Therefore $N_G(E_i)$ must control the fusion on $E_i$.

THEOREM 2.2. *$2^6$ divides the order of $G$.*

PROOF. It is clear that $N_G(T_0) \leqslant N_G(E_i)$, $i = 1, 2$. $A_G(E_i)$ has a Sylow-2-subgroup of order 2 and hence must be solvable. The involution $t$ is conjugate into $S_0$ by an element of $N_G(T_0)$. We conclude that $t$

has 3 or 7 conjugates in $E_i$. If $t$ has 7 conjugates, then $A_G(E_i)$ has order $2\cdot 3\cdot 7$, which contradicts the structure of $GL(3, 2)$. Therefore $t$ has 3 conjugates in $E_1$ and 3 conjugates in $E_2$, which all must be elements of $Z(T_0)$. But this is impossible.

## 3. The case $2^6 T\,|G|$.

LEMMA 3.1. Put $T := N_G(T_0)$. Then $T$ is a Sylow-2-subgroup of $G$, $Z(T)$ has order 4, $t$ is conjugate to $ti_4$ in $T$, and $t$ does not fuse in $G$ to any other class of involutions in $H$.

PROOF. It is clear that $Z(T)$ is elementary of order 4. So at least 3 involutions of $Z(T_0)$ cannot be conjugate to $t$, and we see that $T_0$ does not admit a 3-automorphism. Therefore $t$ is conjugate to precisely one other involution of $Z(T_0)$ and we have shown that $t$ centralizes either 16 or 46 of its conjugates in $G$. We assume that $G$ has no subgroup of index 2, so $t$ operates on its conjugacy class as an even permutation. If $t$ centralizes 16 of its conjugates, then there are $O$ (mod 4) conjugates of $t$ in $G$. But $2^5$ divides $|C_G(t)|$, so $2^7$ divides the order of $G$, contrary to the assumption of this paragraph. We have proved that $t$ has 46 conjugates in $H$, so $t$ must be conjugate to $ti_4$. The lemma is proved.

LEMMA 3.2. $E_i$ is normal in $T$, for $i = 1, 2$.

PROOF. Put $T = T_0\langle y\rangle$ and suppose $E_1^y = E_2$. By lemma 3.1., $t$ has 1 or 4 conjugates in $E_i$ under the action of $N_G(E_i)$. The order of $N_H(E_i)$ is $2^5\cdot 3$, so if $t$ has 4 conjugates in $E_i$ under the action of $N_G(E_i)$, then $2^7$ divides the order of $N_G(E_i)$, which is not possible, Therefore $N_G(E_i) = N_H(E_i)$, and $t$ has an orbit of length 1 under $N_G(E_i)$. Applying lemma 3.1., we see that we must have $t^y = t$. This is a contradiction.

THEOREM 3.3. The order of $G$ is divisible by $2^7$.

PROOF. We have shown that $2^6$ divides the order of $N_G(E_i)$. Applying lemma 3.1. again we see that $t$ has 4 conjugates in $E_i$ under the action of $N_G(E_i)$. But then $2^7$ must divide the order of $N(E_i)$ Theorem 3.3. is proved.

## 4. The case $|N_G(T_0):T_0|=2$.

LEMMA 4.1. Set $T_1:=N_G(T_0)=T_0\langle y\rangle$. Then $T_1=J(T_1)$, $E_1$ and $E_2$ are normal in $T_1$, $t$ is fused to precisely one other involution of $Z(T_0)$, and there are precisely 16 $G$-conjugates of $t$ in $H$.

PROOF. As $2^7$ divides the order of $G$ and, by assumption of this paragraph, the normalizer of $T_0$ has order $2^6$, $T_0$ cannot be characteristic in $T_1$. Therefore $T_1=J(T_1)$. It is clear from some remarks in § 1 that $t$ can only be conjugate to one more class of involutions of $H$. From the fact that $2^7$ divides $|G|$, we conclude that $t$ cannot have 46 conjugate in $H$; so $t$ must be conjugate to 16 involutions of $H$. As $T_1=J(T_1)$, we conclude, using our definition of $J(T_1)$, that we cannot have $E_1^y=E_2$. Lemma 4.1. is proved.

LEMMA 4.2. Set $Z(T_1)=:\langle z_1,z_2\rangle$ such that $z_2=i_4$. Then $z_1$ can be chosen such that $t^y=tz_1$. Furthermore, we may assume that $Z(T_1)\cap Z(N_H(E_1))=\langle z_1\rangle$ and that $Z(T_1)\cap Z(N_H(E_2))=\langle z_1z_2\rangle$. Finally

$$|N_G(E_1)|=2^6\cdot 3\,, \qquad \text{and} \qquad |N_G(E_2)|=2^7\cdot 3\,.$$

PROOF. From the structure of $T_0$ we see that $z_2$ $(=i_4)$ is not conjugate to $t$. So we may set $Z(T_1)=\langle z_1,z_2\rangle$ such that $t^y=tz_1$.

As the normalizers of $E_1$ and $E_2$ in $H$ are isomorphic we may alter the notation such that the last part of the lemma holds.

LEMMA 4.3. We can choose $y$ to be an involution and to centralize $\langle z_1,z_2,a_1\rangle$.

PROOF. Set $D_1:=\langle d_1\rangle$ and $N_1:=N_G(E_1)$. Then $D_1$ is a Sylow-3-subgroup of $N_1$. From our information about $N_1$ and using Sylow's theorem, we conclude that a Sylow-2-subgroup $R$ of $N_{N_1}(D_1)$ has order $2^4$ As $a_2$ inverts $d_1$, it follows that $C_R(D_1)$ has order $2^3$. On the other hand, $C_{R\cap H}(D_1)=\langle t,z_1\rangle$, so $C_H(D_1)$ is a dihedral group of order 8. We choose $y$ to be an involution of this group. As $y$ centralizes $d_1$, $y$ operates on $[E_1,d_1]=\langle z_2,a_1\rangle$. By the Thompson $A\times B$-lemma, $y$ centralizes $\langle z_2,z_1\rangle$. Lemma 4.3. is proved.

LEMMA 4.4. $[y,a_2]\in\langle z_1\rangle$.

PROOF. Using lemma 4.3., we can choose a Sylow-2-subgroup of $N_{N_1}(D_1)$: $R = \langle z_1, t, y, a_2 \rangle$. It is clear that $\langle z_1, t, y \rangle = C_R(D_1)$ and $\langle z_1, t, a_2 \rangle = R \cap H$ are normal in $R$; we conclude that $[y, a_2] \in \langle z_1, t \rangle$ is an involution and therefore must be centralized by $y$. The lemma is proved.

LEMMA 4.5. Set $N_2 := N_G(E_2)$ and take $T_2$ to be the Sylow-2-subgroup of $N_2$ which contains $T_1$. Further set $Q_2 := O_2(N_2)$. The $N_2/E_2$ is isomorphic to $\Sigma_4$, and elementary subgroups of $T_2$ are contained in $T_1$ or in $Q_2$.

PROOF. The orbit of $t$ in $E_2$ under the action of $N_2$ is $\{t, tz_1, tz_1z_2a_2, tz_1a_2\}$. Call these elements $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, respectively. Then $d_2$ acts as $(\underline{2}, \underline{3}, \underline{4})$ and $y$ acts as $(\underline{1}, \underline{2})$. So $d_2$ and $y$ generate the full symmetric group on the orbit of $t$, therefore $T_2/E_2$ is dihedral of order 8 having elementary subgroups $T_1/E_2$ and $Q_2/E_2$. The lemma is proved.

LEMMA 4.6. $E_2 = J(Q_2)$.

PROOF. We can see from the proof of lemma 4.5. that $a_1 y \in Q_2$ Set $f_2 := (ya_1)^{d_2}$. It follows that $[ya_1, E_2] = \langle z_1, z_2 \rangle$ and that $[f_2, E_2] = \langle z_1z_2, a_2 \rangle$. Therefore $Z(Q_2) = \langle z_1z_2 \rangle$. Now it is very easy to see that $E_2$ is the only elementary subgroup of order 16 of $Q_2$, so $E_2 = J(Q_2)$. The lemma is proved.

LEMMA 4.7. $Z(T_1) = \langle z_1, z_2 \rangle =: Z$. $T_1$ contains precisely 4 elementary subgroups of order 16, namely $E_1, E_2, E_3 = Z\langle a_1, y \rangle$ and $E_4 = Z\langle ta_2, y \rangle$.

PROOF. From the proof of lemma 4.5. it follows that $[y, a_2] = z_1$. Therefore $[y, ta_2] = 1$. On the other hand, $y$ centralizes $a_1$ by lemma 4.3. We conclude that $T_1 = \langle a_1, a_2t \rangle \times \langle y, t \rangle$ is the direct product of two dihedral groups of order. 8 Now it is immediate that $T_1$ contains precisely 4 elementary subgroups of order 16, namely those listed above. The lemma is proved.

LEMMA 4.8. $E_2$ is characteristic in $T_2$.

PROOF. First of all, note that $E_1^{f_2} = E_4$, and from the order of $N_G(E_1)$ we conclude that neither $E_1$ nor $E_4$ is conjugate to $E_2$ in $G$. Suppose that $E_3$ is conjugate to $E_2$. The involution $t$ operates on $E_3$ centralizing a hyperplane of $E_3$, but no involution of $Q_2 - E_2$ centralizes a hyperplane of $E_2$. Therefore $E_3t$ must be conjugate to $E_2a_1$.

In particular, $t$ is conjugate to an involution of $E_2 a_1$. Involutions of $E_2 a_1$ are contained in $\langle t, z_1, z_2\rangle a_1 = \langle t, z_1\rangle a_1 \cup \langle t, z_1\rangle a_1 z_2$. These last sets consisting of 4 elements are conjugate under $d_1$ to $\langle t, z_1\rangle z_2$. But lemma 4.2. says that $t$ is not conjugate to any element of $\langle t, z_1\rangle z_2$. This is a contradiction. So $E_3$ cannot be conjugate to $E_2$.

Using lemmas 4.5. and 4.6. we see that $T_1 = J(T_2)$, so every automorphism of $T_2$ operates on the set $\{E_1, E_2, E_3, E_4\}$, and, as we have shown above, fixes $E_2$: The lemma is proved.

