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A mixed boundary value problem for the Laplace equation in an angle (1st part)

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A Mixed Boundary Value Problem for the Laplace Equation in an Angle (1st Part).

Alfredo Lorenzi (*)

Summary - We study, in the context of Sobolev spaces, the smoothness near the corner of functions harmonic in an angle and verifying mixed boundary conditions.

1. Introduction and statement of the problem.

In this work, which, because of its length, we are forced to divide into two parts for printing reasons, we are interested in solving a mixed boundary value problem for the Laplace equation in an angle with an arbitrary width. The boundary conditions are these: Dirichlet datum on a side of the angle, while on the other side an oblique derivative is assigned. More particularly the angle under consideration is

$$\Omega_\alpha = \{(r\cos\theta, r\sin\theta):\ 0<r,\ 0<\theta<\alpha\}$$

where

(1.1) $$0<\alpha<2\pi$$

and the oblique direction is given by $e^{i\omega}$, where

(1.2) $$0<\omega<\pi\,.$$

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In other words we look for a function u such that

$$(1.3)\qquad \begin{cases} \Delta u = 0 & \text{in } \Omega_\alpha \\ u(re^{i\alpha}) = a(r) & r > 0 \\ \dfrac{\partial u}{\partial(e^{i\omega})}(re^{i0}) \equiv \cos\omega \dfrac{\partial u}{\partial x}(re^{i0}) + \sin\omega \dfrac{\partial u}{\partial y}(re^{i0}) = b(r) & r > 0 \end{cases}$$

which has a prescribed degree of smoothness up to the origin: more precisely we search for solutions to problem (1.3) such that

$$(1.4)\qquad u \in \boldsymbol{W}^{1,p}(\Omega_\alpha) \qquad \text{and} \qquad Du \in W^{s-1,p}(\Omega_\alpha)\,,$$

Du denoting the gradient of u, s being any (fixed) positive integer greater than or equal to 2 and p being any real verifying the inequality

$$(1.5)\qquad 1 < p < +\infty\,.$$

We recall that $\boldsymbol{W}^{1,p}(\Omega_\alpha)$ is the completion of $C_0^\infty(\bar{\Omega}_\alpha)$ (*) with respect to the norm

$$\|u\|_{\boldsymbol{W}^{1,p}(\Omega_\alpha)} = \|Du\|_{L^p(\Omega_\alpha)} \text{ (**)}$$

and that its members are functions in $L^p_{\text{loc}}(\bar{\Omega}_\alpha)$ (***), whose gradients belong to $L^p(\Omega_\alpha)$, when $p \in (1, 2)$ ($*_**$), while when $p \in [2, +\infty)$, the elements of $\boldsymbol{W}^{1,p}(\Omega_\alpha)$ are equivalence classes of functions with the aforementioned properties, two functions being equivalent if they differ almost everywhere in an additive constant.

Further we recall that $W^{s-1,p}(\Omega_\alpha)$ denotes the Sobolev space of all functions in $L^p(\Omega_\alpha)$, that have distributional derivatives up to the order $s-1$ belonging to $L^p(\Omega_\alpha)$. $W^{s-1,p}(\Omega_\alpha)$ is a Banach space with

(*) $C_0^\infty(\bar{\Omega}_\alpha)$ denotes the space of the restrictions of functions in $C_0^\infty(R^2)$ to $\bar{\Omega}_\alpha$.

(**) For the properties of $\boldsymbol{W}^{s,p}$ spaces see Shamir [13], [14] or Peetre [12].

(***) $L^p_{\text{loc}}(\bar{\Omega}_\alpha)$ denotes the space of all functions belonging to $L^p(K)$ for every compact $K \subseteq \bar{\Omega}_\alpha$.

($*_**$) See, for instance, estimates (5.37) at the end of the proof of Theorem 1.

respect to the norm

$$\|v\|_{W^{s-1,p}(\Omega_\alpha)} = \left\{ \sum_{|\beta|\leq s-1} \int_{\Omega_\alpha} |D^\beta v(x,y)|^p \, dx\, dy \right\}^{1/p}.$$

As far as the boundary conditions are concerned, we assume a and b in the suitable spaces of traces, i.e.:

$$(1.6)\qquad \begin{aligned} & a \in \boldsymbol{W}^{1/p',p}(0,+\infty) \quad \text{and} \quad a' \in W^{s-1-1/p,p}(0,+\infty) \\ & b \in W^{s-1-1/p,p}(0,+\infty). \end{aligned}$$

We recall that $\boldsymbol{W}^{1/p',p}(0,+\infty)$ is the closure of $C_0^\infty([0,+\infty))$ with respect to the norm

$$|a|_{\boldsymbol{W}^{1/p',p}(0,+\infty)} = \left(\int_0^{+\infty} \int_0^{+\infty} \left| \frac{a(x)-a(y)}{x-y} \right|^p dx\, dy \right)^{1/p}$$

and that, when $p \in (1,2)$, its elements are functions, while, when $p \in [2,+\infty)$, they are equivalence classes of functions, two functions being equivalent if they differ almost everywhere in an additive constant. However, in order to deal with functions rather than with classes of functions, we shall assume the datum a to be a function with the aforementioned properties, when $p \in [2,+\infty)$. Moreover we shall require, for the same reason, u to be the representative of the solution to problem (1.3), (1.4) for which the equation $u(re^{i\alpha}) = a(r)$ holds almost everywhere in $(0,+\infty)$, when $p \in [2,+\infty)$.

Moreover we shall suppose that a and b possess the additional properties:

$$(1.7)\qquad \begin{cases} r^{1/p} b \in L^p(0,+\infty) \quad \text{when } p \neq 2 \\ r^{-\eta} a,\ r^{1-\eta} b \in L^2(0,+\infty) \cdot \\ \qquad \cdot \text{for some } \eta \in \left(\max\left(0, \frac{1}{2} - \frac{\pi-\omega}{\alpha}\right), \frac{1}{2}\right) \text{ when } p = 2. \end{cases}$$

We observe that problem (1.3), (1.4) has a negative index. More precisely it admits a unique solution if, and only if, the data verify a certain number of compatibility conditions: such a number depends on the degree of smoothness required for the solution. The main purpose of this paper is to find explicitly such conditions.

We remark that the question of establishing the regularity of the solution to a boundary value problem in a set with corners was studied by many authors (for an exhaustive reference see, for instance, Avantaggiati-Troisi [1] or Grisvard [4]). We limit ourselves to recalling Volkov [16], Grisvard [3, 4], Avantaggiati-Troisi [1], Merigot [8, 9].

Volkov studies the problem of determining the regularity of a function harmonic in a polygone and verifying on the sides Dirichlet or Neumann conditions in the context of the spaces of Hölder continuous functions.

Grisvard deals in [4] with the Dirichlet problem in a polygone for the Laplace equation from the point of view of H^s-solutions, while in [3] he deals with an analogous problem for the Poisson equation in a cone in R^n.

An exhaustive study of the regularity of solutions to mixed boundary value problems in an angle was done in the papers by Avantaggiati-Troisi in the context of Sobolev weight spaces with $p = 2$.

Finally, we want to mention particularly the papers [8] and [9] by Merigot, in which he deals respectively with the Dirichlet problem in an angle and with our problem in the context of $W^{s,p}$ spaces. We remark that his results overlap partially with ours.

By the way, we observe that mixed boundary value problems in an angle are encountered also in applicative questions as free boundary problems for the Laplace equation (see Baiocchi-Comincioli-Magenes-Pozzi [2]) and in hydrodynamical studies of the sea-motions (see van Ouwerkerk-Dijkers [11]).

As we noted at the beginning of this section, the work is divided into two parts: the former states the main results we have obtained and it gives a proof of a representation theorem (theorem 1, section 4) while the latter is devoted to the proof of the regularity theorem (theorem 2, section 4) and it will appear in one of the next issues of this journal.

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2. Preliminaries.

In this section we determine representation formulas for the solution u to problem (1.3): they are obviously the keystone which enables us to treat in an exhaustive enough manner the problem of finding the compatibility conditions on the data a and b in order that the gradient of u belong to $W^{s-1,p}(\Omega_\alpha)$. Such formulas may be obtained formally as follows: if U denotes the Mellin transform with respect to r of u, then U is a solution to the problem

$$(2.1)\qquad \begin{cases} \dfrac{\partial^2 U}{\partial\theta^2} + z^2 U = 0\,, \\[2mm] U(z,\alpha) = A(z)\,, \\[2mm] -zU(z,0)\cos\omega + \dfrac{\partial U}{\partial\theta}(z,0)\sin\omega = B(z)\,, \end{cases}$$

where A and B are the Mellin transforms respectively of a and rb. Hence, after some easy computations, we get that, if

$$(2.2)\qquad \operatorname{Re} z \neq 0 \quad \text{and} \quad \frac{1}{\pi}(\alpha \operatorname{Re} z + \omega) \notin Z\ (^*)\,,$$

U is of the form

$$(2.3)\qquad U = \frac{\sin(z\theta+\omega)}{\sin(z\alpha+\omega)}A - \frac{\sin[z(\alpha-\theta)]}{\sin(z\alpha+\omega)}z^{-1}B\ (^{**})\,.$$

Observe, now, that the coefficients of A and $-z^{-1}B$ are the Mellin transforms with respect to r of the functions $r \to H_0(r,\theta,1)$ and $r \to K_0(r,\theta,1)$, H_0 and K_0 being homogeneous functions of degree -1 in (r,t) defined as follows:

$$(2.4)\qquad \begin{cases} H_0(r,\theta,t) = \\ \quad = -\dfrac{r^{\nu-1}t^{2\beta-\nu}\{r^{2\beta}\sin[(\nu-1-2\beta)\theta-\omega] + t^{2\beta}\sin[(\nu-1)\theta-\omega]}{r^{4\beta}+2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}} \\[3mm] K_0(r,\theta,t) = \\ \quad = -\dfrac{r^{\nu-1}t^{2\beta-\nu}\{r^{2\beta}\sin[(\nu-1-2\beta)\theta-\omega] - t^{2\beta}\sin[(\nu-1)\theta-\omega]}{r^{4\beta}-2r^{2\beta}t^{2\beta}\cos(2\beta\theta)+t^{4\beta}} \end{cases}$$

(*) We denote by Z the set of all relative integers.

