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Number of Indecomposable Summands in Direct Decompositions of Torsion-Free Abelian Groups.

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Since B. Jónsson [4] has discovered that there is no uniqueness in the direct decompositions of torsion-free abelian groups of finite rank into indecomposable summands, several papers have dealt with the pathologies of these decompositions (see e.g. Corner [1], Fuchs and Loonstra [3], Fuchs [2, § 90]). Though the recent result by Lady [5] (viz. a torsion-free abelian group of finite rank can have but a finite number of non-isomorphic direct decompositions) shows that there is some limitation for the direct decompositions, a number of intriguing questions are still left unanswered. Here we wish to explore the problem concerning the numbers of indecomposable summands in direct decompositions of a group. (We write « group » to mean « torsion-free abelian group », and by « indecomposable summand » we always mean a nonzero one).

In [2, Theorem 90.1], it has been shown that for every integer $n \geqslant 2$, there is a group of rank $2n$ which decomposes into the direct sum of two as well as $n+1$ indecomposable summands. A few years ago, L. N. Campbell (then a graduate student at Tulane) constructed a group which can be decomposed into any number of indecomposable

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summands between $m+1$ and $2m$ (for any positive integer m). Combining groups like those in [2, Theorem 90.1] it is easy to find groups having decompositions into as many different numbers of indecomposable summands as desired. Modifying the example to some extent, to any $n \geqslant 2$, a group of rank $2n$ can be constructed which decomposes into the direct sum of k indecomposable summands where k is any integer such that $2 \leqslant k \leqslant n+1$ (Theorem 1).

Our goal here is a bit more ambitious: we want to settle the general question: given any finite set N of integers $\geqslant 2$, does there exist a group of finite rank which has a decomposition into k indecomposable summands exactly if $k \in N$? (For obvious reasons, if $1 \in N$, then N ought to be a singleton, otherwise no such group can exist.) Our Theorem 2 answers this question in the affirmative. This surprising fact is another convincing evidence of the pathological behavior of direct decompositions of finite rank groups.

With some extra effort we can settle the analogous question for groups of countable rank by showing that to every infinite set N of integers $\geqslant 2$ there exists a group of countable rank which has a direct decomposition into k indecomposable summands exactly if $k \in N$.

For unexplained terminology and basic facts on torsion-free groups, we refer to [2, Chapter XIII].

1. Let k, n be positive integers such that $k \leqslant n$, and $p_1, \ldots, p_n$, q, $r_2, \ldots, r_n$ primes such that equality can occur only among the r's. The symbol A_{nk} will throughout denote the following torsion-free group of rank $n+k$:

$$A_{nk} = \langle p_1^{-\infty} a_1 \rangle \oplus \ldots \oplus \langle p_k^{-\infty} a_k \rangle \oplus \langle p_1^{-\infty} b_1, \ldots, p_n^{-\infty} b_n, \\ q^{-1} r_j^{-1}(b_i - b_j) \text{ for } 1 \leqslant i < j \leqslant n \rangle .$$

Thus A_{nk} is contained in the vector space over the rationals with a basis $\{a_1, \ldots, a_k, b_1, \ldots, b_n\}$ and is as a group generated by $p_i^{-m} a_i$ $(i = 1, \ldots, k;\ m = 1, 2, \ldots)$, $p_j^{-m} b_j$ $(j = 1, \ldots, n;\ m = 1, 2, \ldots)$, $q^{-1} r_j^{-1}(b_i - b_j)$ $(1 \leqslant i < j \leqslant n)$. Standard argument can convince the reader at once that the last summand of A_{nk} is indecomposable.

Lemma 1. *If the intersection* $\{r_2, \ldots, r_k\} \cap \{r_{k+1}, \ldots, r_n\}$ *is empty, then there is a decomposition*

$$A_{nk} = C \oplus D$$

where C, D *are indecomposable of rank* n *and* k, *respectively.*

We choose integers s, t, u, v (to be further specified later) such that

$$sv - tu = 1 \tag{1}$$

and set

$$c_i = sa_i + tb_i\,, \qquad d_i = ua_i + vb_i \qquad \text{for } 1 \leqslant i \leqslant k\,.$$