THEOREM 4.9. The hypothesis of this paragraph cannot be satisfied. We have $|N_G(T_0): T_0| = 4$.

PROOF. It follows from lemmas 4.8 and 4.2. that $T_2$ is a Sylow-2-subgroup of $G$. Furthermore, we see that $N_G(E_2)$ controls the fusion of the involutions of $E_2$. We find that $z_1$ is fused to $z_2$, but to no other $H$-class of involutions. So $z_1$ has 60 conjugates in $H$. Let $t$ act on the conjugacy class of $z_1$ in $G$. From the assumption that $G$ has no subgroup of index 2, and from the fact that $2^6$ divides the order to $C_G(z_1)$, we get that $2^8$ divides the order of $G$, which is a contradiction. Theorem 4.9. is proved.

## 5. The case $E_1 \nsim E_2$ in $T_1$, first results.

LEMMA 5.1. $T_1 = N_G(T_0)$ has order $2^7 \cdot E_1$ and $E_2$ are not conjugate in $G$, $t$ has 8 conjugates in $E_i$ under the action of $N_G(E_i)$ and not more conjugates in $E_i$ under the action of $G$. The order of $N_G(E_i)$ is $2^8 \cdot 3$.

PROOF. The first assertion has been proved in § 4. As $E_1$ and $E_2$ are normal in $T_1$, it is immediate that $t$ has 8 conjugates in $E_i$ under the action of $N_G(E_i)$ and that further fusion is impossible. If $E_1$ and $E_2$ were conjugate in $G$, they would be conjugate in $C(t)$, which is not the case. The lemma is proved.

We fix some notation. $D_i := \langle d_i \rangle$, $N_i := N_G(E_i)$, $Q_i := O_2(N_i)$, $Q_{i0} := [Q_i, D_i] E_i$. By a 16-group we always mean an elementary abelian group of order 16. Unless stated otherwise, we use the « bar » convention for the canonical homomorphism $N_i \to A_i = N_i/E_i$ $(i = 1, 2)$.

LEMMA 5.2. $N_{A_i}(\overline{D_i})$ has order 12. There are involutions $y_1$ and $y_2$ such that $y_i$ centralizes $D_i$: Put $Z := Z(T_1)$. Then $Z = \langle z_1, z_2\rangle$. Furthermore, $[y_1, t] = z_1$ and $[y_2, t] = z_2$:

**Proof.** The operator group $A_i$ is a subgroup of $GL(4,2) \simeq A_8$ of order $2^4 \cdot 3$; therefore no group of order 3 is normalized by a group of order 16 in $A_i$: We know that $N_{A_i}(\overline{D_i})$ has order at least 6, so we conclude by Sylows therem that $|N_{A_i}(\overline{D_i})|$ must be 12.

We have shown that $|N_{N_i}(\overline{D_i})| = 2^4 \cdot 3$, and similarly as in lemma 4.3. we see that $C_{N_i}(D_i)$ is the direct product of $D_i$ and a dihedral group of order 8, say $R_i$: This shows that we may choose involutions $y_1$ and $y_2$, which are not contained in $H$, and involutions $z_1$ and $z_2$ from $Z(T_0)$ such that $R_i = \langle z_1, t, y_i \rangle$. Now it is immediate that $[y_i, t] = z_i$:

It is easy to see that $y_1$ and $y_2$ normalize $T_0$, so $T_1/T_0$ is elementary, and $Z(T_1)$ is a four-group. Using the definition of $y_i$ and the Thompson $A \times B$-lemma we see that $Z(T_1) = \langle z_1, z_2 \rangle$.

The lemma is proved.

Now we want to consider the possible structures of $N_i$. As the roles of $E_1$ and $E_2$ are interchangeable so far, we introduce some common notation.

For the permutation representation of $A_i$ on the orbit of $t$ in $E_i$ we use the following numbering as a common reference:

Setting $z := z_1 z_2$, we have the orbit $\{t, tz, ta_i, tza_i, tz_i, tz_iz, tz_ia_i, tzz_ia_i\}$ for $E_i$. Let these elements, listed in this order, correspond to $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$.

The 3-elements $d_1$ resp. $d_2$ operate as $(\mathbf{2}, \mathbf{3}, \mathbf{4})(\mathbf{6}, \mathbf{7}, \mathbf{8})$, the involutions $a_2$ resp. $a_1$ inverting $d_i$ have the action $(\mathbf{3}, \mathbf{4})(\mathbf{7}, \mathbf{8})$, and $y_i$ operates as $(\mathbf{1}, \mathbf{5})(\mathbf{2}, \mathbf{6})(\mathbf{3}, \mathbf{7})(\mathbf{4}, \mathbf{8})$.

Now it is of interest to investigate the action of $y_j$, $j \neq i$, on $E_i$.

**Lemma** 5.3. Let $c_i$ be the involution of the set $\{a_1, a_2\}$ inverting $d_i$. Then we have two possibilities:

I) $c_i$ centralizes $y_i$, $y_i$ operates on the orbit of $t$ in $E_j$, $j \neq i$, as $(\mathbf{1}, \mathbf{6})(\mathbf{2}, \mathbf{5})(\mathbf{3}, \mathbf{8})(\mathbf{4}, \mathbf{7})$.

II) $c_it$ centralizes $y_i$, $y_i$ operates on the orbit of $t$ in $E_j$ as $(\mathbf{1}, \mathbf{6})(\mathbf{2}, \mathbf{5})$.

**Proof.** Again look at $N_{N_i}(D_i)$. Similarly as in lemma 4.4., we see that $[y_i, c_i] = 1$ or $[y_i, c_i] = [y_i, t]$. Now it is straightforward to compute the action of $y_i$ on $E_j$.

**Lemma** 5.4. Assume that we have case I) for $E_i$. Then $Q_{i0}$ is generated by $E_i$, $y_1y_2$, and $(y_1y_2)^{d_i}$.

PROOF. It is immediate from our knowledge about $A_i$ that $O_2(A_i)$ is elementary of order 8 and contains $\overline{y}_i$. Now use the permutation representation of $A_i$ on the orbit of $t$. We compute $y_1y_2 \triangleq (\mathbf{1},\mathbf{2})(\mathbf{3},\mathbf{4})\cdot$ $\cdot(\mathbf{5},\mathbf{6})(\mathbf{7},\mathbf{8})$. It follows that $(y_1y_2)^{a_i} = (\mathbf{1},\mathbf{3})(\mathbf{2},\mathbf{4})(\mathbf{5},\mathbf{7})(\mathbf{6},\mathbf{8})$ and that $(y_1y_2)^{a_i^2} = (y_1y_2)(y_1y_2)^{a_i} \bmod E_i$. The lemma is proved.

LEMMA 5.5. Assume that we have case II) for $E_i$. Then $Q_{i0}$ is generated by $E_i$, $c_iy_j$ $(j \neq i)$, and $(c_iy_j)^{a_i}$.

PROOF. This time we see that the action of $c_iy_j$ on the orbit of $t$ is $(\mathbf{1},\mathbf{6})(\mathbf{7},\mathbf{8})(\mathbf{2},\mathbf{5})(\mathbf{3},\mathbf{4})$. We finish by calculating in the same way as in the proof of lemma 5.4.

LEMMA 5.6. Assume that we have case II) for $E_i$. Set $e_i := c_iy_j$ and $f_i := e_i^{a_i}$. Then $f_i^{a_i} = f_ie_ih_i$ with $h_i \in \langle z_i \rangle$. Furthermore, $[e_i, f_i] = 1$.

PROOF. We easily compute that $[E_i, e_i] = C_{E_i}(e_i) = \langle z_i, z \rangle$. This implies that $C_{E_i}(f_i) = \langle z_i, a_i \rangle$. The commutator $[e_i, f_i]$ is an involution in $E_i$ and therefore centralized by $e_i$ and $f_i$, so $[e_i, f_i] \in \langle z_i \rangle$.

We put $f_i^{a_i} =: f_ie_ih_i$ with $h_i \in E_i$. Now we see that $[e_i, f_i] =$ $= [e_i, f_i]^{a_i} = [f_i, f_ie_ih_i] = [f_i, e_i][f_i, h_i]$; so $h_i$ is centralized by $f_i$, and similarly, $h_i$ is centralized by $e_i$. As $f_ie_ih_i$ must be an involution, $f_ie_i$ is an involution and so $[f_i, e_i] = 1$. The proof is complete.

LEMMA 5.7. Assume that we have case II) for $E_i$. Then either $(A)$ $[y_i, e_i] = 1$ or $(B)$ $[y_i, e_i] = zz_i$.

PROOF. First of all, note that $[a_i, e_i] = zz_j = z_i$. It is clear that $[y_i, e_i] \in \langle z, z_i \rangle$.

Suppose that $[y_i, e_i] = z_i$. Then $[y_i, f_i] = z_i$ and $[y_i, f_ie_ih_i] =$ $= [y_i, f_i][y_i, e_i] = 1$. On the other hand, $[y_i, f_ie_ih_i] = [y_i, f_i]^{a_i} = z_i$ which is a contradiction.

So assume that $[y_i, e_i] = z$. Then $[y_i, f_i] = a_i$ and $[y_i, f_ie_ih_i] =$ $= [y_i, e_i][y_i, f_i][y_i, f_i, e_i] = za_iz_i$, which leads to the same contradiction as above. The lemma is proved.

For the next three paragraphs, put $R_i := T_1\langle f_i \rangle = T_1Q_i$.

Because of the symmetry of $E_1$ and $E_2$, we may split the analysis into the following cases:

I) $[a_1, y_2] = 1$, $[a_2, y_1] = z_1$,

II) $[a_1, y_2] = z_2$, $[a_2, y_1] = z_1$,

III) $[a_1, y_2] = [a_2, y_1] = 1$.

## 6. The non-isomorphic case.

We will deal with these cases in §§ 6-8.

HYPOTHESIS 6.0. $[y_1, a_2] = z_1$ and $[y_2, a_1] = 1$.