(**) For more details see the proof of Theorem 2 in [7].

where

(2.5) $$\beta = \frac{\pi}{2\alpha}$$

(2.6) $$\nu = \frac{\omega}{\alpha} + 1 .$$

In order to verify such a property the following formula may be of use:

(2.7) $$\int_0^{+\infty} \frac{r^{\varepsilon-1}}{r^{2\gamma} - 2r^{\gamma}\cos\delta + 1}\, dr = \begin{cases} \dfrac{\pi \sin[(\pi-\delta)(\varepsilon/\gamma) + \delta]}{\gamma \sin\delta \sin((\varepsilon/\gamma)\pi)} & 0 < \varepsilon < 2\gamma,\ \varepsilon \neq \gamma,\ \delta \neq n\pi,\ n \in Z \\[2ex] \dfrac{\gamma \sin\delta}{\pi - \delta} & \varepsilon = \gamma,\ \delta \neq n\pi,\ n \in Z . \end{cases}$$

Moreover we observe that H_0 and K_0 are imaginary parts respectively of the functions G_1 and G_2, analytic in their first argument, so defined

(2.8) $$\begin{cases} G_1(\zeta, t) = \dfrac{e^{-i\omega}\zeta^{\nu-1} t^{2\beta-\nu}}{\zeta^{2\beta} + t^{2\beta}} \\[2ex] G_2(\zeta, t) = \dfrac{e^{-i\omega}\zeta^{\nu-1} t^{2\beta-\nu}}{\zeta^{2\beta} - t^{2\beta}} . \end{cases} \qquad \zeta = re^{i\theta} .$$

Recalling that (formally) the pre-image of $-z^{-1}B$ is either the function

(2.9) $$B_1(r) = -\int_r^{+\infty} b(t)\, dt \quad \text{if } \operatorname{Re} z > 0$$

or the function

(2.10) $$B_2(r) = \int_0^r b(t)\, dt \quad \text{if } \operatorname{Re} z < 0 ,$$

from well known properties of the Mellin transformation, supposing that conditions (2.2) are satisfied, we infer that u can be represented in polar co-ordinates as follows: either

$$u(re^{i\theta}) = \frac{1}{\alpha}\int_0^{+\infty} H_0(r,\theta,t)a(t)\,dt + \frac{1}{\alpha}\int_0^{+\infty} K_0(r,\theta,t)B_1(t)\,dt \tag{2.11}$$

or

$$u(re^{i\theta}) = \frac{1}{\alpha}\int_0^{+\infty} H_0(r,\theta,t)a(t)\,dt + \frac{1}{\alpha}\int_0^{+\infty} K_0(r,\theta,t)B_2(t)\,dt\,. \tag{2.12}$$

We observe that, if $b \in L^1(0, +\infty)$, such formulas coincide, since B_1 and B_2 differ in an additive constant and

$$\int_0^{+\infty} K_0(r,\theta,t)\,dt = 0 \qquad r \in (0,+\infty),\ \theta \in (0,\alpha)$$

Suppose now, $a \in C^1([0, +\infty))$ and $b \in C([0, +\infty))$ and that they are smooth at $+\infty$: then u defined by (2.11) or (2.12) is a solution to problem (1.3). In fact the harmonicity of u is an immediate consequence of the harmonicity of the kernels H_0 and K_0 and the assumption of the boundary values depends on the fact that u belongs to $C^1(\bar{\Omega}_\alpha - \{(0,0)\})$, that the functions H_0, K_0, G_1, G_2 are homogeneous of degree -1 in (r, t) and on the following relationships:

$$\frac{1}{\alpha}\int_0^{+\infty} H_0(r,\theta,t)\,dt = 1 \qquad r \in (0,+\infty),\ \theta \in (0,\alpha)$$

$$\int_0^{r-\eta} |H_0(r,\theta,t)|\,dt + \int_{r+\eta}^{+\infty} |H_0(r,\theta,t)|\,dt \to 0 \qquad \text{as } \theta \to \alpha,\ r \in (0,+\infty),\ \eta \in (0,r)$$

$$\frac{\partial K_0}{\partial(e^{i\omega})}(r,\theta,t) = -\operatorname{Im}\left[e^{i\omega}\frac{\partial G_2}{\partial r}(re^{i\theta},t)\right] = \operatorname{Im}[e^{i\omega}G_2(re^{i\theta},t)] + \operatorname{Im}\left[e^{i\omega}t\,\frac{\partial G_2}{\partial t}(re^{i\theta},t)\right]$$

$$-\frac{1}{\alpha}\int_0^{+\infty} \operatorname{Im} e^{i\omega}G_2(re^{i\theta},t)\,dt = 1 \qquad r \in (0,+\infty),\ \theta \in (0,\alpha)$$

$$\int_0^{r-\eta} |\operatorname{Im}[e^{i\omega}G_2(re^{i\theta},t)]|\,dt + \int_{r+\eta}^{+\infty} |\operatorname{Im}[e^{i\theta}G_2(re^{i\omega},t)]|\,dt \to 0 \qquad \text{as } \theta \to 0,\ r \in (0,+\infty),\ \eta \in (0,r)\,.$$

This establishes that formulas (2.11) and (2.12), obtained formally, represent actually a solution to problem (1.3). However, since we are interested in solutions whose gradients belong to the Sobolev space $W^{s-1,p}(\Omega_\alpha)$, we have to handle suitably such formulas. We begin by observing that the eigenfunctions of the homogeneous problem (1.3) are linear combinations of functions of the form

$$v_{M_1}(re^{i\theta}) = r^{\nu-1}\sin[(\nu-1)\theta-\omega]\,,$$

ν being now defined by the formula

$$\nu = 2\left(\frac{\omega}{\pi}+M_1\right)\beta+1 \tag{2.13}$$

where M_1 is an (arbitrary) relative integer. Define now M_1 as follows:

$$M_1 = \left[\left(1-\frac{2}{p}\right)\frac{\alpha}{\pi}+\frac{\pi-\omega}{\pi}\right] \quad (*) \tag{2.14}$$

and, then, subtract from (2.11) and (2.12) (that we can rewrite as follows:

$$u(re^{i\theta}) = \frac{1}{\alpha}\int_0^{+\infty} H_0(r,\theta,t)[a(t)-a(0)]\,dt + \frac{1}{\alpha}\int_0^{+\infty} K_0(r,\theta,t)B_p(t)\,dt + a(0)\Big)$$

a suitable linear combination of the functions $v_{M_1-2\beta}$, v_{M_1}, $v_{M_1+2\beta}$, $v_{M_1+4\beta}$. The function u is defined in the following way:

$$u(re^{i\theta}) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r,\theta,t)a(t)\,dt + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r,\theta,t)B_p(t)\,dt & \text{when } p\in(1,2] \\[2ex] \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r,\theta,t)[a(t)-a(0)]\,dt + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r,\theta,t)B_p(t)\,dt + a(0) & \text{when } p\in(2,+\infty) \end{cases} \tag{2.15}$$

(*) $[x]$ denotes the largest integer not exceeding x.

where H and K are defined by the equations

$$\left\{\begin{aligned} H(r, \theta, t) &= (-1)^{M_1} H_0(r, \theta, t) \\ K(r, \theta, t) &= K_0(r, \theta, t) \end{aligned}\right. \tag{2.16}$$

in which M_1 is defined by (2.14) and ν, defined by (2.6) has been replaced in the definition (2.3) of H_0 and K_0 by ν defined by (2.13), (2.14). Finally the function B_p is defined as follows

$$B_p(r) = \begin{cases} -\int_r^{+\infty} b(t)\, dt & \text{if } p \in (1, 2] \\ \int_0^r b(t)\, dt & \text{if } p \in (2, +\infty)\,. \end{cases} \tag{2.17}$$

3. Notations and basic inequalities.

Define the integers m, n_j, $(j = 1, 2, \ldots)$ and q_s, s being a positive (fixed) integer, in the following way:

$$m \equiv m(p, \alpha, \omega) = N_1(p, \alpha, \omega) - M_1(p, \alpha, \omega) \tag{3.1}$$

$$n_j \equiv n_j(p, \alpha, \omega) = N_{j+1}(p, \alpha, \omega) - N_j(p, \alpha, \omega) \tag{3.2}$$

$$q_s \equiv q_s(p, \alpha, \omega) = M_s(p, \alpha, \omega) - N_{s-1}(p, \alpha, \omega) \tag{3.3}$$

where

$$M_j \equiv M_j(p, \alpha, \omega) - \left[\left(j - \frac{2}{p}\right)\frac{\alpha}{\pi} + \frac{\pi - \omega}{\pi}\right] (*) \tag{3.4}$$

$$N_j \equiv N_j(p, \alpha, \omega) = \left[\left(j - \frac{1}{p}\right)\frac{\alpha}{\pi} + \frac{\pi - \omega}{\pi}\right] (*) \tag{3.5}$$

We observe that p, α and ω have the meanings explained in the introduction, while the meanings of the integers m, n_j, q_s (as it shall

---

(*) $[x]$ denotes the largest integer not exceeding x.

be clear from the statement of theorem 2 in section 4 are the following: m is the number of compatibility conditions to be imposed on the data a and b: n_j and q_s are the analogous numbers of conditions to be imposed respectively on $a^{(j)}$, $b^{(j-1)}$ and $a^{(s-1)}$, $b^{(s-2)}$.

Moreover we observe that the following equations hold:

$$
(3.6)\quad m=\begin{cases}\left[\dfrac{\alpha}{p\pi}\right] & \text{if } M_1\leqslant\left(1-\dfrac{2}{p}\right)\dfrac{\alpha}{\pi}+\dfrac{\pi-\omega}{\pi}<M_1+1+\left[\dfrac{\alpha}{p\pi}\right]-\dfrac{\alpha}{p\pi}\\[2ex] \left[\dfrac{\alpha}{p\pi}\right]+1 & \text{if } M_1+1+\left[\dfrac{\alpha}{p\pi}\right]-\dfrac{\alpha}{p\pi}\leqslant\\ & \qquad\leqslant\left(1-\dfrac{2}{p}\right)\dfrac{\alpha}{\pi}+\dfrac{\pi-\omega}{\pi}<M_1+1\end{cases}
$$

$$
(3.7)\quad n_j=\begin{cases}\left[\dfrac{\alpha}{\pi}\right] & \text{if } N_j\leqslant\left(j-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}+\dfrac{\pi-\omega}{\pi}<N_j+1+\left[\dfrac{\alpha}{\pi}\right]-\dfrac{\alpha}{\pi}\\[2ex] \left[\dfrac{\alpha}{\pi}\right]+1 & \text{if } N_j+1+\left[\dfrac{\alpha}{\pi}\right]-\dfrac{\alpha}{\pi}\leqslant\left(j-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}+\dfrac{\pi-\omega}{\pi}<N_j+1\end{cases}
$$

$$
(3.8)\quad q_s=\begin{cases}\left[\dfrac{\alpha}{p'\pi}\right] & \text{if } N_{s-1}\leqslant\left(s-1-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}+\\ & \qquad+\dfrac{\pi-\omega}{\pi}<N_{s-1}+1+\left[\dfrac{\alpha}{p'\pi}\right]-\dfrac{\alpha}{p'\pi}\\[2ex] \left[\dfrac{\alpha}{p'\pi}\right]+1 & \text{if } N_{s-1}+1+\left[\dfrac{\alpha}{p'\pi}\right]-\dfrac{\alpha}{p'\pi}\leqslant\\ & \qquad\leqslant\left(s-1-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}+\dfrac{\pi-\omega}{\pi}<N_{s-1}+1\end{cases}
$$

where p' is the conjugate exponent of p, i.e. $1/p+1/p'=1$. In particular such equations imply the inequalities:

$$
(3.9)\qquad\begin{cases}0\leqslant m\leqslant 2\ ,\\ 0\leqslant n_j\leqslant 2\ ,\\ 0\leqslant q_s\leqslant 2\ .\end{cases}
$$

Moreover, when $\alpha \in (0, \pi)$, the number n_j of compatibility conditions to be imposed on $a^{(j)}$ and $b^{(j-1)}$ is *at most* 1, while, when $\alpha \in (\pi, 2\pi)$, such a number is *at least* 1; analogous remarks may be done for m and q_s.