Then

$$a_i = vc_i - td_i\,, \qquad b_i = -uc_i + sd_i \qquad \text{for } 1 \leqslant i \leqslant k\,,$$

and obviously $\langle p_i^{-\infty} a_i \rangle \oplus \langle p_i^{-\infty} b_i \rangle = \langle p_i^{-\infty} c_i \rangle \oplus \langle p_i^{-\infty} d_i \rangle$ for $1 \leqslant i \leqslant k$. We set

$$C = \langle p_1^{-\infty} c_1, \ldots, p_k^{-\infty} c_k, p_{k+1}^{-\infty} b_{k+1}, \ldots, p_n^{-\infty} b_n;\ q^{-1} r_j^{-1} u(c_i - c_j),$$
$$q^{-1} r_l^{-1}(uc_j + b_l),\ q^{-1} r_m^{-1}(b_l - b_m) \qquad \text{for } 1 \leqslant i < j \leqslant k < l < m \leqslant n \rangle\,,$$

$$D = \langle p_1^{-\infty} d_1, \ldots, p_k^{-\infty} d_k;\ q^{-1} r_j^{-1} s(d_i - d_j) \qquad \text{for } 1 \leqslant i < j \leqslant k \rangle\,,$$

and want to choose s, t, u, v to have C and D as stated.

To ensure $C \leqslant A_{nk}$, the following divisibility relations must hold in A_{nk}:

$$qr_j | u(c_i - c_j) = usa_i - usa_j + ut(b_i - b_j)\,,$$
$$qr_l | uc_j + b_l = usa_j + (ut + 1) b_j + (b_l - b_j)$$

for all $1 \leqslant i < j \leqslant k < l \leqslant n$. Hence $qr_j | us$ and $qr_l | us$, $ut + 1 = vs$, and since by (1), $(u, v) = 1$, we obtain $qr_l | s$. Thus the following conditions must be satisfied:

$$r_j | us\,, \qquad qr_l | s \qquad \text{for } j \leqslant k < l\,. \tag{2}$$

For $D \leqslant A_{nk}$ to hold, we must have the divisibility relation in A_{nk}:

$$qr_j | s(d_i - d_j) = sua_i - sua_j + sv(b_i - b_j) \qquad \text{for } i < j \leqslant k$$

which is satisfied whenever (2) holds.

Conditions (1) and (2) already guarantee the inclusion $A_{nk} \leqslant C + D$, as is clear from

$$q^{-1}r_j^{-1}(b_i - b_j) = -q^{-1}r_j^{-1}u(c_i - c_j) + q^{-1}r_j^{-1}s(d_i - d_j)\,,$$
$$q^{-1}r_l^{-1}(b_j - b_l) = -q^{-1}r_l^{-1}(uc_j + b_l) + q^{-1}r_l^{-1}sd_j$$

where $j \leqslant k < l$. Consequently, $A = C \oplus D$ if (1) and (2) are satisfied. If we set

$$u = [r_2, \dots, r_k]\,, \qquad s = q \cdot [r_{k+1}, \dots, r_n]\,,$$

then (2) holds and $(u, s) = 1$ permits us to choose t, v so as to satisfy (1).

The choice of u, s makes it possible to replace $q^{-1}r_j^{-1}u(c_i - c_j)$ and $q^{-1}r_j^{-1}s(d_i - d_j)$ by $q^{-1}(c_i - c_j)$ and $r_j^{-1}(d_i - d_j)$, respectively, in the definition of C and D. The indecomposability of C and D can be proved in a straightforward manner.

Remark. Lemma 1 continues to hold if from the set of generators of A_{nk} we drop those $q^{-1}r^{-1}(b_i - b_j)$ for which $i > 1$.

This yields a somewhat simpler example to establish the next theorem; however, the A_{nk}'s are needed for Theorem 2.

Theorem 1. *Let n be any positive integer. There exists a torsion-free group of finite rank* (e.g. $2n$) *which can be decomposed into the direct sum of k indecomposable summands, for any k with $2 \leqslant k \leqslant n+1$.*

Pick distinct primes $r_2, \dots, r_n$, and consider $A = A_{nn}$. For every k $(2 \leqslant k \leqslant n+1)$, $A_{n,n-k+2}$ is a summand of A_{nn} which can be written, in view of Lemma 1, as a direct sum of 2 indecomposable groups. This decomposition, together with the summand $\langle p_{n-k+3}^{-\infty}a_{n-k+3}\rangle \oplus \oplus \dots \oplus \langle p_n^{-\infty}a_n\rangle$, yields a decomposition of A into the direct sum of precisely k indecomposable summands.