LEMMA 6.1. Put $B_1 := [Q_1, D_1]$ and $Z_1 := \langle z_1, z, a_1\rangle$. Then $B_1$ is a non-abelian group of order 32; further-more, $Z_1 = Z(B_1)$.

PROOF. First of all, we remark that $Z_1$ is the subgroup of $E_1$ generated by the involutions of $E_1$ which are not conjugate to $t$. Therefore $Z_1$ is normal in $N_1$. From the definition of $y_i$ and from hypothesis 6.0. we conclude that $Z_1 \leqslant Z(Q_1)$.

We try to compute $[Q_1, D_1] =: B_1$. If we look at $Q_1/Z_1$, we see that $B_1$ is contained in a group of order 32 of the form $Z_1\langle e_1, f_1\rangle =: B_{10}$, such that $d_1$ operates non-trivially on $B_{10}/Z_1$. Lemma 5.4. says that modulo $Z_1$, $e_1$ may be chosen to be either $y_1 y_2$ or $t\cdot y_2 y_1$. In either case, $[a_2, e_1] = z_1$, from hypothesis 6.0. This shows that $z_1$ must be contained in $B_1$, as $a_2$ operates on $[Q_1, D_1]$. So $B_1$ must be non-abelian and of order 32, and we have $B_1 = B_{10}$.

LEMMA 6.2. Put $B_1 = Z_1\langle e_1, f_1\rangle$ such that $e_1 \in \{t y_1 y_2, y_1 y_2\}$. Then we have $e_1 = t y_1 y_2$, $[y_1, y_2] = z_1$, $e_1^2 = z_1 z$, $[e_1, f_1] = z_1$, $[e_1, y_1] = 1$, and $f_1^{d_1} = f_1 e_1 z a_1$.

PROOF. Choose $e_1$ from the set $\{y_1 y_2, t y_1 y_2\}$ to be contained in $[Q_1, D_1]$. Then from lemma 6.1., $B_1$ is non-abelian and $B_1'$ must be a $D_1$-invariant group of order 2. Therefore $B_1' = \langle z_1\rangle$ and $[e_1, f_1] = z_1$.

From lemma 5.7. we know that $[y_2, e_2] = [y_2, y_1]$ is contained in $\langle z_1\rangle$ from our definitions, we conclude that $[y_1, e_1] \in \langle z_1\rangle$. The same argument as in the proof of lemma 5.7. shows that we must have $[y_1, e_1] = 1$.

As, by lemma 6.1., $B_1$ is non-abelian, $e_1$ cannot be an involution, which implies that $e_1 = t y_1 y_2$ and that $[y_1, y_2] = z_1$. Also, it is easy to verify that $e_1^2 = z z_1$.

Now $(f_1)^{d_1} = f_1 e_1 e$ for some $e \in Z_1$: We compute $(f_1 e_1 e)^{d_1} = f_1 e_1 e \cdot f_1 \cdot \cdot e^{d_1} = e_1$, hence $(f_1 e_1)^2 [e, d_1] = e_1^2$ and so $[e, d_1] = a_1$. This shows that $e \in z a_1 \langle z_1\rangle$. Interchanging $y_1$ and $y_1 z_1$ if necessary, we may assume that $e = z a_1$. The lemma is proved.

LEMMA 6.3. Put $Z := Z(T_1) = \langle z_1, z_2 \rangle$. The group $T_1$ contains precisely 9 16-groups, namely

$$E_1 = Z\langle a_1, t\rangle, \qquad E_2 = Z\langle a_2, t\rangle, \qquad E_3 = Z\langle a_1, y_2\rangle,$$

$$E_4 = Z\langle a_2, y_2\rangle, \qquad E_5 = Z\langle y_1, ta_2\rangle, \qquad E_6 = Z\langle y_1 y_2 a_1 a_2 t, a_1 y_2\rangle,$$

$$E_n = Z\langle y_1, y_2 a_2\rangle, \qquad E_8 = Z\langle y_1 y_2 a_1 a_2 t, a_1 t\rangle, \qquad E_9 = Z\langle a_1, y_1\rangle .$$

PROOF. First, note that $\langle z_1, y_1\rangle$ is normal in $T_1$. Put $\hat{T}_1 := := T_1/\langle y_1, z_1\rangle$. Then we see from earlier results that $\hat{T}_1$ is extra-special of order 32 with center $\langle \hat{z}_2\rangle$. We can write $\hat{T}_1$ as the central product of two dihedral groups. $\hat{T}_1 = \langle \hat{a}_1, \hat{a}_2\rangle \curlyvee (\hat{t}, \hat{y}_2\rangle$.

Such a group has precisely 6 maximal elementary subgroups all of which have order 8. Take $U_i$, $1 \leqslant i \leqslant 6$, to be their inverse images. Then we have

$$\begin{aligned}
U_1 &= Z\langle y_1, a_1, t\rangle &&= E_1 E_9 ,\\
U_2 &= Z\langle y_1, a_1, y_2\rangle &&= E_3 E_9 ,\\
U_3 &= Z\langle y_1, a_2, t\rangle &&= E_2 E_5 ,\\
U_4 &= Z\langle y_1, a_2, y_2\rangle &&= E_4 E_7 ,\\
U_5 &= Z\langle y_1, a_1 t, a_2 y_2\rangle &&= E_7 E_8 ,\\
U_6 &= Z\langle y_1, a_1 y_2, ta_2\rangle &&= E_5 E_6 .
\end{aligned}$$

It is clear that all maximal elementary subgroups of $T_1$ are contained in some $U_i$. So we have determined all elementary subgroups of order 16, as one can easily verify that $E_i$, $1 < i \leqslant 9$, are elementary abelian. The lemma is proved.

LEMMA 6.4. $N_G(T_1) = T_1\langle f_1, f_2\rangle =: T$ is a group of order $2^{10}$, the factor group $T/T_1$ is dihedral of order 8. There are precisely 3 $G$-classes of 16-groups in $T_1$, namely $\{E_1, E_3, E_5, E_7\}$ $\{E_2, E_4, E_6, E_8\}$, $\{E_9\}$.

PROOF. From lemmas 5.6. and 6.2. we know the action of $f_1$ and $f_2$ on $T_1$. Regard the operation on the set of 16-groups. By easy computations, we find

$$f_1 \triangleq (E_5, E_7)(E_2, E_8)(E_4, E_6), \qquad f_2 \triangleq (E_1, E_5)(E_3, E_7)(E_6, E_8) .$$

Multiplying this action, we get $f_1 f_2 \triangleq (E_1, E_5, E_3, E_7)(E_2, E_6, E_4, E_8)$, and we see that there are precisely 3 $G$-classes of 16-groups with elements as stated. The elements $f_1$ and $f_2$ generate an outer automorphism group on $T_1$ which is dihedral of order 8. To finish, we use the fact that $N(T_1) \cap N(E_1)$ has order $2^8$ and that $E_1$ has precisely 4 conjugates under $N_G(T_1)$ in $T_1$. The lemma is proved.

LEMMA 6.5. The group $Q_1$ contains precisely 5 16-groups namely $E_9$, $E_1$, $E_3$, and the groups $E_{31} = Z_1\langle f_1 y_1 t\rangle$ and $E_{32} = Z_1\langle f_1 y_2\rangle$.

PROOF. We have $Z(Q_1) = Z_1$, and we know the multiplication table of $Q_1$. So we just check which elements of $Q_1/Z_1$ belong to cosets of involutions, and we see that the assertion of the lemma holds.

LEMMA 6.6. $R_1 = T_1\langle f_1\rangle$ is a Sylow-2-subgroup of $N_1$: $T_1$ and $Q_1$ are characteristic in $R_1$. $N_G(R_1) = R_1\langle (f_1 f_2)^2\rangle$ is a group of order $2^9$. Put $S_1 = N_G(R_1)$. Then $S_1 = V(\mathrm{ccl}_G(t); T)$. $N_G(Q_1)$ has order $2^{10}\cdot 3$. Furthermore, $2^{11}$ divides the order of $G$.

PROOF. First of all, regard the 16-subgroups of $R_1$. As $R_1/E_1\langle y_1\rangle$ is dihedral of order 8, we only have to look at $T_1$ and $Q_1$, and so we have determined all 16-subgroups in the lemmas 6.3. and 6.5.

The groups $E_9$, $E_1$, and $E_4$ are normal in $R_1$, the other 6 groups contained in $T_1$ are normalized by $T_1$ but not by $f_1$, and $E_{31}$, $E_{32}$ have normalizer $Q_1$ in $R_1$. We know that $Q_1 = J(Q_1)$ and that $T_1 = J(T_1)$.

This shows that $T_1$ and $Q_1$ are characteristic in $R_1$. Furthermore, $E_1$ can only have 2 conjugates in $N_G(R_1)$, as $t$ cannot be conjugate to any involution of $E_9$; so $N_G(R_1)$ is as described.

The group $S_1$ is generated by $R_1$ and $R_1^{f_2}$, and $R_1 = V(\mathrm{ccl}_G(t); R_1)$. Having the structure of $T/T_1$ in mind, we only have to prove that there are no conjugates of $t$ in $R_2 - T_1$:

As usual, we only have to determine the involutions of $Q_2 - Q_2 \cap T_1$: The factor group $Q_2/Z$ involves a direct factor which is dihedral, so we may reduce to $Q_{21} = \langle z_2, z, a_2, e_2, f_2, ty_2\rangle$. This group has the normal subgroup $\langle z, z_2, a_2, ty_2\rangle$ and $d_2$ permutes the non-trivial cosets of this subgroup.

It suffices to consider one coset which we can choose to be contained in $T_1$, but from lemma 6.3. we conclude that $t$ has no conjugates in the group $\langle z_2, z, a_2, ty_2, a_1 y_1\rangle$. So we have shown that $T_1 = V(\mathrm{ccl}_G(t); R_2)$, which implies that $S_1 = V(\mathrm{ccl}_G(t); T)$.

As to the next assertion, we note that $Q_1$ contains precisely 4 16-subgroups conjugate to $E_1$, and they are conjugate in $N_G(Q_1)$. So $N_G(Q_1)$ must have order $2^{10}\cdot 3$.