Define, now, the real numbers $\nu, \varrho, \sigma_j, \tau_j$, $(j = 1, 2, \ldots)$, σ_s^* as follows:

$$\nu \equiv \nu(p, \alpha, \omega) = 2\left(\frac{\omega}{\pi} + M_1\right)\beta + 1 \tag{3.10}$$

$$\varrho \equiv \varrho(p, \alpha, \omega) = \nu(p, \alpha, \omega) + 2\beta \tag{3.11}$$

$$\sigma_j \equiv \sigma_j(p, \alpha, \omega) = 2\left(\frac{\omega}{\pi} + N_j\right)\beta - j + 1 \tag{3.12}$$

$$\tau_j \equiv \tau_j(p, \alpha, \omega) = \sigma_j(p, \alpha, \omega) + 2\beta \tag{3.13}$$

$$\sigma_s^* \equiv \sigma_s^*(p, \alpha, \omega) = 2\left(\frac{\omega}{\pi} + M_s\right)\beta - s + 1\,, \tag{3.14}$$

β being defined by formula (2.5).

We observe that the meaning of such numbers is connected with the compatibility conditions to be imposed on a and b and their derivatives: more precisely the compatibily conditions express the « orthogonality » of linear combinations of a and b and their derivatives to some powers of the variable: the exponents are just the numbers defined above.

From the definitions of m, n_j, q_s, M_j, N_j it is easy to infer that $\nu, \varrho, \sigma_j, \tau_j, \sigma_s^*$ enjoy the properties listed below:

$$\frac{2}{p'} < \nu \leqslant \min\left\{1 + \frac{1}{p'} - 2(m-1)\beta, \frac{2}{p'} + 2\beta\right\} \tag{3.15}$$

$$\frac{2}{p'} + 2\beta < \varrho \leqslant \min\left\{1 + \frac{1}{p'} + 2(2-m)\beta, \frac{2}{p'} + 4\beta\right\} \tag{3.16}$$

$$\frac{1}{p'} < \sigma_j \leqslant \min\left\{1 + \frac{1}{p'} - 2(n_j - 1)\beta, \frac{1}{p'} + 2\beta\right\} \tag{3.17}$$

$$\max\left\{\frac{2}{p'} - 2q_s\beta, \frac{1}{p'}\right\} < \sigma_{s-1} \leqslant \min\left\{\frac{2}{p'} - 2(q_s - 1)\beta, \frac{1}{p'} + 2\beta\right\} \tag{3.18}$$

$$(3.19)\qquad \frac{1}{p'}+2\beta<\tau_j\leqslant\min\left\{1+\frac{1}{p'}+2(2-n_j)\beta,\frac{1}{p'}+4\beta\right\}$$

$$(3.20)\qquad \max\left\{\frac{2}{p'}-2(q_s-1)\beta,\frac{1}{p'}+2\beta\right\}<\tau_{s-1}\leqslant$$
$$\leqslant\min\left\{\frac{2}{p'}+2(2-q_s)\beta,\frac{1}{p'}+4\beta\right\}$$

$$(3.21)\qquad 1-\frac{2}{p}<\sigma_s^*\leqslant 1-\frac{2}{p}+2\beta\,.$$

REMARK 3.1. Necessary conditions for the equalities to hold in inequalities (3.15) and (3.16), (3.17) and (3.19), (3.18) and (3.20), (3.21) are respectively listed in the following points i), ii), iii), iv):

i) either

$$\left(1-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z$$

or

$$\frac{\alpha}{p'\pi}-\frac{\omega}{\pi}\in Z$$

ii) either

$$\left(j-\frac{1}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z$$

or

$$\left(j+1-\frac{1}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z$$

iii) either

$$\left(s-1-\frac{1}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z$$

or

$$\left(s-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z$$

iv)

$$\left(s-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\in Z\,.$$

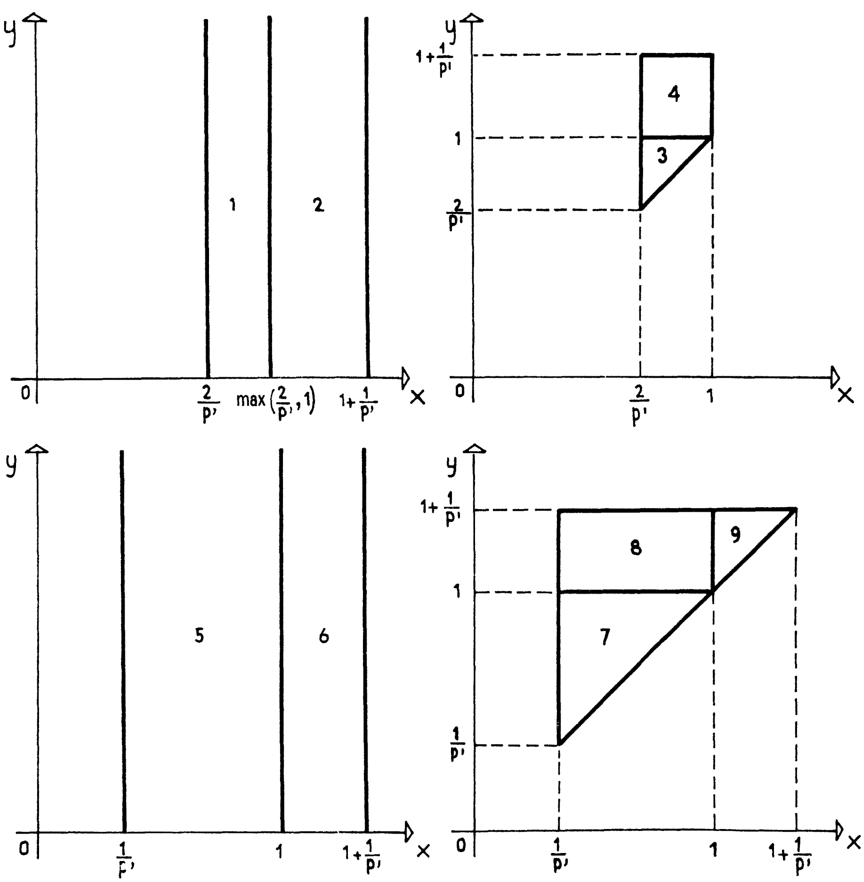
REMARK 3.2. Observe that, when $m=2$, $p\in(1,2)$ and $\nu\in(2/p',1)$. In fact, $m=2$ implies on account of (3.6) that $\alpha\geqslant p\pi$ and, as a consequence of (1.1), that $p\in(1,2)$. Then from inequality (3.15) one infers that

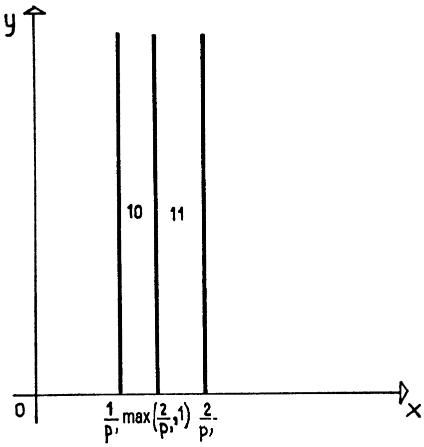
$$\nu<1+\frac{1}{p'}-\frac{\pi}{\alpha}<1+\frac{1}{p'}-\frac{1}{2}<1\,.$$

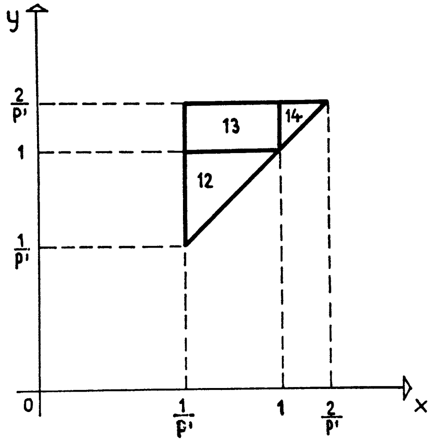
Moreover observe that $q_s = 2$ implies that $\alpha \geqslant p'\pi$ and, as a consequence, that $p \in (2, +\infty)$.

REMARK 3.3. The set of points (p, α, ω) such that $q_s = 2$ and $\tau_{s-1} = 1$ is void: in fact from remark 3.2 we infer that $p > 2$ and from (3.20) we get that $1/p' < 1 - \pi/\alpha$, i.e. $\pi/\alpha < 1/p$, that on account of (1.1) leads to a contradiction.

We go on observing that in the statement of theorem 2 one needs to settle particular relationships among $m, n_j, q_s, \nu, \varrho, \sigma_j, \tau_j$: they can be visualized better by employing the open sets, that we shall denote simply by $1, \ldots, 14$, pictured below:







Finally we list the equations among which the compatibility conditions on a and b are to be picked out:

$C1$ $$\int_0^{+\infty}\left[\frac{1}{\nu-1}\,tb(t)-(-1)^{M_1}a(t)\right]t^{-\nu}\,dt=0$$

$C2$ $$\int_0^{+\infty}\left[\frac{1}{\varrho-1}\,tb(t)+(-1)^{M_1}a(t)\right]t^{-\varrho}\,dt=0$$

$C3$ $$\int_0^{+\infty}\left\{\frac{1}{\nu-1}\,tb(t)-(-1)^{M_1}[a(t)-a(0)]\right\}t^{-\nu}\,dt=0$$

$C4$ $$\int_0^{+\infty}\left\{\frac{1}{\varrho-1}\,tb(t)+(-1)^{M_1}[a(t)-a(0)]\right\}t^{-\varrho}\,dt=0$$

C_j1 $$\int_0^{+\infty}[b^{(j-1)}(t)-(-1)^{N_j}a^{(j)}(t)]\,t^{-\sigma_j}dt=0$$

C_j2 $$\int_0^{+\infty}[b^{(j-1)}(t)+(-1)^{N_j}a^{(j)}(t)]\,t^{-\tau_j}dt=0$$

C_j3 $$\int_0^{+\infty}\{b^{(j-1)}(t)-b^{(j-1)}(0)-(-1)^{N_j}[a^{(j)}(t)-a^{(j)}(0)]\}\,t^{-\sigma_j}dt=0$$

C_j4 $$\int_0^{+\infty}\{b^{(j-1)}(t)-b^{(j-1)}(0)+(-1)^{N_j}[a^{(j)}(t)-a^{(j)}(0)]\}\,t^{-\tau_j}dt=0$$

C_j5 $$b^{(j-1)}(0)-(-1)^{N_j}a^{(j)}(0)=0$$

C_j6 $$b^{(j-1)}(0)+(-1)^{N_j}a^{(j)}(0)=0\,.$$

4. Main results.

In this section we state two theorems: the former assures the existence and the uniqueness in $W^{1,p}(\Omega_\alpha)$ of a solution to problem (1.3), while the latter establishes under what compatibility conditions on a and b the gradient of such a solution belongs to $W^{s-1,p}(\Omega_\alpha)$.