2. Next we proceed to the general problem mentioned in the introduction, namely, to find groups in whose direct decompositions the numbers of indecomposable summands form a prescribed finite set of integers $\geqslant 2$. In our study, it will be relevant to exclude certain decompositions. To this end, we require a survey of direct decompositions of the following group:

$$G = \langle p_1^{-\infty}a_1\rangle \oplus \dots \oplus \langle p_n^{-\infty}a_n\rangle \oplus \langle p_1^{-\infty}b_1, \dots, p_n^{-\infty}b_n, \; q^{-1}r_2^{-1}(b_1 - b_2), \dots, q^{-1}r_n^{-1}(b_1 - b_n)\rangle$$

where $p_1, \dots, p_n, q$ are different primes, and $r_2, \dots, r_n$ are primes different from the preceding ones.

For the sake of convenience, we introduce the notations (used frequently in the sequel):

$$F_i = \langle p_i^{-\infty} a_i \rangle \oplus \langle p_i^{-\infty} b_i \rangle , \qquad F = \bigoplus_{i=1}^{n} F_i \tag{3}$$

which are fully invariant subgroups of G.

LEMMA 2. *The only decompositions of G into indecomposable summands are of the form*

$$G = C \oplus D \oplus E_{k+1} \oplus \dots \oplus E_n$$

for some k $(1 \leqslant k \leqslant n)$ where

(i) *$C, D, E_{k+1}, \dots, E_n$ are indecomposable, the E's are of rank* 1;

(ii) *for every i, there are decompositions*

$$F_i = \langle p_i^{-\infty} c_i \rangle \oplus \langle p_i^{-\infty} d_i \rangle \tag{4}$$

and a permutation $i_2, \dots, i_n$ of $2, \dots, n$ such that

$$\langle p_1^{-\infty} c_1 \rangle \oplus \dots \oplus \langle p_n^{-\infty} c_n \rangle \leqslant C ,$$
$$\langle p_1^{-\infty} d_1 \rangle \oplus \langle p_{i_2}^{-\infty} d_{i_2} \rangle \oplus \dots \oplus \langle p_{i_k}^{-\infty} d_{i_k} \rangle \leqslant D ,$$
$$\langle p_{i_{k+1}}^{-\infty} d_{i_{k+1}} \rangle = E_{k+1}, \dots, \langle p_{i_n}^{-\infty} d_{i_n} \rangle = E_n ;$$

(iii) *$C/(C \cap F)$ is a direct sum of $n-1$ cyclic groups of order q and cyclic groups of orders $r_{i_{k+1}}, \dots, r_{i_n}$, while $D/(D \cap F)$ is a direct sum of cyclic groups of orders $r_{i_2}, \dots, r_{i_k}$;*

(iv) *none of $r_{i_2}, \dots, r_{i_k}$ equals any of $r_{i_{k+1}}, \dots, r_{i_n}$.*

Since $\langle p_i^{-\infty} a_i \rangle$ are summands of G, none of F_i can be contained in an indecomposable summand of G. Therefore, in any decomposition of G into indecomposable summands, F_1 intersects exactly two summands, say C and D. Accordingly, we write $G = C \oplus D \oplus E$ where the summands C, D are indecomposable. ($E = 0$ is not excluded.) If, for some $i \geqslant 2$, $b_i \in E$, then $b_1 \in C \oplus D$, $q^{-1}(b_1 - b_i) \in G$,

$q^{-1}b_1 \notin G$ leads to a contradiction. Thus $b_2, \ldots, b_n \notin E$, whence $(C \oplus D) \cap \cap F_i \neq 0$ for every i. Define a partition $\{2, \ldots, n\} = I_1 \cup I_2$ by putting $i \in I_1$ if $F_i \leqslant C \oplus D$ and $i \in I_2$ otherwise (i.e. if $E \cap F_i \neq 0$); and write $I_1 = \{i_2, \ldots, i_k\}$, $I_2 = \{i_{k+1}, \ldots, i_n\}$.