Regard the center of a Sylow-2-subgroup of $N_G(Q_1)$ containing $S_1$. From the action on the 16-subgroups of $Q_1$ we see that $(f_1f_2)^2 =: g$ is contained in $O_2(N_G(Q_1) \bmod Q_1)$. As $g$ and $f_1$ centralize $Z$, we see that a Sylow-2-subgroup of $N_G(Q_1)$ has a center of order 4. But $Z(T) = \langle z_2\rangle$ is of order 2. Therefore $2^{11}$ must divide the order of $G$. Lemma 6.6. is proved.

LEMMA 6.7. Put $f := f_1f_2$, $g := f^2$ and $U := T_1\langle g\rangle = Z(T \bmod T_1)$. Then $t$ is not conjugate to any involution of $U - T_1$ in $G$.

PROOF. To start, we determine $C_{T_1}(f)$. We compute $z_1^f = z$, $a_1^f = y_1 h_2$ with $h_2 \in \langle z_2\rangle$, $a_2^f \in a_1a_2y_1y_2tZ$, $t^f = y_1a_2tz_2h$, $y_2^f = a_2y_1y_2h$, $(ty_2)_f = zty_2$. From this, it follows that $C_{T_1}(f) = \langle a_1y_1, z_2\rangle$.

As $g^2 = f^4$ we must have $g^2 \in \langle a_1y_1, z_2\rangle$. The element $g$ centralizes $Z$ and normalizes the intersections $E_1 \cap E_3$, $E_5 \cap E_7$, $E_2 \cap E_4$, $E_6 \cap E_8$, hence $g^2$ centralizes these intersections, in particular, $g^2$ centralizes $a_2$. We have proved that $g^2 \in \langle z_2\rangle$.

Put $T_{10} := E_9\langle a_2, ty_2\rangle$. Then we see that $g$ normalizes $T_{10}$, and that $g$ centralizes $T_{10} \bmod Z$. On the other hand, we can compute that $t^g \in yZ$. If $gx$ is an involution, for some $x \in T_1$, then $g^2x^2[g, x] = 1$ and therefore $x \in T_{10}$.

We note that $[g, ty_2] = (ty_2)^2 = z_2$, this implies that if $gx$ is an involution then $gxty_2$ is an involution as well. (It is straightforward to see that $ty_2$ is contained in the center of $T_{10}$). So choose $x \in E_9\langle a_2\rangle$. We have $[g, a_2] = z$.

Assume $g^2 = 1$. If $x^2 = [g, x] = 1$, then $x \in E_9$. But $g$ centralizes $E_9$ and so $E_9\langle g\rangle$ is elementary of rank 5. Hence, if involutions of this type occur, they cannot be conjugate to $t$. If $g^2 = z_2$, then $E_9g$ does not contain any involutions.

The coset $T_{10}g$ contains 64 elements, but we have already excluded 32 elements. Trivial computations show that not all of the remaining 32 elements can be involutions, so there are less than 32 conjugates of $t$ in $U - T_1$. But $T$ normalizes $U$ and $T$ has order $2^{10}$. On the other hand, $C_G(t)$ has Sylow-2-subgroups of order $2^5$. Hence $U$ contains $0 \pmod{32}$ involutions which are $G$-conjugate to $t$. Altogether, this means that there cannot be any conjugates of $t$ in $U - T_1$. Lemma 6.7. is proved.

Lemma 6.8. Take $P_1$ to be the Sylow-2-subgroup of $N_G(Q_1)$ containing $S_1$. Then $g_2 := g^{d_1}$ is contained in $P_1$, and we have $P_1 = Q_1\langle g, g_2, f_1\rangle$. $S = P_1\langle f_2\rangle = N_G(S_1)$, and $S_1 = V(\mathrm{ccl}_G(t); P_1)$.

Proof. Regard the action of $N_G(Q_1)$ on the set $\{E_1, E_3, E_{31}, E_{32}\}$. We know that $N(Q_1)$ induces the full symmetric group on 4 letters. $R_1$ operates as $(E_{31}, E_{32})$ and corresponds to a transposition.

Suppose that $g \triangleq (E_1, E_3)$. Then $R_1 = Q_1\langle a_2\rangle$ is isomorphic to $Q_1\langle g\rangle$, but $Z(R_1) = Z$ and $Z(Q_1\langle g\rangle) = Z_1$, as $g$ centralizes $a_1$, a contradiction. So we must have $g \triangleq (E_1, E_3)(E_{31}, E_{32})$, and $g \in O_2(N(Q_1) \bmod Q_1) = O_2(N(Q_1))$. The 3-element $d_1$ normalizes $Q_1$, so $g^{d_1} \in O_2(N_G(Q_1))$.

The group $S_1$ contains precisely 64 involutions, which are $G$-conjugates of $t$, and $2^{11}$ divides $|N(S_1)|$, so $S = P_1\langle f_2\rangle = N_G(S_1)$.

The factor group $P_1/Q_1$ is dihedral, so, for determining the elementary subgroups of $P_1$, it suffices to determine those of $S_1 = Q_1\langle g, f_1\rangle$ and of $Q_1\langle g, g_2\rangle = O_2(N(Q_1))$. As we have the action of $d_1$, it is enough to consider the group $Q_1\langle g\rangle$.

This group is normalized by $P_1$ which is of order $2^{10}$, so $Q_1\langle g\rangle$ contains $O$ (mod 32) involutions which are conjugate to $t$. On the other hand, $Q_1\langle g\rangle \leqslant S_1$ and $S_1$ contains 64 $t$-conjugates. The involution $a_2 t$ is conjugate to $t$ and lies in $S_1$ but not in $Q_1\langle g\rangle$, so $Q_1\langle g\rangle$ contains precisely 32 conjugates of $t$. But these must already be contained in $Q_1$. This shows that $Q_1 = V(\mathrm{ccl}_G(t); O_2(N(Q_1)))$, ans do it follows that $S_1 = V(\mathrm{ccl}_G(t); P_1)$. Lemma 6.8. is proved.

Lemma 6.9. The elementary group $E_9$ is normal in $S$. Put $C_9 := C_S(E_9) = E_9\langle ty_2, f_1, g, g_2\rangle$. Then $\hat{S} := S/C_9 = \langle t, \hat{a}_2, f_2\rangle$ is dihedral of order 8. The inverse images of the elementary maximal subgroups of $\hat{S}$ are $P_1 = C_9\langle t, a_2\rangle$ and $P_2 = C_9\langle f_2, a_2\rangle$.

Proof. Trivial.

Lemma 6.10. Put $P_3 := C_9\langle a_2\rangle$. Then $P_3$ does not contain any involutions which are conjugate to $t$.

Proof. Obviously, $P_3$ is contained in $P_1$, and we know the conjugates of $t$ in $P_1$. It is obvious from our earlier results that so involution which is conjugate to $t$ and appears in $R_1$, is contained in $P_3$. But $P_3$ is $f_2$-invariant. The lemma is proved.

Lemma 6.11. Put $P_4 := C_9\langle f_2\rangle$. Then $P_4$ does not contain any involutions which are conjugate to $t$.

PROOF. In proofs of some earlier lemmas, we have seen that $C_9$

does not contain any conjugates of $t$. So suppose $xf_2$ to be an involution, $x \in C_9$. Then $x^2 = [x, f_2]$, and $f_2$ centralizes $x$ modulo $E_9$.

Now $(f_1 f_2)^2 = g = f_1^2[f_1, f_2]$, hence $[f_1, f_2] \in gE_9$. $C_{90} := C_{C_9 \bmod E_9}(f_2)$ is a group of order at most $2^7$, and $x$ must be chosen from $C_{90}$. It is immediate that $xf_2$, $z_1 x f_2$, $y_1 x f_2$ and $z y_1 x f_2$ have all different squares, so there are at most $2^5$ involutions in $P_4 - C_9$.

On the other hand, $P_4$ is normalized by a group of order $2^{10}$ and therefore contains $O$ (mod 32) conjugates of $t$. But in the proof of lemma 6.6. we have seen that $f_2$ is not conjugate to $t$. So $P_4$ cannot contain any conjugates of $t$. The lemma is proved.

LEMMA 6.12. $P_2$ does not contain any conjugates of $t$.

PROOF. This is clear from the preceeding lemmas, as $P_2$ is the union of $P_3$, $P_4$, and $P_4^t$.

LEMMA 6.13. $S_1 = V(\mathrm{ccl}_G(t); S)$.

PROOF. This follows from lemmas 6.6, 6.8. and 6.12.

THEOREM 6.14. Hypothesis 6.0. cannot be satisfied.

PROOF. From lemmas 6.8. and 6.13. we conclude that $S$ is a Sylow-2-subgroup of $G$. Lemma 6.12. says that $P_2$, which is a maximal subgroup of $S$, does not contain any conjugates of $t$. By the Thompson transfer lemma, it follows that $G$ has a subgroup of index 2, which is a contradiction.

## 7. $E_1 \nsim E_2$, the « case II » case.

HYPOTHESIS 7.0. $[y_2, a_1] = z_2$, $[y_1, a_2] = z_1$.
We use the notation introduced in lemma 5.6. for $i = 1$ and $i = 2$.

LEMMA 7.1. We have $[e_i, t] = zz_i$, $[e_i, a_i] = z_i$, $[e_i, y_i] = zz_i$, $[y_1, y_2] = z$, for $i = 1, 2$.

PROOF. The first two relations are immediate from lemma 5.6. and the definition of $e_i$.

To prove the other two relations, use lemma 5.7. and assume that $[y_1, e_1] = 1$, which implies that $[y_1, y_2] = z_1$. But then $[y_2, e_2] = [y_2, y_1 a_1] = z_1 z_2$ which contradicts lemma 5.7. for $i = 2$. So we must have $[y_1, y_2] = z$. The lemma is proved.

LEMMA 7.2. Put again $T_1 := T_0\langle y_1, y_2\rangle = N_G(T_0)$. Then $T_1$ contains precisely 8 16-groups, which are:

$$\begin{aligned}
E_1 &= Z\langle a_1, t\rangle\,, & E_2 &= Z\langle a_2, t\rangle\,,\\
E_3 &= Z\langle a_1, y_1\rangle\,, & E_4 &= Z\langle a_2, y_2\rangle\,,\\
E_5 &= Z\langle t y_1 a_2, y_1\rangle\,, & E_6 &= Z\langle t y_2 a_1, y_2\rangle\,,\\
E_n &= Z\langle a_1 y_1, t y_1 y_2\rangle\,, & E_8 &= Z\langle a_2 y_2, t y_1 y_2\rangle\,.
\end{aligned}$$

PROOF. First, check that the groups listed above, are elementary. But this follows from lemma 7.1.