THEOREM 1. *Suppose that a is a function in $W^{1/p',p}(0,+\infty)$ and that $b\in L^p(0,+\infty)$ and that they possess also properties* (1.7). *Moreover suppose that (p,α,ω) is such that*

$$\left(1-\frac{2}{p}\right)\frac{\alpha}{\pi}-\frac{\omega}{\pi}\notin Z\,. \tag{4.1}$$

Then problem (1.3) *admits a unique solution u belonging to $W^{1,p}(\Omega_\alpha)$, which satisfies the estimate*

$$\|Du\|_{L^p(\Omega_\alpha)}\leqslant C\big[|a|_{W^{1/p',p}(0,+\infty)}+\|r^{1/p}b\|_{L^p(0,+\infty)}\big] \tag{4.2}$$

where C is a positive constant depending only on (p,α,ω).

Moreover u is represented by formula (2.15).

THEOREM 2. *Suppose that $s\geqslant 2$ is an assigned integer and that a and b possess properties* (1.6) *and* (1.7). *Moreover suppose that (p,α,ω,s) is such that*

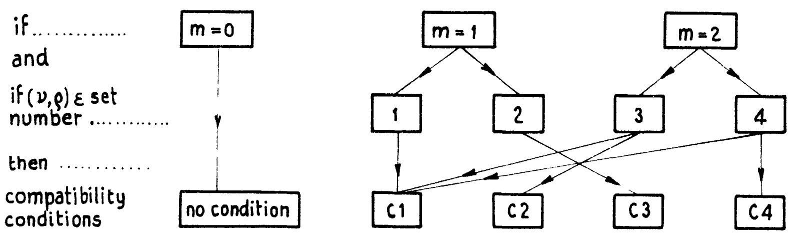
$$\begin{cases}\left(1-\dfrac{2}{p}\right)\dfrac{\alpha}{\pi}-\dfrac{\omega}{\pi}\notin Z\\[2ex] \left(j-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}-\dfrac{\omega}{\pi}\notin Z \qquad j=1,2,\dots,s-1\,.\\[2ex] \left(s-\dfrac{2}{p}\right)\dfrac{\alpha}{\pi}-\dfrac{\omega}{\pi}\notin Z\end{cases} \tag{4.3}$$

Suppose also that

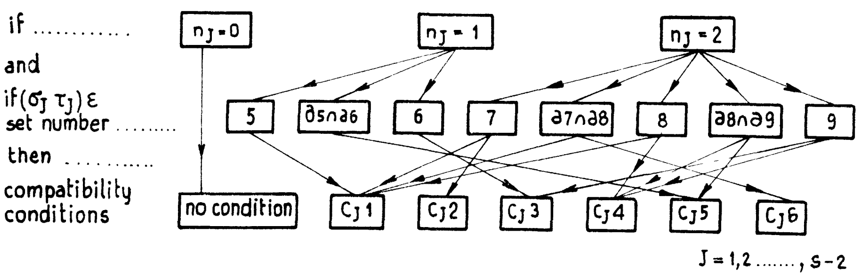
$$(4.4) \qquad \begin{cases} \left(j-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}\notin\dfrac{1}{2}N \qquad j=1,\ldots,s-1 \\ \\ \left(s-\dfrac{1}{p}\right)\dfrac{\alpha}{\pi}\notin\dfrac{1}{2}N \end{cases} \qquad \textit{if } \alpha\in(0,\pi)\ (^*)\,.$$

Then problem (1.3), (1.4) *admits a unique solution, if, and only if, the data a and b verify the compatibility conditions listed in the three following graphs:*

graph 1 (**)



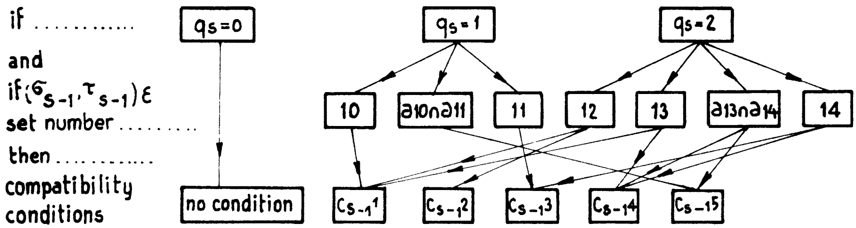
graph 2 (**)



(*) N denotes the set of all positive integers.

(**) $\partial 1$, $\partial 2$ and so on denote the boundaries of sets 1, 2 and so on.

graph 3 (*)



REMARK 4.1. We have to explain in what sense we intend that the equation $(\partial u/\partial(e^{i\omega}))(re^{i0}) = b(r)$ holds in theorem 1. The sense is the following: there exists a sequence $\{u_n\} \subseteq C^\infty(\overline{\Omega}_\alpha - \{(0,0)\}) \cap W^{1,p}(\Omega_\alpha)$ such that $u_n \to u$ in $W^{1,p}(\Omega_\alpha)$ and

$$(1+r)^{1/p} \frac{\partial u_n}{\partial(e^{i\omega})}(re^{i0}) \to (1+r)^{1/p} b \text{ in } L^p(0, +\infty)$$

as $n \to +\infty$. The u_n's may be chosen in the following way: if $\{a_n\}$, $\{b_n\}$ are two sequences in $C_0^\infty([0, +\infty))$ such that $a_n \to a$ in $W^{1/p',p}(0, +\infty)$ and $(1+r)^{1/p} b_n \to (1+r)^{1/p} b$ in $L^p(0, +\infty)$ as $n \to +\infty$, u_n is defined by (2.15) with (a, b) replaced by (a_n, b_n).

REMARK 4.2. We observe that from theorem 2 and definitions (3.1), (3.2), (3.3) it is easy to infer that the index i of problem (1.3), (1.4) is negative and is given by the formula

$$i = -\left(m + \sum_{j=1}^{s-2} n_j + q_s\right) = M_1 - M_s = \left[\left(1 - \frac{2}{p}\right)\frac{\alpha}{\pi} - \frac{\omega}{\pi}\right] - \left[\left(s - \frac{2}{p}\right)\frac{\alpha}{\pi} - \frac{\omega}{\pi}\right].$$

5. Proof of theorem 1.

We observe that, in order to prove that u defined by formula (2.15) is a solution to problem (1.3) belonging to $W^{1,p}(\Omega_\alpha)$, it suffices to show

(*) $\partial 1$, $\partial 2$ and so on denote the boundaries of sets 1, 2 and so on.

that Du belongs to $L^p(\Omega_\alpha)$ and verifies estimate (4.2). In fact, if such properties hold, from the fact that formula (2.15) implies that u is harmonic in Ω_α (since the kernels H and K are so) and therefore, u is a C^∞-function, it is easy to infer that u is the representative of a class in $W^{1,p}(\Omega_\alpha)$ and it assumes the boundary values, recalling the observations in section 2, remark 4.1 and estimate (4.2).

We shall show that $Du \in L^p(\Omega_\alpha)$ by studying the properties of the traces of u on the half lines going out of the origin. We shall denote such traces by u_θ, the parameter θ varying in $[0, \alpha]$; they are so defined:

if $p \in (1, 2]$

$$(5.1)\qquad u_\theta(r) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, 0, t)a(t)\,dt + \dfrac{1}{\alpha}\int_0^{+\infty} K(r, 0, t)B_p(t)\,dt + B_p(r)\cos\omega & \text{if } \theta = 0 \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t)a(t)\,dt + \dfrac{1}{\alpha}\int_0^{+\infty} K(r, \theta, t)B_p(t)\,dt & \text{if } \theta \in (0, \alpha) \\ a(r) & \text{if } \theta = \alpha \end{cases}$$

if $p \in (2, +\infty)$

$$(5.1)\qquad u_\theta(r) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, 0, t)[a(t) - a(0)]\,dt + \\ \quad + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r, 0, t)B_p(t)\,dt + B_p(r)\cos\omega + a(0) & \text{if } \theta = 0 \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t)[a(t) - a(0)]\,dt + \dfrac{1}{\alpha}\int_0^{+\infty} K(r, \theta, t)B_p(t)\,dt + a(0) & \\ & \text{if } \theta \in (0, \alpha) \\ a(r) & \text{if } \theta = \alpha\,. \end{cases}$$

We recall that the functions H, K and B_p are defined respectively by (2.16) and (2.17). Moreover, as far as the function u_0 is concerned, we have to observe two matters: firstly, the second integral appearing in the right hand side of the first equation in (5.1) is to be considered in a Cauchy principal value sense, i.e.

$$(5.2)\qquad \int_0^{+\infty} K(r,0,t)B_p(t)\,dt = \lim_{\varepsilon\to 0} \int_{\{t>0:|r^\beta - t^\beta|>\varepsilon\}} K(r,0,t)B_p(t)\,dt$$

where the singular kernel $K(\cdot, 0, \cdot)$ is given by the formula

$$(5.3)\qquad K(r,0,t) = \frac{r^{\nu-1}t^{2\beta-\nu}}{r^{2\beta}-t^{2\beta}}\sin\omega$$

where ν is defined by (2.13) and (2.14) and β by (2.5); secondly, if a and b belong to $C([0, +\infty))$ and behave well at $+\infty$, then $u_0(r_0)$ is the limit of $u(re^{i\theta})$ as $re^{i\theta}\to r_0e^{i0}$ for every $r_0\in(0, +\infty)$.

The existence of the limit in (5.2) is guaranteed by the identity

$$\int_0^{+\infty} K(r,0,t)B_p(t)\,dt = r^\xi\int_0^{+\infty} K(r,0,t)\frac{t^\xi}{r^\xi}\frac{B_p(t)}{t^\xi}\,dt\,,$$

where

$$(5.4)\qquad \xi = \begin{cases} \dfrac{1}{p'} & \text{if } p\neq 2\\ \eta & \text{if } p=2\,,\end{cases}$$

by the hypotheses made on b, by property (4.1) (which implies, on account of inequality (3.15) and remark 3.1, that $\nu\in(2/p', 2/p'+2\beta)$) and by lemmas 5.1, 5.2, 5.3 and corollary 5.4 stated below.

LEMMA 5.1. *Suppose that* $\beta\in(\frac{1}{4}, +\infty)$ *and* $\gamma\in(1/p', 1/p'+2\beta)$. *Then, if* $f\in L^p(0, +\infty)$, *the function*

$$(5.5)\qquad F(r) = \int_0^{+\infty}\frac{r^{\gamma-1}t^{2\beta-\gamma}}{r^{2\beta}-t^{2\beta}}f(t)\,dt$$

belongs to $L^p(0, +\infty)$ *and satisfies the estimate*

$$\|F\|_{L^p(0,+\infty)}\leqslant C_1\|f\|_{L^p(0,+\infty)}\,,$$

where C_1 *is a positive constant depending only on* (p, β, γ).

PROOF. It follows easily from the change of variable $t \to t^{1/\beta}$ and from the fact that singular integrals of the form

$$G(x) = \int_{-\infty}^{+\infty} \frac{|x/t|^s}{\pi(x-t)} g(t)\, dt ,$$

where g is an even function in $L^p(-\infty, +\infty)$, belong to $L^p(-\infty, +\infty)$ for every $s \in (-1-1/p, 1/p')$ and the linear mapping $g \to G$ is continuous from $L^p(0, +\infty)$ into itself.