From what has been said it is clear that in the decomposition

$$F_i = (C \cap F_i) \oplus (D \cap F_i) \oplus (E \cap F_i) \qquad (i = 1, \ldots, n)$$

precisely one summand is 0 and the other two are of rank 1. Note that any decomposition of F_i is of the form (4) where c_i, d_i can be chosen such that

$$c_i = s_i a_i + t_i b_i\,, \qquad d_i = u_i a_i + v_i b_i$$

and s_i, t_i, u_i, v_i are integers satisfying $s_i v_i - t_i u_i = 1$. We quickly note that

$$a_i = v_i c_i - t_i d_i\,, \qquad b_i = -u_i c_i + s_i d_i\,.$$

Hence for suitable choices of c_i, d_i, the following inclusions are evident:

$$\langle p_1^{-\infty} c_1\rangle \oplus \langle p_{i_2}^{-\infty} c_{i_2}\rangle \oplus \ldots \oplus \langle p_{i_k}^{-\infty} c_{i_k}\rangle \leqslant C\,,$$
$$\langle p_1^{-\infty} d_1\rangle \oplus \langle p_{i_2}^{-\infty} d_{i_2}\rangle \oplus \ldots \oplus \langle p_{i_k}^{-\infty} d_{i_k}\rangle \leqslant D\,,$$
$$E' = \langle p_{i_{k+1}}^{-\infty} d_{i_{k+1}}\rangle \oplus \ldots \oplus \langle p_{i_n}^{-\infty} d_{i_n}\rangle \leqslant E\,,$$

while each of $\langle p_i^{-\infty} c_i\rangle$ $(i \in I_2)$ is contained either in C or in D.

Our next goal is to show that $E' = E$. It suffices to verify that $G' = C \oplus D \oplus E'$ contains all the generators of G. Manifestly, $F \leqslant G'$ holds. Now, if $i \in I_1$, then F_1, $F_i \leqslant C \oplus D$ implies that the pure subgroup generated by $F_1 + F_i$ is contained in $C \oplus D$, so $q^{-1} r_i^{-1}(b_1 - b_i) \in \in C \oplus D \leqslant G'$. If $i \in I_2$, then $b_1 - b_i = -u_1 c_1 + s_1 d_1 + u_i c_i - s_i d_i$ with $-u_1 c_1 + s_1 d_1 + u_i c_i \in C \oplus D$, $-s_i d_i \in E$ shows that

$$q^{-1} r_i^{-1}(-u_1 c_1 + s_1 d_1 + u_i c_i) \in C \oplus D \qquad \text{and} \qquad -s_i d_i \in E'.$$

Hence $q^{-1} r_i^{-1}(b_1 - b_i) \in G'$ becomes clear. This establishes $E' = E$ and (i).

We proceed to show that $c_i \in C$, $c_j \in D$ with $i, j \in I_2$ leads to a contradiction. In fact, in this case $F_i \leqslant C \oplus \langle p_i^{-\infty} d_i\rangle$, $F_j \leqslant D \oplus \langle p_j^{-\infty} d_j\rangle$, and it is easy to see that then $q^{-1}(b_i - b_j) = q^{-1}(b_1 - b_j) - q^{-1}(b_1 - b_i) \in G$

can not belong to the direct sum $C \oplus D \oplus E$. Thus either C or D, say C, intersects every F_i; this proves (ii).

Suppose I_2 is not empty (i.e. $E \neq 0$), and let $i \in I_2$. Then $qr_i | b_1 - b_i = -u_1c_1 + u_ic_i + s_1d_1 - s_id_i$ (where $-u_1c_1 + u_ic_i \in C$, $s_1d_1 \in D$, $s_id_i \in E$) implies $qr_i | s_1$, s_i and $q^{-1}r_i^{-1}(-u_1c_1 + u_ic_i) \in C$. The presence of the last element in C is sufficient to guarantee that $qr_i | b_1 - b_i$ whence it must be of order qr_i mod $C \cap F$. Therefore neither q nor r_i can divide u_1 or u_i.

If I_1 is empty, then D is of rank 1, $D \oplus E$ is a subgroup of F, and since $G/(G \cap F)$ is the direct sum of $n-1$ cyclic groups of order q and cyclic groups of orders $r_2, \dots, r_n$, (iii) is trivial in this case.

If I_1 is not empty either, then for $i \in I_1$ we have $qr_i | b_1 - b_i = -u_1c_1 + u_ic_i + s_1d_1 - s_id_i$ whence

$$x_i = q^{-1}r_i^{-1}(-u_1c_1 + u_ic_i) \in C \,, \qquad y_i = q^{-1}r_i^{-1}(s_1d_1 - s_id_i) \in D$$

follows. Since C and D are indecomposable, neither x_i nor y_i can belong to F. From $q|s_1$ we infer $q|s_i$, so x_i is of order q and y_i is of order r_i mod F. Now (iii) follows at once in this case.