Now we determine the maximal elementary subgroups of $T_1$ with the aid of a suitable factor group. The factor group $\hat{T}_1 = T_1/\langle z, a_1 a_2 t\rangle = \langle \hat{y}_1, \widehat{a_1 t}\rangle\langle \hat{y}_2, \widehat{a_2 t}\rangle$ is the central product of two dihedral groups of order 8, so $T_1$ has precisely 6 maximal elementary subgroups which are all of order 8. Take $U_i$, $1 \leqslant i \leqslant 6$, to be their inverse images in $T_1$: Then we have

$$\begin{aligned}
U_1 &= Z\langle a_1 a_2 t, y_1, y_2\rangle\,, &&\\
U_2 &= Z\langle a_1, a_2 t, y_1\rangle &&= E_3 E_5\,,\\
U_3 &= Z\langle a_2, a_1 t, y_2\rangle &&= E_4 E_6\,,\\
U_4 &= Z\langle a_1, a_2, t\rangle &&= E_1 E_2\,,\\
U_5 &= Z\langle a_1 a_2 t, y_1 y_2, a_1 a_2\rangle\,, &&\\
U_6 &= Z\langle a_1 y_1, a_2 y_2, y_1 y_2 t\rangle &&= E_7 E_8\,.
\end{aligned}$$

It is easy to check that $U_1$ and $U_5$ do not contain elementary groups of order 16. The lemma is proved.

LEMMA 7.3. Neither $E_3$ nor $E_4$ is conjugate to $E_1$ or $E_2$ in $G$.

PROOF. We show that $E_3$ and $E_4$ contain more than 7 involutions which are not conjugate to $t$.

As to $E_3$, we have $C(e_2) = C(a_1y_1) \geqslant Z\langle a_1, y_1, f_2, ty_2\rangle$, so $e_2$ is centralized by a group of order $2^6$ and cannot be conjugate to $t$. As $a_1$ is not conjugate to $t$ either, we are done for $E_3$. Take the involution $e_1$ for $E_4$, and use the same argument.

LEMMA 7.4. Let $EE$ be the set $\{E_1, E_5, E_7, E_2, E_6, E_8\}$ Then we have the $G$-orbits $\{E_1, E_5, E_7\}$ and $\{E_2, E_6, E_8\}$. Furthermore, $N_G(T_1)$ has order $2^8 \cdot 3$.

PROOF. Regard the action of $f_1$ and $f_2$ on $EE$. We find

$$f_1 \triangleq (E_5, E_7)(E_2, E_6) \qquad \text{and} \qquad f_2 \triangleq (E_1, E_5)(E_6, E_8)\,.$$

Having the non-fusion of $E_1$ and $E_2$ in mind, we see that the orbits are as described.

We have seen that $E_1$ has precisely 3 conjugates under the action of $N_G(T_1)$. As $N_G(T_1) \cap N_G(E_1) = T_1\langle f_1\rangle$ has order $2^8$, we have determined the order of $N_G(T_1)$ and finished the proof of lemma 7.4.

LEMMA 7.5. $J(Q_1) = E_1E_3$, $J(Q_2) = E_2E_4$.

PROOF. Regard $\hat{Q}_1 := Q_1/Z\langle e_1, ty_1\rangle = \langle \hat{a}_1, \hat{t}_1\hat{f}_1\rangle$, which is dihedral of order 8. As usual, take the inverse images of the maximal elementary subgroups. We get

$$Q_{11} = Z\langle e_1, ty_1, a_1, t\rangle \qquad \text{and} \qquad Q_{12} = Z\langle e_1, ty_1, a_1, f_1\rangle\,.$$

The group $Q_{11}$ is contained in $T_1$, and with the aid of lemma 7.2. it follows that $E_1E_3 = J(Q_{11})$.

Turn to $Q_{12}$. We can write $Q_{12} = \langle z_2, f_1\rangle \curlyvee \langle a_1, e_1\rangle \curlyvee \langle ty_1\rangle$, so $Q_{12}$ is the central product of two dihedral groups of order 8 and a cyclic group of order 4. It is straightforward that such a group has 2-rank 3. This proves the lemma.

LEMMA 7.6. $R_1$ is a Sylow-2-subgroup of $G$.

PROOF. Lemma 7.5. implies that $T_1 = J(R_1)$. On the other hand, we know the order of $N_G(T_1)$, and $R_1$ must be a Sylow-2-subgroup of $N_G(T_1)$.

THEOREM 7.7. Hypothesis 7.0. cannot be satisfied.

PROOF. By lemma 7.6., $R_1$ is conjugate to $R_2$. Regard the normal elementary subgroups of $R_i$ of order 16. We find that $\{E_1, E_3\}$ is conjugate to $\{E_2, E_4\}$. But this cannot happen. Theorem 7.7. is proved.

## 8. $E_1 \nsim E_2$, final.

HYPOTHESIS 8.0. $[y_1, a_2] = [y_2, a_1] = 1$.

LEMMA 8.1. Put $B_i := [Q_i, D_i]$. Then $B_1$ and $B_2$ are homocyclic abelian groups of order 16 and of the same type. We may write $B_i = \langle z, a_i, e_i, f_i \rangle$ such that $e_i \in y_1 y_2 \langle z_i \rangle$ and $f_i = e_i^{d_i}$. There are two possibilities:

I) $B_1$ and $B_2$ have exponent 4, $[y_1, y_2] = z$,

II) $B_1$ and $B_2$ are elementary, $[y_1, y_2] = 1$.

PROOF. First of all, we have $B_i \leqslant Q_{i0}$ for $i = 1, 2$, It is clear that $Z(Q_{i0}) = \langle z_1, z_2, a_i \rangle$. Put

$$B_{i0} := [Q_{i0}, D_i] Z(Q_{i0}) = Z(Q_{i0}) \langle e_i, f_i \rangle$$

such that $e_i \in \{y_1 y_2, t y_1 y_2\}$ and $f_i = e_i^{d_i}$.

The involution $a_j$, $j \neq i$, inverts $d_i$ and centralizes $e_i$ for either choice of $e_i$. Put $f_1^{d_1} =: f_i e_i q_i$, $q_i \in Z(Q_{i0})$. Then $(f_i e_i q_i)^{a_j} = f_i = f_i e_i q_i \cdot e_i \cdot q_1^{a_j}$, hence $e_1^2 = [q_i, a_j] \in \langle z \rangle$. Looking at the square of $f_i e_i q_i$ we see that we have $[e_i, f_i] = 1$ and that $B_{i0}$ is abelian.

Interchange $e_i$ and $e_i z_i$, if necessary, to see that we can write $B_i$ as asserted.

As $(t y_1 y_2)^2 = z (y_1 y_2)^2$ we see that $[y_1, y_2] \in \langle z \rangle$. Suppose that $e_1 \in y_1 y_2 t \langle z_1 \rangle$. Then $[y_1, e_1] \in \{z_1 z, z_1\}$ which is not compatible with the operation of $d_1$ on $Q_1$. So $e_1 \in y_1 y_2 \langle z_1 \rangle$. The same argument holds for $e_2$, and our lemma is proved.

LEMMAS 8.3.-8.5. will be proved under

HYPOTHESIS 8.2. $B_1$ and $B_2$ have exponent 4.

LEMMA 8.3. Let $T_1$ and $Z$ be as usual. Then $T_1$ contains 12 16-groups, namely (if $u = a_1a_2y_1y_2$)

$$\begin{aligned} E_1 &= Z\langle a_1, t\rangle, & F_1 &= Z\langle a_1, y_1\rangle, \\ E_2 &= Z\langle a_2, t\rangle, & F_2 &= Z\langle a_2, y_1\rangle, \\ E_3 &= Z\langle u, a_2t\rangle, & F_3 &= Z\langle u, a_2y_1\rangle, \\ E_4 &= Z\langle a_1, y_1y_2t\rangle, & F_4 &= Z\langle a_1, y_2\rangle, \\ E_5 &= Z\langle a_2, y_1y_2t\rangle, & F_5 &= Z\langle a_2, y_2\rangle, \\ E_6 &= Z\langle u, a_1t\rangle, & F_6 &= Z\langle u, a_1y_1\rangle. \end{aligned}$$

PROOF. Again we use our « factor group method ». The group $\langle z, a_1, a_2, y_1y_2\rangle$ is normal in $T_1$ and the factor group is dihedral of order 8. We get $T_{11} = Z\langle a_1, a_2, y_1y_2, t\rangle = \langle a_1, a_2\rangle \curlyvee (y_1y_2t, t)\langle z_1\rangle$ and $T_{12} = Z\langle a_1, a_2, y_1y_2, y_1\rangle = \langle a_1, a_2\rangle \curlyvee \langle y_1, y_2\rangle \times \langle z_1\rangle$. Both maximal subgroups are the direct product of an extraspecial group of type $D_8 \curlyvee D_8$ and a group of order 2, the list of elementary subgroups now is immediate.

LEMMA 8.4. Put $T_K := Z\langle a_1, a_2, y_1y_2\rangle$. For any 16-subgroup of $T_1$, $E$ say, put $K(E) := E \cap T_K$. Then $T_K$ is characteristic in $T_1$. Furthermore, $\langle z\rangle$ is characteristic in $T_1$.

PROOF. Regard the intersections of the 16-subgroups of $T_1$, which are or order 8. The only ones occuring more then once are $Z\langle a_1\rangle$, $Z\langle a_2\rangle$, and $Z\langle u\rangle$. These three groups of order 8 generate $T_K$, hence $T_K$ is characteristic in $T_1$, and so is $\langle z\rangle = T'_K$.

LEMMA 8.5. $2^9$ divides the order of $G$.