LEMMA 5.2. *Suppose that* $(1+r)^{1-\xi} b \in L^p(0, +\infty)$, ξ *being defined by* (5.4). *Then the function* B_p *belongs to* $W^{1/p',p}(0, +\infty)$ *and satisfies the estimates*

$$(5.6) \qquad |B_p(r)| \leqslant r^{\xi - 1/p} \|(1+r)^{1-\xi} b\|_{L^p(0,+\infty)} \; p \in (1, 2] , \qquad r > 0$$

$$(5.7) \qquad |B_p(r) - B_p(s)| \leqslant \|(1+r)^{1/p} b\|_{L^p(0,+\infty)} |s-r|^{1/p'} , \qquad r, s > 0$$

$$(5.8) \qquad |B_p|_{W^{1/p',p}(0,+\infty)} \leqslant \frac{2^{1/p} p}{p-1} \|r^{1/p} b\|_{L^p(0,+\infty)}$$

$$(5.9) \qquad \|r^{-\eta} B_2\|_{L^2(0,+\infty)} \leqslant \frac{2}{1-2\eta} \|(1+r)^{1-\eta} b\|_{L^2(0,+\infty)} .$$

PROOF. Estimates (5.6), (5.7) can be easily obtained by Hölder's inequality, while estimates (5.8), (5.9) are consequences of Hardy's inequality (see [6]): we prove only the former.

$$|B_p|_{W^{1/p',p}(0,+\infty)} = 2^{1/p} \left(\int_0^{+\infty} dr \int_r^{+\infty} \left| \frac{B_p(s) - B_p(r)}{s-r} \right|^p ds \right)^{1/p} =$$

$$= 2^{1/p} \left(\int_0^{+\infty} dr \int_r^{+\infty} \left| \frac{1}{s-r} \int_r^s b(t)\, dt \right|^p ds \right)^{1/p} \leqslant \frac{2^{1/p} p}{p-1} \left(\int_0^{+\infty} dr \int_r^{+\infty} |b(t)|^p dt \right)^{1/p} =$$

$$= \frac{2^{1/p} p}{p-1} \|r^{1/p} b\|_{L^p(0,+\infty)} .$$

Finally we observe that a function with property (5.8) is easily seen to belong to $W^{1/p',p}(0, +\infty)$.

LEMMA 5.3. (Grisvard [5]) *Suppose* $f \in C_0^\infty([0, +\infty))$ *and* $\sigma \in (0, 1)$: *then the following estimates hold*:

$$(5.10) \qquad \left(\int_0^{+\infty} |f(t)|^p t^{-\sigma p} dt\right)^{1/p} \leq \frac{2-\sigma p}{2(1-\sigma p)} |f|_{W^{\sigma,p}(0,+\infty)} \qquad \textit{when } \sigma \in \left(0, \frac{1}{p}\right);$$

$$(5.11) \qquad \left(\int_0^{+\infty} |f(t)-f(0)|^p t^{\sigma p} dt\right)^{1/p} \leq \frac{\sigma p}{2|1-\sigma p|} |f|_{W^{\sigma,p}(0,+\infty)} \quad \textit{when } \sigma \in \left(\frac{1}{p}, 1\right)$$

where

$$|f|_{W^{\sigma,p}(0,+\infty)} = \left(\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy\right)^{1/p}.$$

COROLLARY 5.4. *Estimates* (5.10) *and* (5.11) *hold also for functions* $f \in W^{\sigma,p}(0, +\infty)$.

The next step consists in showing that the functions u_θ satisfy the estimates

$$(5.12)_1 \quad \|r^{-1/p'} u_\theta\|_{L^p(0,+\infty)} \leq C_2[|a|_{W^{1/p',p}(0,+\infty)} + \|r^{1/p} b\|_{L^p(0,+\infty)}] \quad \text{if } p \in (1, 2)$$

$$(5.12)_2 \quad \|r^{-1/p'}[u_\theta - a(0)]\|_{L^p(0,+\infty)} \leq C_2[|a|_{W^{1/p',p}(0,+\infty)} + \|r^{1/p} b\|_{L^p(0,+\infty)}] \quad \text{if } p \in (2, +\infty)$$

$$(5.13) \quad |u_\theta|_{W^{1/p',p}(0,+\infty)} \leq C_3[|a|_{W^{1/p',p}(0,+\infty)} + \|r^{1/p} b\|_{L^p(0,+\infty)}]$$

for every $\theta \in [0, \alpha]$ and for every p, C_2 and C_3 being positive constants depending only on $(\theta, p, \alpha, \omega)$.

We observe that $(5.12)_1$, (5.13) are easy consequences of corollary 5.4, of lemmas 5.1, 5.2 and 5.5, 5.6 (these latter are stated below) and of the following properties of the kernels H and K

$$(5.14) \qquad \begin{cases} H(1, \theta, t)\, t^{1-2/p} \in L^1(0, +\infty) & \text{for every } \theta \in [0, \alpha) \\ K(1, \theta, t)\, t^{1-2/p} \in L^1(0, +\infty) & \text{for every } \theta \in (0, \alpha) \end{cases}$$

(which hold, since $\nu \in (2/p', 2/'p + 2\beta)$, as we have already remarked).

The previous arguments and the formula

$$u_\theta(0) = a(0) \quad \text{for every } \theta \in [0, \alpha] \text{ when } p > 2 \tag{5.15}$$

prove estimate $(5.12)_2$.

LEMMA 5.5. *Let g be a complex-valued function defined in the first quadrant of the plane, let it be homogeneous of degree -1 and such that*

$$C_\sigma = \int_0^{+\infty} |g(1, t)| t^{\sigma - 1/p} \, dt < +\infty$$

for some $\sigma \in [0, 1)$. Then the function F so defined

$$F(r) = \int_0^{+\infty} g(r, t) f(t) \, dt \tag{5.16}$$

verifies the estimates:

$$\|F\|_{L^p(0,+\infty)} \leqslant C_0 \|f\|_{L^p(0,+\infty)} \tag{5.17}$$

$$|F|_{W^{\sigma,p}(0,+\infty)} \leqslant C_\sigma |f|_{W^{\sigma,p}(0,+\infty)} . \tag{5.18}$$

PROOF. It follows easily using the change of variable $t \to rt$ and Minkowski's inequality for integrals.

LEMMA 5.6. *Suppose that $\beta \in (\frac{1}{4}, +\infty)$, $\sigma \in (0, 1)$ and $\gamma \in (\sigma + 1/p', \sigma + 1/p' + 2\beta)$. Then the function F defined by* (5.7) *verifies the estimate*

$$|F|_{W^{\sigma,p}(0,+\infty)} \leqslant c |f|_{W^{\sigma,p}(0,+\infty)} , \tag{5.19}$$

c being a positive constant depending only on $(p, \beta, \gamma, \sigma)$.

We postpone the proof of lemma 5.6 and we prove, on the contrary, formula (5.15): to this purpose we consider the two cases $\theta \neq 0$ and $\theta = 0$. Suppose, first, that $\theta \neq 0$: then (5.15) can be easily derived from (5.1) by recalling properties (5.14) and by taking into account that now the functions a and B_p are Hölder continuous with exponent $1 - 2/p$, since $p > 2$ and they belong to $W^{1/p',p}(0, +\infty)$.

Suppose, now, $\theta = 0$: taking advantage of the fact that B_p is Hölder continuous with exponent $1 - 2/p$, of the fact that $B_p(0) = 0$

and of the following estimates

$$(5.20)\quad \begin{cases} |K(1,0,t)\,B_p(rt)| \leqslant c_1 |B_p|_{W^{1/p',p}(0,+\infty)} r^{1-2/p} |K(1,0,t)|\, t^{1-2/p} \\ |K(1,0,t)\,[B_p(rt)-B_p(t)]| \leqslant \\ \qquad \leqslant c_2 |B_p|_{W^{1/p',p}(0,+\infty)} r^{1-2/p} |K(1,0,t)|\, |t-1|^{1-2/p} \end{cases}$$

(where c_1 and c_2 are positive constants), it is easy to infer the identities

$$(5.21)\quad \int_0^{+\infty} K(r,0,t)\,B_p(t)\,dt = \int_0^{+\infty} K(1,0,t)\,B_p(rt)\,dt =$$
$$= \int_0^{(\frac{1}{2})^{1/\beta}} K(1,0,t)\,B_p(rt)\,dt + \int_{(\frac{1}{2})^{1/\beta}}^{(\frac{3}{2})^{1/\beta}} K(1,0,t)[B_p(rt)-B_p(r)]\,dt +$$
$$+ \int_{(\frac{3}{2})^{1/\beta}}^{+\infty} K(1,0,t)\,B_p(rt)\,dt + B_p(r)\int_{(\frac{1}{2})^{1/\beta}}^{(\frac{3}{2})^{1/\beta}} K(1,0,t)\,dt.$$

We observe that the first three integrals in the last member in (5.21) are ordinary Lebesgue integrals, while the last one is a Cauchy principal value integral.

Taking the limit in (5.21) as $r \to 0$ and recalling (5.20), we obtain that

$$\lim_{r\to 0} \int_0^{+\infty} K(r,0,t)\,B_p(t)\,dt = .0;$$

hence formula (5.15) is proved, the relation

$$\lim_{r\to 0} \int_0^{+\infty} H(r,0,t)[a(t)-a(0)]\,dt = 0$$

being obvious.

Now we are going to show that the gradient of u belongs to $L^p(\Omega_\alpha)$ when $p \neq 2$; then we shall treat the case $p = 2$.

When $p \neq 2$, we have to consider separately the two cases $\alpha \in [\pi, 2\pi)$, $\alpha \in (0, \pi)$.

1-st case: $\alpha \in [\pi, 2\pi)$, $p \neq 2$. We define the functions h_θ depending on the parameter $\theta \in [0, \alpha - \pi]$ (the traces of u along the lines $y = x \operatorname{tg} \theta$) as follows:

$$h_\theta(x) = \begin{cases} u_\theta(x) & x > 0 \\ u_{\theta+\pi}(-x) & x < 0 \,. \end{cases} \tag{5.22}$$

Observe that, on account of estimates (5.12), the functions $x \to |x|^{-1/p'} h_\theta(x)$ and $x \to |x|^{-1/p'}[h_\theta(x) - a(0)]$ belong to $L^p(-\infty, +\infty)$ for every $\theta \in [0, \alpha - \pi]$ respectively for any $p \in (1, 2)$ and any $p \in (2, +\infty)$. Moreover h_θ satisfies the estimate

$$|h_\theta|_{W^{1/p',p}(-\infty,+\infty)} \leqslant C_5\big[|a|_{W^{1/p',p}(0,+\infty)} + \|r^{1/p} b\|_{L^p(0,+\infty)}\big] \tag{5.23}$$

for every $\theta \in [0, \alpha - \pi]$, C_5 being a positive constant depending only on $(\theta, p, \alpha, \omega)$. We remark that (5.23) is a consequence of the estimate

$$\begin{aligned} |h_\theta|_{W^{1/p',p}(-\infty,+\infty)} \leqslant 2|u_\theta|_{W^{1/p',p}(0,+\infty)} + 2|u_{\theta+\pi}|_{W^{1/p',p}(0,+\infty)} + \\ + 2(p-1)^{-1/p'}\|(u_\theta - u_{\theta+\pi}) r^{-1/p'}\|_{L^p(0,+\infty)} \end{aligned} \tag{5.24}$$

and of (5.12).