If I_2 is empty, then $E = 0$ and as in the preceding paragraph, we conclude that neither x_i nor y_i belongs to F, so one is of order q, the other is of order r_i mod F. If, say y_i is of order r_i for some i, then $q|s_1$, s_i for this i and hence for every i. We are in the situation (iii) with $k = n$.

It remains only verify (iv) which is vacuous unless neither I_1 nor I_2 is empty. By way of contradiction, assume $i \in I_1$, $j \in I_2$ and $r_i = r_j$. Then $r_i | b_i - b_j = -u_ic_i + u_jc_j + s_id_i - s_jd_j$ implies $r_i | s_i$ which is absurd since y_i is of order r_i mod F.

This completes the proof of Lemma 2.

Recall that the proof of [2, Theorem 90.1] shows that if $p_1, \dots, p_n$, q, r are different primes, then the group

$$G = \langle p_1^{-\infty} a_1 \rangle \oplus \dots \oplus \langle p_n^{-\infty} a_n \rangle \oplus \langle p_1^{-\infty} b_1, \dots, p_n^{-\infty} b_n, \\ q^{-1}r^{-1}(b_1 - b_2), \dots, q^{-1}r^{-1}(b_1 - b_n) \rangle$$

can be decomposed into the direct sum of two indecomposable groups. From Lemma 2 we see that G can not have decompositions into k indecomposable summands unless $k = 2$ or $k = n + 1$.

3. In order to get a survey on the direct decompositions of the group $A = A_{nn}$, we require several fully invariant subgroups of A. These are described by the next lemma.

LEMMA 3. *Let $p_1, \ldots, p_n$, q be different primes which are different from the primes $r_2, \ldots, r_n$, and let*

$$(5) \qquad A = \langle p_1^{-\infty} a_1\rangle \oplus \ldots \oplus \langle p_n^{-\infty} a_n\rangle \oplus \langle p_1^{-\infty} b_1, \ldots, p_n^{-\infty} b_n;\ q^{-1} r_j^{-1}(b_i - b_j) \ \textit{for}\ 1 \leqslant i < j \leqslant n\rangle .$$

Then the subgroup

$$F = \bigoplus_{i=1}^{n} F_i \qquad \textit{where}\ F_i = \langle p_i^{-\infty} a_i\rangle \oplus \langle p_i^{-\infty} b_i\rangle$$

is fully invariant and so are all subgroups of A obtained from F by adjoining a subset of the generators $\{q^{-1}(b_i - b_j), r_j^{-1}(b_i - b_j)$ for $i < j\}$.

Since a subgroup generated by fully invariant subgroups is again fully invariant, it suffices to verify the full invariance of $\langle F, q^{-1}(b_i - b_j)\rangle$ and $\langle F, r_j^{-1}(b_i - b_j)\rangle$ for an index pair $i < j$. An endomorphism η of A maps each F_i into itself, thus

$$\eta(b_i - b_j) = s_i a_i + t_i b_i - s_j a_j - t_j b_j \qquad (i < j)$$

with s_i, t_i, s_j, t_j rationals whose denominators are powers of p_i and p_j, respectively. But $qr_j | b_i - b_j$ in A implies that the numerators of s_i, s_j and $t_i - t_j$ are divisible by qr_j. In this case,

$$\eta q^{-1}(b_i - b_j) \in \langle F, q^{-1}(b_i - b_j)\rangle \qquad \text{and} \qquad \eta r_j^{-1}(b_i - b_j) \in \langle F, r_j^{-1}(b_i - b_j)\rangle ,$$

whence the assertion follows.

4. We can now get a full description of the direct decompositions of the group A (in Lemma 3) into indecomposable summands.

LEMMA 4. *Each direct decomposition of A (in Lemma 3) into indecomposable summands is of the following form:*

$$A = C \oplus D \oplus E$$

where, for some $1 \leqslant k \leqslant n$, *we have*

(i) C *is indecomposable of rank* n, $C \geqslant \langle p_1^{-\infty} c_1, \ldots, p_n^{-\infty} c_n \rangle$;

(ii) D *is indecomposable of rank* k, $D \geqslant \langle p_1^{-\infty} d_1, \ldots, p_k^{-\infty} d_k \rangle$;

(iii) E *is completely decomposable of rank* $n-k$, $E = \langle p_{k+1}^{-\infty} d_{k+1} \rangle \oplus \oplus \ldots \oplus \langle p_n^{-\infty} d_n \rangle$;

(iv) $\langle p_i^{-\infty} c_i \rangle \oplus \langle p_i^{-\infty} d_i \rangle = F_i$ *for every* i;