PROOF. Suppose that $R_1$ is a Sylow-2-subgroup of $G$. We choose a maximal subgroup $R_{11} = T_K\langle f_1, y_1t\rangle$. We see that $Z(R_{11}) \geqslant Z(T_K) = Z\langle y_1y_2\rangle$. Regarding cosets of involutions of $R_{11}/Z(T_K)$, we easily see that $T_K = \Omega_1(R_{11})$. As $t$ is not conjugate to any element of $T_K$, $t$ is not conjugate into $R_{11}$, hence $G$ has a subgroup of index 2, which cannot be the case. The lemma is proved.

THEOREM 8.6. Hypothesis 8.2. cannot be satisfied.

PROOF. $\{E_1, E_4, F_1, F_4\}$ is the set of normal 16-groups in $R_1$; $E_1$ and $F_1$ are normal in $N_1$. On the other hand, $R_2$ has the normal 16-subgroups $E_2$, $E_5$, $F_2$, $F_5$, $E_2$ and $F_5$ are normal in $N_2$.

The action of $T_1\langle f_1, f_2\rangle$ on the set of 16-subgroups of $T_1$ causes the orbits $\{E_1, E_3, E_5\}$, $\{E_2, E_4, E_6\}$, $\{F_1, F_3, F_5\}$ and $\{F_2, F_4, F_6\}$.

As $N_G(R_1) > R_1$, $E_1$ is conjugate to $F_1$ or to $F_4$. Suppose that $E_1$ is conjugate to $F_1$. Then $E_1$ is conjugate to $F_5$, hence $N_1$ is conjugate to $N_2$, and as $E_1$ and $F_1$ are conjugate, $E_2$ and $F_5$ must be conjugate in $G$. But this is a contradiction.

Therefore $E_1$ is conjugate to $F_4$ and $F_1$ is conjugate to $E_4$. Again we see that $N_1$ and $N_2$ are conjugate. We have $E_1 \nsim F_1$, but $E_2 \sim F_5$. This again is a contradiction. The theorem is proved.

We have proved that we are in case II) of lemma 8.1., so $B_1$ and $B_2$ are elementary abelian, and, in particular, $[y_1, y_2] = 1$.

LEMMA 8.7. $T_1$ possesses 4 elementary subgroups of order 16 which contain conjugates of $t$. They are $E_1 = Z\langle a_1, t\rangle$, $E_2 = Z\langle a_2, t\rangle$, $E_3 = Z\langle a_1 y_1 y_2, a_2 t\rangle$, $E_4 = Z\langle a_2 y_1 y_2, a_1 t\rangle$. $N_G(T_1) = T_1\langle f_1, f_2\rangle =: T$ is a group of order $2^9$.

PROOF. The factor group $T_1/\langle z, a_1, a_2, y_1 y_2\rangle$ is dihedral. We get $T_{11} = Z\langle a_1, a_2, y_1, y_2\rangle$ and $T_{12} = Z\langle a_1, a_2, y_1 y_2, t\rangle$. $T_{11}$ has a center of order 16 and does not contain any conjugates of $t$. $T_2$ can be written in the form $D_8 \curlyvee D_8 \times Z_2$ and contains precisely 6 elementary 16-groups. Two of them are contained in $T_{11}$, and the other ones are listed above.

It is easy to see that we have $f_1 \triangleq (E_2, E_4)$ and $f_2 \triangleq (E_1, E_3)$. As $f_1$ and $f_2$ both normalize $T_1$, we see in the usual way that the normalizer of $T_1$ in $G$ must be as described.

LEMMA 8.8. $T$ is a Sylow-2-subgroup of $G$.

PROOF. We will show that $T_1 = V(\mathrm{ccl}_G(t); T)$, then our assertion will follow immediately.

First of all, we show that $T_1 = V(\mathrm{ccl}_G(t); R_1)$. To this end, we show that conjugates of $t$ which are contained in $Q_1$, also are contained in $T_1$. In fact, $Q_1$ contains an elementary group of order 64, $Q_{11} = \langle z, z_1, a_1, y_1, y_2, f_1\rangle$, and the only involutions in $Q_1 - Q_{11}$ are the conjugates of $t$ in $E_1$ and contained in $T_1$.

In the same way, we see that $T_1 = V(\mathrm{ccl}_G(t); R_2)$. Put $R_3 := T_1\langle f\rangle$ where $f := f_1 f_2$. To finish, we have to show that $T_1 = V(\mathrm{ccl}_G(t); R_3)$.

We get from easy calculations that $C_{T_1}(f) = Z\langle y_1, y_2\rangle$ so $f^2 \in Z\langle y_1, y_2\rangle$. If $fx$ is an involution, $x \in T_1$, then $f^2 x^2 = [f, x] \in Z\langle y_1, y_2\rangle$.

Put $T_{11} = Z\langle y_1, y_2, a_1, a_2\rangle$ as above. Then $x$ must be in $T_{11}$. But $Z(T_{11}) = Z\langle y_1, y_2\rangle$, hence the centralizer of $fx$ has 2-rank at least 5. Therefore $fx$ cannot be conjugate to $t$. The lemma is proved.

THEOREM 8.9. $E_1$ and $E_2$ are conjugate in $N(T_0)$.

PROOF. Take $M = Z\langle y_1, y_2, a_1, a_2, f_1, f_2\rangle$, which is a maximal subgroup of $T$. $Z(M) = Z\langle y_1, y_2\rangle$ is a 16-group, so $t$ cannot be conjugate into $M$, and $G$ has a subgroup of index 2, a contradiction. Hence hypothesis 8.0. cannot be satisfied, and we have proved that $E_1$ and $E_2$ are conjugate in $G$.

## 9. The case of conjugation.

In this section we finish the proof of theorem A. We fix some notation. As before, $T_1 = N_G T_0)$ is a group of order $2^7$. Put $T_2 = T_1 \cap \cap N_G(E_1)$ and $Z = Z(T_2)$. We keep the notation $Z = \langle z_1, z_2\rangle$ such that

$$\langle z_i\rangle = Z \cap Z(N_H(E_i)) \qquad \text{and} \qquad z = z_1 z_2 = [a_1, a_2]\,.$$

Furthermore, $N_i := N_G(E_i)$, $Q_i := O_2(N_i)$, $Z_i := Z(Q_i)$, $i = 1, 2$.

LEMMA 9.1. Put $T_2 = T_0\langle y\rangle$. Then we have $t^y = zt$. Furthermore, $|N_i| = 2^7 \cdot 3$.

PROOF. Suppose that $t$ has 8 conjugates in $N_G(E_i)$, then $E_1$ and $E_2$ are conjugate in $C(t)$, which is not the case. So the conjugates of $t$ in $E_i$ split into 2 orbits with 4 elements each under the action of $N(E_i)$. From the structure of $N_H(E_i)$ we easily conclude that we must have $t^y = zt$. As $t$ has 4 conjugates under the action of $N_G(E_i)$, we must have $|N_G(E_i)| = 2^7 \cdot 3$.

LEMMA 9.2. We can choose $y$ to be an involution and to centralize $Z\langle a_1, a_2\rangle$.

PROOF. $T_2$ is a maximal subgroup of a Sylow-2-subgroup of $N_1$. If $T_0$ is characteristic in $T_2$, $T_1$ must be a Sylow-2-subgroup of $N_1$, but this is not the case. This implies that $T_2 = J(T_2)$ and in particular that $T_2 = \Omega_1(T_2)$. So we may choose $y$ to be an involution and to centralize a hyperplane of a 16-group of $T_0$, and we may assume taht $y$ centralizes a hyperplane of $E_1$. But then we have $[T_2, E_1] = \langle z \rangle$. As $T_1$ interchanges $E_1$ and $E_2$, we get $[T_2, E_2] = \langle z \rangle$ and $y$ centralizes a hyperplane of $E_2$ as well. Hence, in each of the sets $\{ya_1, tya_1\}$ and $\{ya_2, tya_2\}$ there is precisely one involution. Suppose that $[y, a_1] = z$. Then replace $y$ by $ya_2$ or $yta_2$. Therefore we may assume that $[y, a_1] = 1$. Replacing $y$ by $ya_1$ if necessary, we may assume that $y$ also centralizes $a_2$. Our lemma is proved.

LEMMA 9.3. $A_G(E_i) \simeq \Sigma_4$, for $i = 1, 2$.

PROOF. Regard the action of $A(E_i)$ on the orbit of $t$ in $E_i$ which is the set $\{t, tz, ta_i, tza_i\}$; the element $d_i$ operates as $(tz, ta_i, tza_i\}$; the element $d_i$ operates as $(tz, ta_i, tza_i)$, $y$ acts as $(t, tz)(ta_i, tza_i)$ and $a_j$, $j \neq i$, interchanges $ta_i$ and $tza_i$. So we have the full symmetric group on this orbit.

LEMMA 9.4. $T_1/T_0$ is cyclic, $T_1 = T_0\langle x \rangle$, where $x$ can be chosen to have the following properties:

(1) $a_1^x = a_2$, $a_2^x = a_1$,

(2) $t^x = tz_1$.

Let $y_0 := x^2$. Then $y$ can be chosen to be the unique involution of the set $\{y_0, ty_0\}$.

PROOF. From lemma 9.3. it follows that $\langle z_i \rangle = Z(N_i)$, hence $T_1$ interchanges $z_1$ and $z_2$, $Z(T_1) = \langle z \rangle$ is of order 2. Suppose that $T_1/T_0$ is elementary. Then there are 3 maximal subgroups of $T_1$ containing $T_0$. We know that 4 elements of $Z(T_0)$ are conjugate to $t$, and that a group $M$ with $T_0 < M < T_1$ has a center of order 4, so we must have $Z(M) = Z$ for any choice of $M$ and therefore $Z(T_1) = Z$, a contradiction. We have proved that $T_1/T_0$ is cyclic.

It is clear that $x$ acts transitively on $Z(T_0) - Z$. Replacing $x$ by $x^{-1}$ if necessary, we may assume that condition (2) holds.

Suppose that $a_1^x = za_2$. Then replace $x$ by $a_2x$. On the other hand, if $a_2^x = za_1$, then replace $x$ by $xa_2$. So $x$ can be chosen to satisfy condition (1) as well.