Now in order to show that the gradient of u is in $L^p(\Omega_\alpha)$ we consider the Poisson integrals of h_0 and $h_{\alpha-\pi}$ related to the half-planes Ω_π and $\Omega_{\alpha-\pi,\alpha} = \{(r\cos\theta, r\sin\theta): 0 < r, \alpha - \pi < \theta < \alpha\}$ and we denote them respectively by v_0 and $v_{\alpha-\pi}$. The above properties for h_θ assure that v_0 and $v_{\alpha-\pi}$ have their gradients in L^p and satisfy the estimates

$$\begin{cases} \|Dv_0\|_{L^p(\Omega_\pi)} \leqslant C_6 |h_0|_{W^{1/p',p}(-\infty,+\infty)} \\ \|Dv_{\alpha-\pi}\|_{L^p(\Omega_{\alpha-\pi,\alpha})} \leqslant C_7 |h_{\alpha-\pi}|_{W^{1/p',p}(-\infty,+\infty)} \end{cases} \tag{5.25}$$

where C_6 and C_7 are positive constants depending respectively only on p and (p, α).

Finally, to conclude, it suffices to observe that u coincides with the v's in the intersections of the domains. This result can be obtained easily by substituting the expressions of h_0 and $h_{\alpha-\pi}$ in the formulas that define v_0 and $v_{\alpha-\pi}$ and interchanging the integrations. In the first case the Poincaré-Bertrand formula (see [10]) is to be used, since singular integrals occur. The result depends on the following formulas, that may be proved by taking, for instance, the Mellin transforms of both members:

$$\frac{1}{\pi}\int_0^{+\infty}\frac{r\sin\theta}{r^2-2rt\cos\theta+t^2}\begin{Bmatrix}H(t,0,s)\\ K(t,0,s)+\cos\omega\end{Bmatrix}dt+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\frac{r\sin\theta}{r^2+2rt\cos\theta+t^2}\begin{Bmatrix}H(t,\pi,s)\\ K(t,\pi,s)\end{Bmatrix}dt=\begin{Bmatrix}H(r,\theta,s)\\ K(r,\theta,s)\end{Bmatrix},$$

$$\frac{1}{\pi}\int_0^{+\infty}\frac{r\sin(\alpha-\theta)}{r^2+2rt\cos(\alpha-\theta)+t^2}\begin{Bmatrix}H(t,\alpha-\pi,s)\\ K(t,\alpha-\pi,s)\end{Bmatrix}dt=$$

$$=\begin{Bmatrix}H(r,\theta,s)-\dfrac{r\sin(\alpha-\theta)}{r^2-2rs\cos(\alpha-\theta)+s^2}\\ K(r,\theta,s)\end{Bmatrix}.$$

To this purpose it is useful to observe that the integrals in the left sides of the previous equations are multiplicative convolutions and that the hypothesis $\alpha\in[\pi,2\pi)$ implies $\nu>0$: this property permits, in turn, to take Mellin transforms with a « weight » belonging to $(-1,1)$.

Finally we remark that from estimates (5.22), (5.25) we get easily estimate (4.2).

PROOF OF LEMMA 5.6. Consider the identity

$$F(r)=\lim_{\varepsilon\to0}\int_{\{t:\,t>0,|1-t^\beta|>\varepsilon r^{-\beta}\}}\frac{t^{2\beta-\gamma}}{1-t^{2\beta}}f(rt)\,dt=\lim_{\varepsilon\to0}\int_{\{t:\,t>0,|1-t^\beta|>\varepsilon\}}\frac{t^{2\beta-\gamma}}{1-t^{2\beta}}f(rt)\,dt$$

which implies

$$\frac{F(r)-F(rv)}{|1-v|^{\sigma+1/p}}r^{-\sigma}=\lim_{\varepsilon\to0}\int_{\{t:\,t>0,|1-t^\beta|>\varepsilon\}}\frac{r^{-\sigma}t^{2\beta-\gamma}}{1-t^{2\beta}}\frac{f(rt)-f(rtv)}{|1-v|^{\sigma+1/p}}dt=\tag{5.26}$$

$$=\lim_{\varepsilon\to0}\int_{\{t:\,t>0,|r^\beta-t^\beta|>\varepsilon r^\beta\}}\frac{r^{\gamma-\sigma-1}t^{2\beta-\gamma+\sigma}}{r^{2\beta}-t^{2\beta}}\frac{f(t)-f(tv)}{|1-v|^{\sigma+1/p}}t^{-\sigma}dt=$$

$$=\lim_{\varepsilon\to0}\int_{\{t:\,t>0;|r^\beta-t^\beta|>\varepsilon\}}\frac{r^{\gamma-\sigma-1}t^{2\beta-\gamma+\sigma}}{r^{2\beta}-t^{2\beta}}\frac{f(t)-f(tv)}{|1-v|^{\sigma+1/p}}t^{-\sigma}dt.$$

Since $|f|_{W^{\sigma,p}(0,+\infty)} < +\infty$, it follows that the function

$$(t, v) \to \frac{f(t) - f(tv)}{|1-v|^{\sigma+1/p}} t^{-\sigma}$$

belongs to $L^p((0, +\infty)\times(0, +\infty))$ and satisfies the equation

$$\int_0^{+\infty}\int_0^{+\infty} \frac{|f(t)-f(tv)|^p}{|1-v|^{1+\sigma p}} t^{-\sigma p}\, dt\, dv = |f|^p_{W^{\sigma,p}(0,+\infty)} .$$

From the hypothesis $\gamma \in (\sigma + 1/p', \sigma + 1/p' + 2\beta)$, from (5.26) and lemma 5.1 we get the estimate

$$\int_0^{+\infty} \frac{|F(r)-F(rv)|^p}{|1-v|^{1+\sigma p}} r^{-\sigma p}\, dr \leqslant c^p \int_0^{+\infty} \frac{|f(t)-f(tv)|^p}{|1-v|^{1+\sigma p}} t^{-\sigma p} dt$$

where c is a constant depending only on $(p, \beta, \gamma, \sigma)$.

Integrating both members of this inequality with respect to v over $(0, +\infty)$ we get (5.19).

2-nd case: $\alpha \in (0, \pi)$, $p \neq 2$. Consider the Dirichlet problem

$$\text{(5.27)} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega_\alpha \\ v(re^{i\alpha}) = a(r) & r > 0 \\ v(re^{i0}) = u_0(r) & r > 0 \\ v \in W^{1,p}(\Omega_\alpha) . \end{cases}$$

An application of theorem 3 in appendix shows that such a problem has a unique solution v satisfying the estimate

$$\text{(5.28)} \qquad \|Dv\|_{L^p(\Omega_\alpha)} \leqslant C_8\left[|a|_{W^{1/p',p}(0,+\infty)} + |u_0|_{W^{1/p',p}(0,+\infty)}\right] ,$$

where C_8 is a constant depending only on (p, α). In fact, under our hypotheses, $(1-2/p)(\alpha/\pi) \notin Z$, a, $u_0 \in W^{1/p',p}(0, +\infty)$ and, on account of formula (5.15), $u_0(0) = a(0)$ when $p > 2$. Moreover from estimates (5.13) and (5.28) we infer that

$$\|Dv\|_{L^p(\Omega_\alpha)} \leqslant C\left[|a|_{W^{1/p',p}(0,+\infty)} + \|r^{1/p} b\|_{L^p(0,+\infty)}\right] ,$$

where C is a positive constant depending only on (p, α, ω). There fore we have only to prove that $v = u$.

If we denote by $H(r, \theta, t, \omega)$ and $K(r, \theta, t, \omega)$ the kernels related to problem (1.3) and by $H(r, \theta, t, 0)$ and $K(r, \theta, t, 0)$ the kernels related to the Dirichlet problem (see appendix), then v is given by the formula

$$(5.29)\qquad v(re^{i\theta}) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t, 0)a(t)\,dt + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r, \theta, t, 0)u_0(t)\,dt \\ \hfill \text{when } p \in (1, 2] \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t, 0)[a(t) - a(0)]\,dt + \\ \qquad + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r, \theta, t, 0)[u_0(t) - u_0(0)]\,dt + a(0) \\ \hfill \text{when } p \in (2, +\infty) \end{cases}$$

Substituting the expression of u_0 in the previous formula and interchanging the integrations (using, when necessary, the Poincaré-Bertrand formula) we obtain the wanted equation $v = u$. In fact, taking advantage, for instance, of the Mellin transformation it is possible to prove the equations:

$$\frac{1}{\alpha}\int_0^{+\infty} K(r, \theta, t, 0)H(t, 0, s, \omega)\,dt = H(r, \theta, s, \omega) - H(r, \theta, s, 0)$$

$$\frac{1}{\alpha}\int_0^{+\infty} K(r, \theta, t, 0)K(t, 0, s, \omega)\,dt = K(r, \theta, s, \omega) - K(r, \theta, s, 0)\cos\omega\,.$$

3-rd case: $\alpha \in (0, 2\pi)$, $p = 2$. Similarly to the previous case, we consider the Dirichlet problem (5.27). Theorem 3 in appendix establishes that such a problem with $p = 2$ has a unique solution v belonging to $W^{1,2}(\Omega_\alpha)$ if, and only if,

$$(5.30)\qquad \int_0^{+\infty} |u_0(r) - a(r)|^2 \frac{dr}{r} < +\infty\,.$$

If such a condition is satisfied we can verify as before that $v = u$. Therefore we have to prove only inequality (5.30). To this purpose we observe that the functions $t \to H(r, 0, t)$ and $t \to K(r, 0, t)$ are integrable over $(0, +\infty)$ (*) and the following equations hold

$$\left.\begin{aligned} &\frac{1}{\alpha}\int_0^{+\infty} H(r, 0, t)\,dt = 1 \\ &\frac{1}{\alpha}\int_0^{+\infty} K(r, 0, t)\,dt = -\cos\omega \end{aligned}\right\} \qquad r \in (0, +\infty)\,.$$

Hence we get, using a simple change of variable in the integrals, the chain of equations

$$\begin{aligned} (5.31) \qquad r^{-\frac12}[u_0(r) - a(r)] &= \frac{r^{-\frac12}}{\alpha}\int_0^{+\infty} H(r, 0, t)[a(t) - a(r)]\,dt + \\ &+ \frac{r^{-\frac12}}{\alpha}\int_0^{+\infty} K(r, 0, t)[B_2(t) - B_2(r)]\,dt = \frac{r^{-\frac12}}{\alpha}\int_0^{+\infty} H(1, 0, t)[a(rt) - a(r)]\,dt + \\ &+ \frac{r^{-\frac12}}{\alpha}\int_0^{+\infty} K(1, 0, t)[B_2(rt) - B_2(r)]\,dt\,. \end{aligned}$$

We recall that from estimates (5.6) and (5.7) in lemma 5.2 it follows that also the last integral is an ordinary Lebesgue integral. Applying Minkowski's inequality for sums and integrals, from (5.31) we infer that

$$\begin{aligned} \left(\int_0^{+\infty} |u_0(r) - a(r)|^2 \frac{dr}{r}\right)^{\frac12} &\le \frac{1}{\alpha}\int_0^{+\infty} |H(1, 0, t)| \left(\int_0^{+\infty} |a(rt) - a(r)|^2 \frac{dr}{r}\right)^{\frac12} dt + \\ &+ \frac{1}{\alpha}\int_0^{+\infty} |K(1, 0, t)| \left(\int_0^{+\infty} |B_2(rt) - B_2(r)|^2 \frac{dr}{r}\right)^{\frac12} dt\,. \end{aligned}$$

(*) The latter function is integrable in the Cauchy principal value sense.