(v) *no prime belongs both to* $\{r_2, \ldots, r_k\}$ *and to* $\{r_{k+1}, \ldots, r_n\}$ *simultaneously.*

The subgroup F_1 intersects precisely two indecomposable summands of A, in any direct decomposition into indecomposable summands. We can thus write $A = C \oplus D \oplus E$ with C, D indecomposable and $C \cap F_1 = \langle p_1^{-\infty} c_1 \rangle$, $D \cap F_1 = \langle p_1^{-\infty} d_1 \rangle$ satisfying (iv). If $E = 0$, then $k = n$, and the assertion is clear. So suppose $E \neq 0$. Owing to Lemma 3, the subgroup

$$G = \langle F, q^{-1} r_2^{-1}(b_1 - b_2), \ldots, q^{-1} r_n^{-1}(b_1 - b_n) \rangle$$

is fully invariant in A, thus $G = (C \cap G) \oplus (D \cap G) \oplus (E \cap G)$. From Lemma 2 we conclude that, say, C is of rank n and $E \cap G$ is completely decomposable. This proves (i). Suppose $j \geqslant 2$ is the largest index with $\langle p_j^{-\infty} d_j \rangle \leqslant D$. Apply Lemma 2 again, this time to the fully invariant subgroup

$$H = \langle F, q^{-1} r_j^{-1}(b_i - b_j) \text{ for } i < j,\ q^{-1} r_l^{-1}(b_j - b_l) \text{ for } j < l \rangle$$

and its decomposition $H = (C \cap H) \oplus (D \cap H) \oplus (E \cap H)$ to conclude that $(C \cap H)/(C \cap F)$ contains the direct sum of $n-1$ cyclic groups of order q and cyclic groups of orders $r_{j+1}, \ldots, r_n$, $(D \cap H)/(D \cap F)$ contains a direct sum of cyclic groups of orders $r_j (\neq r_{j+1}, \ldots, r_n)$, so their number must be $j-1$. This can happen only if $j = k$ and D contains $d_1, \ldots, d_k$. This proves (ii), while (iv) is clear.

If we form H as above with any $2 \leqslant j \leqslant k$, then Lemma 2 will ensure $r_j \neq r_{k+1}, \ldots, r_n$, i.e. (v) holds.

To check the complete decomposability of E, it will be sufficient to prove that $A' = C \oplus D \oplus \langle p_{k+1}^{-\infty} d_{k+1} \rangle \oplus \ldots \oplus \langle p_n^{-\infty} d_n \rangle$ is equal to A. Since C, D are pure in A, it is evident that all generators of A belong to A' trivially, except for $r_j^{-1}(b_i - b_j)$ $(i \leq k < j)$. Again form H as above, now for $j > k$, and apply Lemma 2 to this H. Evidently, $(C \cap H)/(C \cap F)$ must contain the r_j-components of $H/(H \cap F)$, whence $r_j^{-1}(b_i - b_j) \in A'$, and so $A' = A$. This completes the proof.

We have now come to our main theorem on decompositions of finite rank groups.

THEOREM 2. *Let N be any finite set of integers ≥ 2. There exists a torsion-free group A (of rank $\leq 2n$ where $n+1$ is the largest integer in N) such that A has a decomposition into the direct sum of k indecomposable summands exactly if $k \in N$.*

It suffices to prove this for $2 \in N$ and $n \geq 2$. Let N consist of the integers $2 = n_0 < n_1 < \ldots < n_m = n+1$, and set $k_i = n - n_i + 2$ $(i = 0, \ldots, m-1)$. Define the group A as in Lemma 3 with the primes $r_2, \ldots, r_n$ subject to the conditions

$$(6) \qquad r_2 = \ldots = r_{k_{m-1}}, \qquad r_{k_{m-1}+1} = \ldots = r_{k_{m-2}}, \ldots, r_{k_1+1} = \ldots = r_{k_0},$$

but $r_{k_{m-1}}, \ldots, r_{k_0}$ are all different. As the last summand B (of rank n) in (5) is indecomposable, A has a decomposition into $n+1$ indecomposable summands. Because of (6), from Lemma 1 we infer that the direct sum of the first k_i $(i \leq m-1)$ groups $\langle p_i^{-\infty} a_i \rangle$ and B can be decomposed into the direct sum of two indecomposable summands, yielding a decomposition of A into $n - k_i + 2 = n_i$ $(i \leq m-1)$ indecomposable summands. By Lemma 4, A can not have, in view of (6), any decomposition into any other number of indecomposable summands. Q.E.D.