Put $y_0 := x^2$. Then $y_0$ centralizes $Z\langle a_1, a_2 \rangle$, and so does $ty_0$. We have seen in the proof of lemma 9.2. that $\Phi(T_2) = \langle z \rangle$, so either $y_0$ or $ty_0$ is an involution. The lemma is proved.

LEMMA 9.5. $T_2$ contains precisely 6 elementary subgroups of order 16

$$E_1 = Z\langle a_1, t\rangle , \qquad E_2 = Z\langle a_2, t\rangle ,$$
$$E_3 = Z\langle a_1, y\rangle , \qquad E_4 = Z\langle a_2, y\rangle ,$$
$$E_5 = Z\langle a_1 y, a_2 t\rangle , \qquad E_6 = Z\langle a_1 t, a_2 y\rangle .$$

PROOF. We write $T_2 = \langle a_1, a_2\rangle \curlyvee \langle y, t\rangle \langle z_1\rangle$ in order to «see» the elementary subgroups as usual.

LEMMA 9.6. Put $R_i := T_2 Q_i$. Then $R_i$ is a Sylow-2-subgroup of $N_i$. Set $f_i := y^{d_i}$, $f_i y q_i := f^{d_i}$, $q_i \in E_i$. Then $[f_i, a_j] = yq_i$.
There are two cases:

I) $q_i \in \langle z_i\rangle$, $[y, f_i] = 1$, $E_3$ and $E_4$ are not conjugate to $E_1$.

II) $q_i \in a_i t\langle z_i\rangle$, $[y, f_i] = a_i z$, all 16-subgroups of $T_2$ are conjugate in $N_G(T_2)$.

PROOF. First of all, $(f_i y q_i)^{a_j} = f_i y q_i \cdot y \cdot q_i^{a_j} = f_i$, therefore we get the relation $[y, q_i] = [q_i, a_j]$ and we find two cases:

*Case I.* $q_i \in Z$,

*Case II.* $q_i \in Z a_i t$.

We shall prove that we always are in the same case for $i = 1$ and $i = 2$. But for the first part of this proof, this does not matter.
Now use the action of $d_i$. We get

$$[y, f_i]^{d_i} = [f_i, f_i y q_i] = [f_i, q_i][f_i, y] .$$

In case I, we get $[f_i, q_i] = 1$ and therefore $[y, f_i] \in E_i \cap C(y) \cap \cap C(d_i) = \langle z_i\rangle$. In case II we have $[f_i, q_i] = a_i$ and $[y, f_i] \in z a_i \langle z_i\rangle$.
Furthermore, $(f_i y q_i)^{d_i} = y = f_i y q_i \cdot f_i \cdot q_i^{d_i}$, which implies the relation $[y, f_i][f_i, q_i][q_i, d_i] = 1$. In case I, we conclude that $[y, f_i] = 1$ and that $[q_i, d_i] = 1$; whereas in case II $[y, f_i]$ must be $z a_i$, so $[q_i, d_i] = z$ and $q_i \in a_i t\langle z_i\rangle$.
From the definition of $x$, we get $E_1^x = E_2$, $E_3^x = E_4$ and $E_5^x = E_6$.
Now suppose that for $i = 1$ or $i = 2$ we are in case I. Then $C(y) \geqslant \geqslant Z\langle a_1, a_2, y, f_i\rangle$, hence $y$ is centralized by a group of order $2^6$ and cannot be conjugate to $t$. As $a_1$ and $a_2$ are not conjugate to $t$ either, we see that $E_3$ and $E_4$ cannot be conjugate to $E_1$.

On the other hand, if for $i = 1$ or for $i = 2$ we have case II, then, if $i = 1$ we have $E_4^{f_1} = E_5$ and if $i = 2$ we have $E_3^{f_2} = E_6$. In either case, we have $E_2^{f_1} = E_6$ resp. $E_1^{f_2} = E_5$. So if we have « case II » for $i = 1$ or $i = 2$, then all 16-groups of $T_2$ are conjugate. This proves that we must have case I resp. case II simultaneously for $i = 1$ and $i = 2$. The lemma is proved.

We will deal with these two cases separately. Lemmas 9.8-9.10. will be proved under

HYPOTHESIS 9.7. We have case I of lemma 9.6.

LEMMA 9.8. $N_G(T_2) = T_2\langle x, f_1, f_2\rangle =: T$ has order $2^9$. The factor group $T/T_2$ is dihedral or order 8.

PROOF. Put $EE := \{E_1, E_2, E_5, E_6\}$. It follows from lemma 9.6. that $N_G(T_2)$ operates on $EE$, and as $N_G(T_2) \cap N_G(E_1) \cap N_G(E_2) = T_2$, $N_G(T_2)/T_2$ acts faithfully on $EE$.

We compute $f_1 \triangleq (E_2, E_6)$, $f_2 \triangleq (E_1, E_5)$, and $x \triangleq (E_1, E_2)(E_5, E_6)$, there elements generate a dihedral group of order. 8. Furthermore, $N_G(T_2) \cap N_G(E_1)$ has order $2^7$, hence the order of $N_G(T_2)$ must be $2^9$; the lemma is proved.

LEMMA 9.9. $T_2 \leqslant V(\mathrm{ccl}_G(t); T) \leqslant T_2\langle f_1 f_2\rangle$.

PROOF. First we prove that $T_2 = V(\mathrm{ccl}_G(t); R_1)$. Involutions of $R_1$ are contained in $T_2$ or in $Q_1$. It is clear that $Z(Q_1) = Z\langle a_1\rangle$. Determine the elements of $Q_1/Z(Q_1)$ corresponding to cosets of involutions. We find 4 nontrivial cosets with representatives $t$, $y$, $f_1$, and $f_1 y$. Note that $Z(Q_1(\langle y\rangle$, $Z(Q_1)\langle f_1\rangle$ and $Z(Q_1)\langle f_1 y\rangle$ are conjugate under $d_1$. So it is sufficient to prove that no involution of $E_3$ is conjugate to $t$. But $E_3\langle f_1\rangle$ is elementary of order 32, and we are done.

Regard the inverse images of the involutions of $T/T_2$. We have see that $T_1/T_0$ is cyclic, hence $T_2 = \Omega_1(T_1)$. Above we have excluded $R_1 - T_2$. The elements $x$ and $f_1$ correspond to representatives of the 2 non-central classes of involutions in $T/T_2$, therefore only the inverse image of $Z(T/T_2)$ is left.

LEMMA 9.10. $T$ is a Sylow-2-subgroup of $G$.

PROOF. Put $U := T\langle f_1 f_2\rangle = Z(T \bmod T_2)$ and write $f := f_1 f_2$. It is immediate that $Z = Z(U)$. Regard $\hat{U} := U/Z(U)$. We get from lemma 9.6. that $[\hat{f}, \hat{a}_1] = [\hat{f}, \hat{a}_2] = \hat{y}$ and that $[\hat{f}, \hat{t}] = \hat{a}_1\hat{a}_2\hat{y}$. It is clear that $\hat{T}_2$ is an elementary 16-group. Suppose that $\hat{U}$ contains a further maximal group which is elementary. Then $\hat{U}'$ must be of order 2. But we have seen that $\hat{U}'$ is of order 4, so $\hat{T}_2$ is the only elementary 16-group contained in $\hat{U}$, therefore $\hat{T}_2$ is characteristic in $\hat{U}$ and $T_2$ is characteristic in $U$.

We get that $T_2$ is characteristic in $T$. Indeed, if $T_2 = V(\mathrm{ccl}_G(t); T)$ this is obvious. On the other hand, if $U = V(\mathrm{ccl}_G(t); T)$ then $U$ is characteristic in $T$; as $T_2$ is characteristic in $U$, we get that $T_2$ is characteristic in $T$ again.

Now it follows directly that $T$ is a Sylow-2-subgroup of $G$.

THEOREM 9.11. We have case II of lemma 9.6. for $i = 1$ and $i = 2$.

PROOF. Suppose not. Then we shall derive a contradiction with the aid of the Thompson transfer lemma.

$M := Z\langle a_1, a_2, y, f_1, f_2, x\rangle$ is a maximal subgroup of $T$. We conclude from the structure of $T/T_2$ that $T_f := \langle a_1, a_2, y, f_1, f_2, t\rangle = \Omega_1(T)$.

Assume that $t$ is conjugate into $M$. Then $t$ is conjugate into $\Omega_1(M) \leqslant M \cap T, =: M_0$. We have $M_0 = Z\langle y, a_1, a_2, f_1, f_2\rangle$ and that $Z(M_0) = Z\langle y\rangle$.

We know that $[a_2, f_1] = q_1 y$ and $[a_1, f_2] = q_2 y$. Because of $(xt)^2 = y_0 z_1$, we may interchange $y$ and $yz_1$ such that $q_1 = 1$.

Now calculate $t^f = (a_1 t)^{f_2} = q_2 y a_1 a_2 t$, so

$$t^{f^2} = q_2 y \cdot q_2 y a_1 \cdot y a_2 \cdot q_2 y a_1 a_2 t = (a_1 a_2)^2 q_2 t\ ,$$

but $f^2 \in T_2$, so we must have $q_2 = 1$ and $t^{f^2} = zt$. On the other hand, $f^2 \in C_{T_2}(f) = Z\langle y, a_1 a_2\rangle$ and $f^2$ centralizes $Z\langle y, a_1, a_2\rangle$, so we must have $f^2 = yv,\ v\ {}_7 Z$.

We look for involutions in $M_0$ which can be conjugate to $t$. As we have seen in the proof of lemma 9.9., we only have to regard $M_{00} := Z\langle y, a_1, a_2, f\rangle$. Let $fx$ be an involution, $x \in Z\langle y, a_1, a_2\rangle$.

Then $f^2 x^2 = [f, x] = y$, and $x \in Z(M_0) a_1 \cup Z(M_0) a_2$. But

$$Z(M_0)\langle a_1 f_1, f_2\rangle \quad \text{and} \quad Z(M_0)\langle f_1, f_2 a_2\rangle$$

are elementary of order 32. So we have shown that $t$ cannot be conjugate into $M_{00}$. But this implies that $t$ cannot be conjugate into $M$, and we can apply the Thompson transfer lemma. Our theorem is proved.