Now we observe that (5.30) is an immediate consequence of lemmas 5.7 and 5.8 stated below and of the fact that the functions $t \to H(1, 0, t) \cdot (1 + |\ln t|)$ and $t \to K(1, 0, t) \ln t$ belong to $L^1(0, +\infty)$. Moreover, from such lemmas we can derive the estimate

$$\left(\int_0^{+\infty} |u_0(r) - a(r)|^2 \frac{dr}{r}\right)^{\frac{1}{2}} \leqslant C_9\left[|a|_{W^{\frac{1}{2},2}(0,+\infty)} + \|r^{\frac{1}{2}} b\|_{L^2(0,+\infty)}\right]$$

where C_9 is a positive constant depending only on (α, ω).

Finally from estimate (6.6) in theorem 3 (see appendix) we infer easily estimate (4.2).

LEMMA 5.7. *Let f be any function in $C_0^\infty([0, +\infty))$. Then the following estimate holds:*

$$\left(\int_0^{+\infty} |f(rt_1) - f(rt_2)|^2 \frac{dr}{r}\right)^{\frac{1}{2}} \leqslant 2^{-\frac{1}{2}} |f|_{W^{\frac{1}{2},2}(0,+\infty)} \left(2 + \left|\ln \frac{t_2}{t_1}\right|\right) \tag{5.32}$$

for every $t_1, t_2 \in (0, +\infty)$.

LEMMA 5.8. *Suppose that $r^{\frac{1}{2}} b \in L^2(0, +\infty)$ and that B_2 is defined by* (2.18). *Then the function*

$$\varphi(t) = \left(\int_0^{+\infty} |B_2(rt) - B_2(r)|^2 \frac{dr}{r}\right)^{\frac{1}{2}} \qquad t \in (0, +\infty)$$

satisfies the estimate

$$|\varphi(t_1) - \varphi(t_2)| \leqslant \|r^{\frac{1}{2}} b\|_{L^2(0,+\infty)} |\ln t_1 - \ln t_2| \tag{5.33}$$

for every $t_1, t_2 \in (0, +\infty)$.

PROOF OF LEMMA 5.7. Consider the identity

$$f(rt_1) - f(rt_2) = F(rt_2) - F(rt_1) + \int_{rt_2}^{rt_1} F(s) \frac{ds}{s} \tag{5.34}$$

where

$$F(r) = \frac{1}{r} \int_0^r [f(s) - f(r)]\, ds .$$

The latter function verifies the following estimate;

$$\|r^{-\frac{1}{2}}F\|_{L^2(0,+\infty)} \leqslant 2^{-\frac{1}{2}}|f|_{W^{1/2,2}(0,+\infty)}; \tag{5.35}$$

in fact using Schwarz' inequality we get that

$$\int_0^{+\infty} r^{-3}\left|\int_0^r [f(s)-f(r)]\,ds\right|^2 dr \leqslant \int_0^{+\infty} r^{-2}\,dr \int_0^r |f(s)-f(r)|^2\,ds \leqslant$$

$$\leqslant \int_0^{+\infty} dr \int_0^r \left|\frac{f(r)-f(s)}{r-s}\right|^2 ds = \frac{1}{2}\,|f|^2_{W^{\frac{1}{2},2}(0,+\infty)}\,.$$

Now simple changes of variables in the integrals and an application of Minkowski's inequality for integrals yield the chain of inequalities

$$\left(\int_0^{+\infty}\left|r^{-\frac{1}{2}}\int_{rt_2}^{rt_1} F(s)\frac{ds}{s}\right|^2 dr\right)^{\frac{1}{2}} = \left(\int_0^{+\infty}\left|r^{-\frac{1}{2}}\int_{t_2}^{t_1} F(rs)\frac{ds}{s}\right|^2 dr\right)^{\frac{1}{2}} \leqslant \tag{5.36}$$

$$\leqslant \left|\int_{t_2}^{t_1}\left(\int_0^{+\infty}|F(rs)|^2\frac{dr}{r}\right)^{\frac{1}{2}}\frac{ds}{s}\right| = \|r^{-\frac{1}{2}}F\|_{L^2(0,+\infty)}\left|\int_{t_2}^{t_1}\frac{ds}{s}\right| \leqslant 2^{-\frac{1}{2}}|f|_{W^{\frac{1}{2},2}(0,+\infty)}\left|\ln\frac{t_2}{t_1}\right|.$$

Finally from (5.34), (5.35), (5.36) it follows easily estimate (5.32).

PROOF OF LEMMA 5.8. From Minkowski's inequality for sums and integrals and from simple changes of variables in the integrals we get the chain of inequalities

$$|\varphi(t_1)-\varphi(t_2)| \leqslant \left(\int_0^{+\infty}|B_2(rt_1)-B_2(rt_2)|^2\frac{dr}{r}\right)^{\frac{1}{2}} = \left(\int_0^{+\infty}\left|\int_{rt_1}^{rt_2} b(s)\,ds\right|^2\frac{dr}{r}\right)^{\frac{1}{2}} =$$

$$= \left(\int_0^{+\infty}\left|\int_{t_1}^{t_2} r^{\frac{1}{2}}b(rs)\,ds\right|^2 dr\right)^{\frac{1}{2}} \leqslant \left|\int_{t_1}^{t_2}\left(\int_0^{+\infty}|b(rs)|^2 r\,dr\right)^{\frac{1}{2}} ds\right| =$$

$$= \|r^{\frac{1}{2}}b\|_{L^2(0,+\infty)}\left|\int_{t_1}^{t_2}\frac{ds}{s}\right| = \|r^{\frac{1}{2}}b\|_{L^2(0,+\infty)}|\ln t_2 - \ln t_1|$$

which proves (5.33).

Finally, as far as the uniqueness of the solution to problem (1.3) is concerned, we begin by observing that, if u is a function in $W^{1,p}(\Omega_\alpha)$, then the following estimates hold:

$$
(5.37)\quad
\begin{cases}
\left(\displaystyle\int_{\Omega_\alpha} |u(x, y)|^p \frac{dx\,dy}{(x^2+y^2)^{p/2}}\right)^{1/p} \leqslant \dfrac{p}{2-p}\, \|Du\|_{L^p(\Omega_\alpha)} & \text{when } p \in (1, 2) \\[2ex]
\left(\displaystyle\int_{\Omega_\alpha} |u(x, y)-u(0,0)|^p \frac{dx\,dy}{(x^2+y^2)^{p/2}}\right)^{1/p} \leqslant \dfrac{p}{p-2}\, \|Du\|_{L^p(\Omega_\alpha)} & \text{when } p \in (2, +\infty)\,.
\end{cases}
$$

They are easy consequences of the fact that every function in $W^{1,p}(\Omega_\alpha)$ admits a trace at $(0, 0)$ for $p > 2$ and of the representation formulas

$$
u(x, y) =
\begin{cases}
-\displaystyle\int_1^{+\infty} \left[x \frac{\partial u}{\partial x}(tx, ty) + y \frac{\partial u}{\partial y}(tx, ty)\right] dt & \text{when } p \in (1, 2) \\[2ex]
\displaystyle\int_0^1 \left[x \frac{\partial u}{\partial x}(tx, ty) + y \frac{\partial u}{\partial y}(tx, ty)\right] dt + u(0, 0) & \text{when } p \in (2, +\infty)\,.
\end{cases}
$$

On the contrary, when $p = 2$, taking advantage of properties (1.7) we can derive from formula (2.15) that the function $r \to r^{-\eta} u(re^{i\theta})$ belongs to $L^2(0, +\infty)$ for every $\theta \in (0, \alpha)$, η being the same as in formula (1.7).

Now, if u is a solution to problem (1.3) with the aforementioned properties, it is possible to define the Mellin transform U with respect to r respectively of u, when $p \in (1, 2]$, and of $u - u(0, 0)$, when $p \in (2, +\infty)$. It is also possible to show in a strict way that, if (p, α, ω) satisfies property (4.1), then U is given by formula (2.3), where A denotes the Mellin transform respectively of a, when $p \in (1, 2]$, and of $a - a(0)$, when $p \in (2, +\infty)$. Such a formula implies immediately the uniqueness of a solution to problem (1.3) belonging to $W^{1,p}(\Omega_\alpha)$.

6. Appendix.

In this section we state a regularity theorem for the solution to a Dirichlet problem in an angle. Since it overlaps partially a similar theorem by Merigot [8], we omit its proof.

Our problem is to find a function u such that

$$
(6.1)\qquad \begin{cases} \Delta u = 0 & \text{in } \Omega_\alpha \\ u(re^{i\alpha}) = a(r) & r > 0 \\ u(re^{i0}) = b(r) & r > 0 \\ u \in W^{1,p}_{L}(\Omega_\alpha) \\ Du \in W^{s-1,p}(\Omega_\alpha) \end{cases}
$$

where $s \geqslant 1$ is an assigned integer.

As far as the boundary conditions are concerned, we assume that a and b are functions enjoying the following properties:

$$
(6.2)\qquad a, b \in W^{1/p',p}(0, +\infty);
$$

if $s \geqslant 2$, we assume also that

$$
(6.3)\qquad a', b' \in W^{s-1-1/p,p}(0, +\infty).
$$

Moreover, if $p = 2$, we suppose that

$$
(6.4)\qquad r^{-\eta}a, r^{-\eta}b \in L^2(0, +\infty) \quad \text{for some } \eta \in \left(\max\left(0, \frac{1}{2} - \frac{\pi}{\alpha}\right), \frac{1}{2}\right).
$$

We observe that, also for problem (6.1), the numbers m, n_j, q_s, ν (and so on), defined in section 3, are of use, provided that in their definition ω is replaced by 0. Also sets $1, \ldots, 14$ are the same, while the compatibility conditions on a and b are somewhat different. We list them:

$$
C1 \qquad \int_0^{+\infty} [b(t) - (-1)^{M_1} a(t)] t^{-\nu}\, dt = 0
$$

$$
C2 \qquad \int_0^{+\infty} [b(t) + (-1)^{M_1} a(t)] t^{-\varrho}\, dt = 0
$$

$$
C3 \qquad \int_0^{+\infty} \{b(t) - b(0) - (-1)^{M_1}[a(t) - a(0)]\} t^{-\nu}\, dt = 0
$$

$C4 \qquad \int_0^{+\infty} \{b(t) - b(0) + (-1)^{M_1}[a(t) - a(0)]\} t^{-\varrho} dt = 0$

$C_j 1 \qquad \int_0^{+\infty} [b^{(j)}(t) - (-1)^{N_j} a^{(j)}(t)] t^{-\sigma_j} dt = 0$

$C_j 2 \qquad \int_0^{+\infty} [b^{(j)}(t) + (-1)^{N_j} a^{(j)}(t)] t^{-\tau_j} dt = 0$