5. Turning our attention to the case of groups of countable rank, we start now with the following group where k is a nonnegative integer:

$$(7) \qquad A_k = \langle p_{k+1}^{-\infty} a_{k+1} \rangle \oplus \langle p_{k+2}^{-\infty} a_{k+2} \rangle \oplus \ldots$$

$$\oplus \langle p_1^{-\infty} b_1, \ldots, p_k^{-\infty} b_k, p_{k+1}^{-\infty} b_{k+1}, \ldots; q^{-1} r_i^{-1}(b_i - b_j) \text{ for } 1 \leq i < j \rangle .$$

Here $p_1, \ldots, p_{k+1}, \ldots, q$ are different primes which are different from the primes $r_1, r_2, \ldots$. It is easy to see that all the summands of A_k in (7) are indecomposable.

LEMMA 5. *If $r_1, \ldots, r_k$ are all different from $r_{k+1}, r_{k+2}, \ldots$, and if $r_{k+1} \leqslant r_i$ for $i \geqslant k+1$, then there is a decomposition $A_k = C \oplus D$ where both C and D are indecomposable of rank $\aleph_0$.*

Let s_i, t, u_i, v (for all $i > k$) be integers to be further specified later such that

$$s_i v - t u_i = 1 \qquad (i > k) \tag{8}$$

and set

$$c_i = s_i a_i + t b_i\,, \qquad d_i = u_i a_i + v b_i \qquad (i > k).$$

Thus

$$a_i = v c_i - t d_i\,, \qquad b_i = -u_i c_i + s_i d_i\,.$$

We now define

$$C = \langle p_1^{-\infty} b_1, \ldots, p_k^{-\infty} b_k, p_{k+1}^{-\infty} c_{k+1}, \ldots; \quad q^{-1} r_i^{-1}(b_i - b_j) \text{ for } i < j \leqslant k;\; q^{-1} r_i^{-1}(b_i + u_l c_l) \text{ for } i \leqslant k < l;\; q^{-1} r_l^{-1}(u_l c_l - u_m c_m) \text{ for } k < l < m\rangle\,,$$

$$D = \langle p_{k+1}^{-\infty} d_{k+1}, p_{k+2}^{-\infty} d_{k+2}, \ldots; q^{-1} r_l^{-1}(s_l d_l - s_m d_m) \text{ for } k < l < m\rangle\,,$$

and want to find conditions in order to have $A_k = C \oplus D$.

Just as in the proof of Lemma 1, it follows that $C \leqslant A_k$ is equivalent to the conditions

$$q r_i | s_l \qquad \text{for } i \leqslant k < l\,, \tag{9}$$

$$q r_l | u_m s_m \qquad \text{for } k < l \leqslant m\,, \tag{10}$$

$$q r_l | t(u_l - u_m) \qquad \text{for } k < l < m\,, \tag{11}$$

where (9) follows from $q r_i | u_l s_l$, $1 + u_l t = s_l v$, on using $(u_l, v) = 1$. Analogously, $D \leqslant A_k$ is equivalent to (10) plus

$$q r_l | v(s_l - s_m) \qquad \text{for } k < l < m\,. \tag{12}$$

On the other hand, if (8)-(12) are fulfilled, then all the generators of A_k are contained in $C+D$, so $A_k = C \oplus D$. Note that (12) is superfluous, since (8) implies $v(s_l - s_m) = t(u_l - u_m)$ which is, by (11), divisible by qr_l.

Thus our problem consists in finding integers s_i, t, u_i, v satisfying (8)-(11) such that C and D are indecomposable. (9) implies that s_i $(i > k)$ must be of the form $s_i = q\cdot[r_1, \dots, r_k]\cdot x_i$ for some integer x_i. If we choose $u_i = [r_{k+1}, \dots, r_i]\cdot y_i$ for some integer y_i, then (10) will be satisfied and (11) reduces to $q|t(u_l - u_m)$ whence $(s_i, t) = 1$, $q|s_i$ implies the further reduction

$$(11^*) \qquad q|u_l - u_m \qquad (k < l < m)\,.$$

We can thus concentrate on (8) and (11*).