LEMMA 9.12. The order of $N_G(T_2)$ is $2^8 \cdot 3$. $N_G(T_2)/T_2$ is isomorphic to $\Sigma_3 \times Z_2$.

PROOF. Put $EE = \{E_1, E_2, E_3, E_4, E_5, E_6\}$. Then, by lemma 9.6. and other facts, we get

$$f_1 \triangleq (E_2, E_6)(E_4, E_5)\,, \qquad f_2 \triangleq (E_1, E_5)(E_3, E_6)\,,$$

$x \triangleq (E_1, E_2)(E_3, E_4)(E_5, E_6)$. This implies that $c := f_1 x = (E_1, E_2, E_5, E_3, E_4, E_6)$. Furthermore, we compute $d := c^2 = (E_1, E_5, E_4)(E_2, E_3, E_6)$ and $e := c^3 = (E_1, E_3)(E_2, E_4)(E_5, E_6)$.

Now it is obvious that order and structure of $N_G(T_2)$ are as described.

LEMMA 9.13. The order of $N_G(Q_1)$ is $2^9 \cdot 3$.

PROOF. There are precisely 4 16-groups in $Q_1$, namely $E_1$, $E_3$, $E_{31} = Z(Q_1)\langle f_1 \rangle$, and $E_{32} = Z(Q_1)\langle f_1 yt \rangle$. The group $R_1 = T_2 Q_1$ contains 8 16-groups. We see that $E_1$ and $E_3$ are normal in $R_1$, $E_{31}$ and $E_{32}$ have normalizer $Q_1$, finally $E_2$, $E_4$, $E_5$, and $E_6$ have normalizer $T_2$ in $R_1$. As $T_2$ and $Q_1$ are non-isomorphic, it follows that $Q_1$ and $T_2$ are characteristic in $R_1$.

From lemma 9.12. we conclude that $N_G(R_1) = R_1\langle e \rangle$, hence e normalizes $Q_1$, and all 16-subgroups of $Q_1$ are conjugate in $N(Q_1)$. As $N_G(E_1) \leqslant N_G(Q_1)$, we must have that the order of $N_G(Q_1)$ is $2^9 \cdot 3$.

LEMMA 9.14. Put $U := T_2\langle e \rangle$. Then $T_2 = J(U)$.

PROOF. $T := T_2\langle e, f_1 \rangle$ is a Sylow-2-subgroup of $N_G(T_2)$; as $Z(T_1) = \langle z \rangle$, we must have $Z(T) = \langle z \rangle$. On the other hand, $\langle z \rangle = T_2'$ is normal in $N_G(T_2)$, therefore $d$ centralizes $Z$ and so does $f_1$. This shows that $z_1^e = zz_1$.

Take $e_0$ to be any involution of $U - T_2$. Then $e_0$ normalizes $E_1 \cap E_3$, $E_2 \cap E_4$, and $E_5 \cap E_6$, so $e_0$ normalizes $T_{20} := Z\langle a_1, a_2, yt \rangle$. On the other hand, $(E_1 \cap E_2)^{e_0} = (E_3 \cap E_4)$, therefore $(Zt)^{e_0} = Zy$. As $e_0$ centralizes $T_{20}$ modulo $Z$, we then must have $C_{T_2}(e_0) \leqslant T_{20}$.

If $U$ contains a 16-group which does not lie in $T_2$, choose $e_0$ from such a group and outside $T_2$. Then $e_0$ centralizes an elementary group of order 8 in $T_{20}$. But the only groups of this type in $T_{20}$ are $Z\langle a_1 \rangle$, $Z\langle a_2 \rangle$, and $Z\langle a_1 a_2 ty \rangle$, and no one of these groups can be centralizes by $e_0$, as $e_0$ does not centralize $Z$. Our lemma is proved.

Now we are able to derive a final contradiction. To this end, we want to prove that $R_1 = J(T)$.

There is a group $T_2 \leqslant T_3 \leqslant T$ such that $T_3$ is isomorphic to $T_1$, therefore $T/T_2$ cannot be covered by an elementary abelian group. So any 16-group of $T$ is contained in $R_1$, $T_3$ or $U$. But, as we have seen, $T_2 = J(U) = J(T_3)$. Hence $R_1 = J(T)$.

On the other hand, we have noted in the proof of lemma 9.13. that $T_2$ is characteristic in $R_1$, hence $T$ is a Sylow-2-subgroup of $N_G(R_1)$. This, however, implies that $T$ is a Sylow-2-subgroup of $G$. But $T$ has order $2^8$ and we have seen in lemma 9.13. that $2^9$ divides the order of $G$. This is the desired contradiction. Theorem A is proved.

## 10. Proof of theorem B.

Let $G$ be a finite group having no subgroup of index 2, containing an involution $t$ such that $H = C_G(t) = \langle t \rangle \times \Sigma$, $\Sigma \cong \Sigma_7$.

We choose a fixed Sylow-2-subgroup of $H$, $T_0$, which can be taken to correspond to the one introduced in § 1, when we regard $\Sigma_6$ as a subgroup of $\Sigma_7$. We use the notation introduced in § 1.

Lemma 10.1. In $H$, $i_2$ has 21 conjugates, $i_4$ has 105 conjugates and $i_6$ has 105 conjugates.

Proof. The symmetric group on 7 letters contains $\binom{7}{2} = 21$ transpositions. There are $\binom{7}{4} \cdot 3$ involutions operating on 4 letters. The subgroup $\Sigma_6$ contains 15 involutions operating on 6 letters, so $\Sigma_7$ must contain $7 \cdot 15 = 105$ involutions of this type.

Lemma 10.2. In $H$, $d_1$ has 70 conjugates, and $d_2$ has 280 conjugates. In particular, $\langle t, i_6 \rangle$ is a Sylow-2-subgroup of $C_H(d_2)$.

Proof. There are $\binom{7}{3} \cdot 2$ 3-elements operating on 3 letters. Regard 3-elements operating on 6 letters. In $\Sigma_6$, we find $\binom{5}{2} \cdot 4 = 40$ elements of this type in $\Sigma_7$. The structure of $C_H(d_2)$ is obvious.

We remark that $N_H(E_1)$ and $N_H(E_2)$ have the same structure as in the case $\Sigma_6$. So we can take § 2 literally to see that $2^6$ divides the order of $G$.

Lemma 10.3. $2^7$ divides the order of $G$. Furthermore,

$$|N_G(T_0) : T_0| = 4 .$$

PROOF. The first assertion obviously follows from the second one. So assume that $|N_G(T_0): T_0) = 2$. If $t$ is conjugate to any other $H$-class of involution, $t$ is conjugate to its representative in $Z(T_0)$ under the action of $N(T_0)$. Our assumption implies that $t$ is conjugate to just one other class, therefore $t$ has 22 or 106 conjugates in $H$, and as $G$ does not have a subgroup of index 2, $T = N_G(T_0) = T_0\langle y\rangle$ is a Sylow-2-subgroup of $G$.

Suppose that $E_1$ and $E_2$ are normal in $T$. Then $t$ must have 2 conjugates in $E_1$ and $E_2$. But this is impossible.

Suppose that $E_1^y = E_2$. Then $T_0$ must be a Sylow-2-subgroup of $N_G(E_i)$, $t$ is isolated in $N_G(E_i)$, therefore $N_G(E_i) = N_H(E_i)$, $Z(N_G(E_1)) = \langle t, i_2\rangle$ and $Z(N_G(E_2)) = \langle t, i_6\rangle$. We conclude that $\langle t, i_2\rangle^y = \langle t, i_6\rangle$. But now either $t^y = t$ or $t$ is conjugate to at least 3 elements of $Z(T_0)$. Both is not possible. The lemma is proved.

THEOREM 10.4. Suppose that $K$ is a finite group of even order, $t \in K$ is an involution and $C_K(t)$ has a Sylow-2-subgroup which is elementary of order 4. Then the Sylow-2-subgroups of $K$ are dihedral or semi-dihedral. In particular, the 2-rank of $K$ is 2.

PROOF. Let $\langle s, t\rangle$ be a Sylow-2-subgroup of $C_K(t)$ and $S$ be a Sylow-2-subgroup of $K$ containing $\langle s, t\rangle$. Suppose that the order of $S$ is $2^n$. Then $t$ has $2^{n-2}$ conjugates in $S$. The commutator subgroup of $S$ has order at most $2^{n-2}$, so $t$ cannot be contained in $S'$. On the other hand, $S'\langle t\rangle$ is normal is $S$. This forces $|S'| = 2^{n-2}$. By [3], theorem 5.4.5., $S$ is dihedral, semi-dihedral or generalized quaternion. As $S$ contains at least 3 involutions, it cannot be quaternion. The theorem is proved.

LEMMA 10.5. $d_1$ and $d_2$ are not conjugate in $G$.

PROOF. It follows from theorem 10.4. that $C_G(d_2)$ has Sylow-2-subgroups of 2-rank 2. Suppose that $d_1$ and $d_2$ are conjugate in $G$. Then $t$ centralizes $350 = 2 \pmod 4$ conjugates of $d_2$, hence a Sylow-2-subgroup of $G$ can be at most twice as big as a Sylow-2-subgroup of $C_G(d_2)$. But $G$ has 2-rank at least 4. This is a contradiction.

LEMMA 10.6. $E_1$ and $E_2$ are not conjugate in $G$.

PROOF. Suppose they are. Then, as before, we see that $t$ has precisely 4 conjugates in $E_i$ under the action of $N_G(E_i)$, and that the order of $N_G(E_i)$ is $2^7 \cdot 3$. In particular, $\langle d_1\rangle$ is a Sylow-3-subgroup of $N_G(E_1)$, and $\langle d_2\rangle$ is a Sylow-3-subgroup of $N_G(E_2)$. If $E_1$ and $E_2$ are conjugate, then their normalizers are conjugate in $G$ as well, and so are any two Sylow-3-subgroups. But that contradicts lemma 10.5. Our lemma is proved.

Now we are in a position to make use of §§ 5-8, where the lengths of conjugacy classes in $H$ do not matter at all. This remark finishes the proof of theorem B.

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Manoscritto pervenuto in redazione l'8 maggio 1974.