$C_j 3 \qquad \int_0^{+\infty} \{b^{(j)}(t) - b^{(j)}(0) - (-1)^{N_j}[a^{(j)}(t) - a^{(j)}(0)]\} t^{-\sigma_j} dt = 0$

$C_j 4 \qquad \int_0^{+\infty} \{b^{(j)}(t) - b^{(j)}(0) + (-1)^{N_j}[a^{(j)}(t) - a^{(j)}(0)]\} t^{-\tau_j} dt = 0$

$C_j 5 \qquad b^{(j)}(0) - (-1)^{N_j} a^{(j)}(0) = 0$

$C_j 6 \qquad b^{(j)}(0) + (-1)^{N_j} a^{(j)}(0) = 0 .$

THEOREM 3. *Suppose that* $s = 1$, *that* a *and* b *possess properties* (6.2), (6.4) *and that*

$$\left(1 - \frac{2}{p}\right)\frac{\alpha}{\pi} \notin Z \qquad \text{when } p \neq 2 .$$

Then, if $p \in (1, 2)$, *problem* (6.1) *admits a unique solution for every pair* (a, b); *while, if* $p \in [2, +\infty)$, *it admits a unique solution if, and only if, the following compatibility conditions are verified*:

$$\int_0^{+\infty} |a(r) - b(r)|^2 \frac{dr}{r} < +\infty \qquad \text{when } p = 2$$

$$a(0) = b(0) \qquad \text{when } p \in (2, +\infty) .$$

If such conditions are satisfied, then u *verifies the estimates*:

if $p \neq 2$,

$$\|Du\|_{L^p(\Omega_\alpha)} \leqslant C_1[|a|_{W^{1/p',p}(0,+\infty)} + |b|_{W^{1/p',p}(0,+\infty)}] \tag{6.5}$$

if $p = 2$,

$$\|Du\|_{L^2(\Omega_\alpha)} \leqslant C_2 \left[|a|_{W^{\frac12,2}(0,+\infty)} + |b|_{W^{\frac12,2}(0,+\infty)} + \left(\int_0^{+\infty} |a(r) - b(r)|^2 \frac{dr}{r}\right)^{\frac12}\right] \tag{6.6}$$

where C_1 and C_2 are positive constants depending respectively only on (p, α) and α.

Moreover the solution u can be represented in polar co-ordinates as follows:

$$u(re^{i\theta}) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t) a(t)\, dt + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r, \theta, t) b(t)\, dt & \text{when } p \in (1, 2] \\ \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} H(r, \theta, t)[a(t) - a(0)]\, dt + \\ \qquad + \dfrac{1}{\alpha}\displaystyle\int_0^{+\infty} K(r, \theta, t)[b(t) - b(0)]\, dt + a(0) & \text{when } p \in (2, +\infty) \end{cases}$$

where the kernels H and K are defined by formulas (2.16) *with ω replaced by* 0.

THEOREM 4. *Suppose that $s \geq 2$ is an assigned integer and that a and b possess properties* (6.2), (6.3), (6.4). *Suppose also that (p, α) is such that*

$$\begin{aligned} &\left(1 - \frac{2}{p}\right)\frac{\alpha}{\pi} \notin Z \quad \text{when } p \neq 2 \\ &\left(j - \frac{1}{p}\right)\frac{\alpha}{\pi} \notin N \quad j = 1, \ldots, s-1 \\ &\left(s - \frac{2}{p}\right)\frac{\alpha}{\pi} \notin N\,. \end{aligned} \tag{6.7}$$

Moreover suppose that

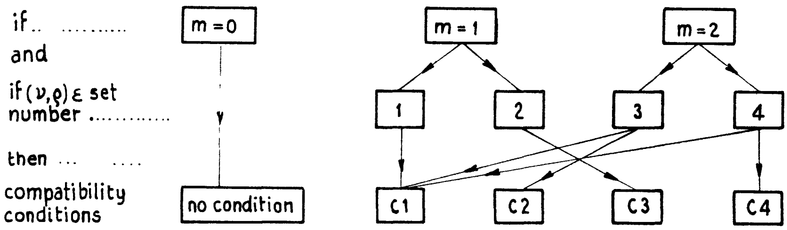
$$\left.\begin{aligned} &\left(j - \frac{1}{p}\right)\frac{\alpha}{\pi} \notin \frac{1}{2} N \quad j = 1, \ldots, s-1 \\ &\left(s - \frac{2}{p}\right)\frac{\alpha}{\pi} \notin \frac{1}{2} N \end{aligned}\right. \quad \text{if } \alpha \in (0, \pi)\,. \tag{6.8}$$

Then problem (6.1) *admits a unique solution, if, and only if,*

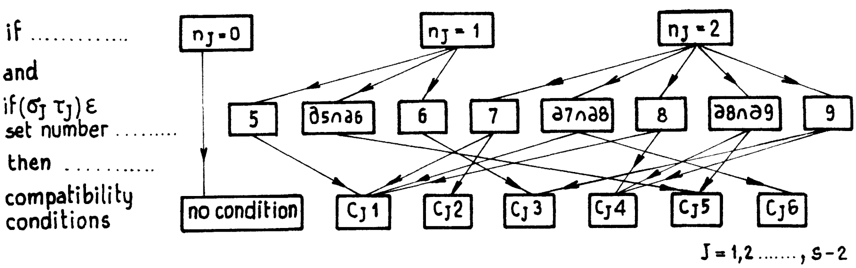
$$a(0) = b(0)$$

and, in addition, the functions a and b verify the compatibility conditions listed respectively in the three following graphs:

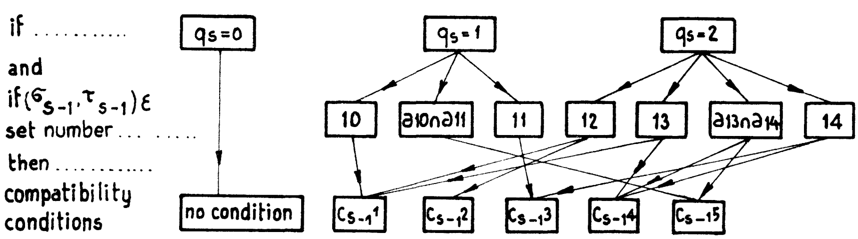
graph 1 (*)



graph 2 (*)

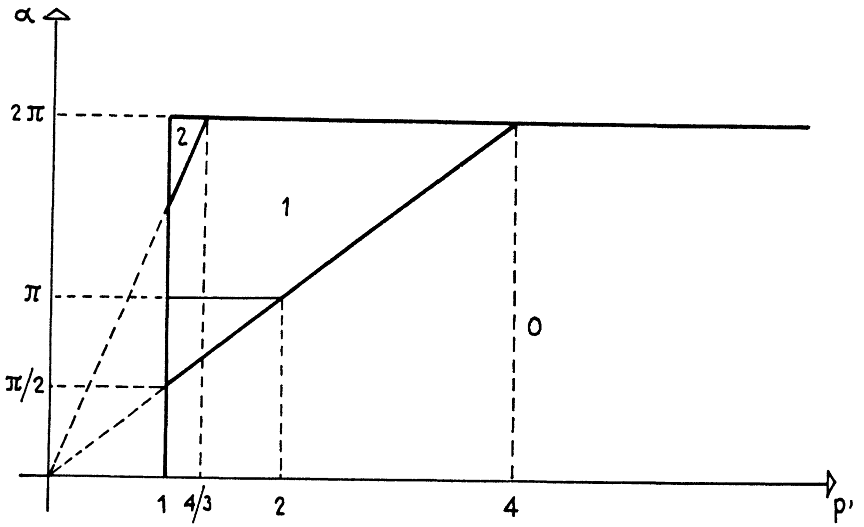


graph 3 (*)



(*) $\partial 1$, $\partial 2$ and so on denote the boundaries of sets 1, 2 and so on.

REMARK 6.1. Consider the three open sets 0, 1, 2 defined by the picture drawn below. The meaning of such sets is the following: the



regions 0, 1, 2 contain the set of points (p', α) such that (p, α) satisfies properties (6.7) and (6.8) with $s = 2$ and the Dirichlet problem (6.1), corrisponding to $s = 2$, Ω_α and p, admits a unique solution, if, and only if, the data a and b satisfy respectively 0, 1, 2 compatibility conditions.

REFERENCES

[1] A. AVANTAGGIATI - M. TROISI, *Spazi di Sobolev con peso e problemi ellittici in un angolo - I*, Ann. Mat. Pura Appl., **95** (1973), pp. 361-408; *II*, Ann. Mat. Pura Appl., **97** (1973), pp. 207-252; *III*, Ann. Mat. Pura Appl., **99** (1974), pp. 1-64.

[2] C. BAIOCCHI - V. COMINCIOLI - E. MAGENES - G. A. POZZI, *Free boundary problems in the theory of fluid flow through porous media: existence and uniqueness theorems*, Ann. Mat. Pura Appl., **97** (1973), pp. 1-82.

[3] P. GRISVARD, *Problème de Dirichlet dans un cone*, Ricerche Mat., **20** (1971), pp. 175-192.

[4] P. GRISVARD, *Alternative de Fredholm relative au problème de Dirichlet dans un polygone ou un polyèdre*, Boll. Un. Mat. Ital. (4), **5** (1972), pp. 132-164.

[5] P. GRISVARD, *Spazi di tracce e applicazioni*, Rend. Mat., **5** (1972) pp. 637-729.

[6] HARDY - LITTLEWOOD - POLYA, *Inequalities*, Cambridge (1964).

[7] A. LORENZI, *A mixed problem for the Laplace equation in a right angle with an oblique derivative given on a side of the angle*, Ann. Mat. Pura Appl., **100** (1974), pp. 259-306

[8] M. MERIGOT, *Régularité de dérivées de la solution du problème de Dirichlet dans un secteur plan*, Matematiche, **27** (1972), pp. 324-359.

[9] M. MERIGOT, *Etude du problème $\Delta u = f$ dans un polygone plan. Inegalités a priori* Boll. Un. Mat. Ital. (4) **10** (1974), pp. 577-597.

[10] N. I. MUSKHELISHVILI, *Singular integral equations*, P. Noordhoff (1953).

[11] M. P. VAN OUVERKERK - DIJKERS, *Solutions of the equation of Helmholtz in a quarter plane with an oblique boundary condition*, Mathematisch Centrum Amsterdam (1970).

[12] J. PEETRE, *Espaces d'interpolation et théorème de Sobolev*, Ann. Inst. Fourier Grenoble, **16**, 1 (1966), pp. 279-317.

[13] E. SHAMIR, *Mixed boundary value problems for elliptic equations in the plane. The L^p theory*, Ann. Scuola Norm. Sup. Pisa, **17** (1963), pp. 117-139.

[14] E. SHAMIR, *Reduced Hilbert transforms and singular integral equations* J. Analyse Math., **12** (1964), pp. 277-305.

[15] E. C. TITCHMARSH, *Theory of Fourier integrals* (Oxford University Press (1937).

[16] E. A. VOLKOV, *Differentiability properties of solutions of boundary value problems for the Laplace equation on a polygon*, Trudy Mat. Inst. Steklov, **77** (1965), pp. 113-142; English translation: Proc. Steklov Inst, Math. (1967), pp. 127-159.

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