Set

$$s_{k+1} = q\cdot[r_1, \dots, r_k]\,, \qquad u_{k+1} = r_{k+1}$$

and pick t, v so as to satisfy (8) for $i = k+1$, and in addition, $0 < v < r_{k+1}$. Suppose s_i, u_i have been chosen for $k < i \leqslant m$ satisfying the relevant conditions for indices up to m. From the theory of linear Diophantine equations we know that, because of (8), we must have $s_{m+1} = s_m + tw$ and $u_{m+1} = u_m + vw$ for some integer w. Our conditions imply that w must be of the form $w = q\cdot[r_1, \dots, r_k]\cdot[r_{k+1}, \dots, r_m]\cdot z$ with some integer z.

Now if r_{m+1} is equal to one of $r_{k+1}, \dots, r_m$, then we can choose $s_{m+1} = s_m$, $u_{m+1} = u_m$ (i.e. $z = 0$) and all conditions are satisfied where the indices do not exceed $m+1$. If r_{m+1} is a new prime, then we choose integers y_{m+1} and z such that

$$r_{m+1} y_{m+1} = y_m + vq[r_1, \dots, r_k]z\,;$$

this can be done, since r_{m+1} is prime to $q[r_1, \dots, r_k]$ and to v (because $0 < v < r_{k+1} < r_{m+1}$). Putting $u_{m+1} = [r_{k+1}, \dots, r_{m+1}]y_{m+1}$, $s_{m+1} = s_m + tw$, all conditions with indices at most $m+1$ are fulfilled. Consequently, s_i, t, u_i, v can be selected to satisfy all of (8)-(11). Since $q \nmid u_i$ and $r_i \nmid s_l$ $(k < i, l)$, the groups C and D are indecomposable, indeed.

6. We can now verify our main result on groups of countable rank.

THEOREM 3. *Let N be an infinite set of integers $\geqslant 2$. There exists a torsion-free group A of countable rank which can be decomposed into the direct sum of a finite number k of indecomposable summands if and only if $k \in N$.*

Let N consist of the integers $n_0 = 2 < n_1 < \ldots < n_j < \ldots$. Choose the group A as A_0 in (7) with the primes r_i subject to the following condition:

$$r_1 = \ldots = r_{n_1-2} < r_{n_1-1} = \ldots = r_{n_2-2} < r_{n_2-1} = \ldots < \ldots .$$

Since, for every $j \geqslant 1$, the primes $r_1, \ldots, r_{n_j-2}$ are different from the rest and none of the rest is smaller than r_{n_j-1}, we infer from Lemma 5 that A_{n_j-2} decomposes into the direct sum of two indecomposable summands. These, together with the summands $\langle p_i^{-\infty} a_i \rangle$ for $i = 1, \ldots, n_j - 2$, yield a decomposition of A into exactly n_j indecomposable summands.

Suppose now that A has a decomposition into the direct sum of k indecomposable summands:

$$A = X_1 \oplus \ldots \oplus X_k \qquad (X_j \neq 0) .$$

Define the pure subgroups $B_n = \langle F_1 \oplus \ldots \oplus F_n \rangle_*$ for $n \geqslant 1$, and select n_0 such that $B_{n_0} \cap X_i \neq 0$ for $i = 1, \ldots, k$. Then

$$B_n = (B_n \cap X_1) \oplus \ldots \oplus (B_n \cap X_k)$$

where no component vanishes for $n \geqslant n_0$. Since B_n is a group like A in Lemma 3 (with reversed order of indices), from Lemma 4 we conclude that—after rearranging the X_i's—$B_n \cap X_1 \oplus \ldots \oplus B_n \cap X_{k-2}$ is completely decomposable intersecting precisely $F_1, \ldots, F_j$ for some j, while $B_n \cap X_{k-1}$ intersects $F_{j+1}, \ldots, F_n$ and $B_n \cap X_k$ is indecomposable of rank n, intersecting all of $F_1, \ldots, F_n$. Passing to a larger n, only the last two summands of B_n change. Therefore, for a large n, the first $k-2$ summands $B_n \cap X_i$ are equal to X_i and hence of rank 1, and the last two summands are indecomposable of ranks $n - k + 2$ and n, respectively ($B_n \cap X_{k-1}$ can be decomposable only if it has a summand of rank 1 contained in $F_{j+1} = F_{k-1}$; but it is not a summand of X_{k-1}, so it is neither of $B_n \cap X_{k-1}$ if n is large enough).

Consequently, k must be the number of indecomposable summands of B_n for large n. By Lemma 4, k must therefore be equal to some n_j ($j = 0, 1, \ldots$) whence the result follows.

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