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1. Class groups.

Throughout this article let R denote a Dedekind domain with quotient field K, and let Λ be an R-order in a separable K-algebra A. We shall give another approach to some results of Jacobinski [2] on hereditary orders. As general references for the background material needed for this article, the reader is referred to [3]-[5].

A (left) *Λ-lattice* is a left Λ-module M which is finitely generated and torsionfree as R-module. For each maximal ideal P of R, let R_P, M_P (and so on) denote P-adic completions. Two Λ-lattices M, N are in the same *genus* (notation: $M \vee N$) if for each P, $M_P \cong N_P$ as Λ_P-lattices. If $M \vee \Lambda$, we call M *locally free* (of rank 1). Two Λ-lattices X, Y are *stably isomorphic* if $X \dotplus \Lambda^{(k)} \cong Y \dotplus \Lambda^{(k)}$ for some k. Let $[X]$ denote the stable isomorphism class of X.

The R-order Λ is *hereditary* if every left ideal of Λ is Λ-projective. We state without proof (see references):

(1) THEOREM. The following are equivalent:

i) Λ is hereditary.

ii) Every left Λ-lattice is projective.

iii) Every right Λ-lattice is projective.

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The *locally free class group* $\mathrm{Cl}\,\Lambda$ is defined to be the abelian group generated by $\{[M]: M \vee \Lambda\}$, with addition given by

$$[M] + [M'] = [M''] \quad \text{whenever } M \dotplus M' \cong \Lambda \dotplus M''.$$

Thus $[M] = 0$ in $\mathrm{Cl}\,\Lambda$ if and only if M is stably isomorphic to Λ.

If Λ' is a maximal R-order in A, then the class group $\mathrm{Cl}\,\Lambda'$ can be computed explicitly, especially when K is an algebraic number field, or when K is a function field and A satisfies the Eichler condition relative to R. In these cases, $\mathrm{Cl}\,\Lambda'$ turns out to be a ray class group of the center of Λ'. Given any order Λ, we may find a maximal order Λ' in A with $\Lambda \subset \Lambda'$. It is obviously desirable to relate the two class groups $\mathrm{Cl}\,\Lambda$ and $\mathrm{Cl}\,\Lambda'$.

Whether or not K is a global field, each inclusion of orders $\Lambda \subset \Lambda'$ gives rise to a well defined homomorphism

$$\beta: \mathrm{Cl}\,\Lambda \to \mathrm{Cl}\,\Lambda', \quad \beta[X] = [\Lambda' \otimes_\Lambda X] \quad \text{for } [X] \in \mathrm{Cl}\,\Lambda.$$

We now prove

(2) THEOREM. Let Λ be a hereditary order, and let Λ' be any R-order in A containing Λ. Then $\beta: \mathrm{Cl}\,\Lambda \cong \mathrm{Cl}\,\Lambda'$ is an isomorphism.

PROOF. (We do not assume that Λ' is a maximal order, nor that K is a global field.) Given any $[X'] \in \mathrm{Cl}\,\Lambda'$, we may assume without loss of generality that X' is a Λ'-lattice in A. Let $u(A)$ denote the group of units of A. Since $X' \vee \Lambda'$, for each P we may write

$$X'_P = \Lambda'_P x_P \quad \text{for some } x_P \in u(A_P).$$

Furthermore, $X'_P = \Lambda'_P$ a.e. (« almost everywhere »), so we may choose $x_P = 1$ a.e. Now set

$$X = A \cap \Big\{\bigcap_P \Lambda_P x_P\Big\}.$$

Then X is a Λ-lattice in A such that $X_P = \Lambda_P x_P$ for all P, whence $[X] \in \mathrm{Cl}\,\Lambda$. We have $\Lambda' X = X'$, since this holds locally at each P. Then $\beta[X] = [X']$, since $\Lambda' \otimes_\Lambda X \cong \Lambda' X$. This proves that β is an epimorphism.

Now let $S = \{P_1, \dots, P_n\}$ be the set of all P at which $\Lambda_P \neq \Lambda'_P$.

If $X \vee \Lambda$, then by Roiter's Lemma there is a Λ-exact sequence

$$0 \to X \to \Lambda \to T \to 0 , \tag{3}$$

where T is a left Λ-module such that $T_P = 0$ for all $P \in S$. Now Λ' is a right Λ-lattice, hence is Λ-projective by Theorem 1, since Λ is hereditary. Therefore the functor $\Lambda' \otimes_\Lambda \cdot$ is exact. Applying it to the sequence (3), we obtain a Λ'-exact sequence

$$0 \to \Lambda' X \to \Lambda' \to \Lambda' \otimes_\Lambda T \to 0 , \tag{4}$$

where we have identified $\Lambda' \otimes_\Lambda X$ with $\Lambda' X$.

We claim next that $\Lambda' \otimes_\Lambda T \cong T$ as left Λ-modules. Indeed, both are R-torsion modules, so we need only show that

$$\Lambda'_P \otimes_{\Lambda_P} T_P \cong T_P \quad \text{for all } P.$$

This is clear when $P \in S$, while for $P \notin S$ it follows from the fact that $\Lambda'_P = \Lambda_P$. This shows that $\Lambda' \otimes_\Lambda T \cong T$, so (4) may be rewritten as

$$0 \to \Lambda' X \to \Lambda' \to T \to 0 . \tag{5}$$

Comparing (3) and (5) and using Schanuel's Lemma, we obtain

$$X \dotplus \Lambda' \cong \Lambda \dotplus \Lambda' X \quad \text{as } \Lambda\text{-lattices.} \tag{6}$$

Now let $[X] \in \ker \beta$, so $[\Lambda' X] = 0$ in $\mathrm{Cl}\,\Lambda'$. Thus $\Lambda' X$ is stably isomorphic to Λ', so there exists a free Λ'-lattice F' such that

$$\Lambda' F \dotplus F' \cong \Lambda' \dotplus F' .$$

It follows from (6) that

$$X \dotplus (\Lambda' \dotplus F') \cong \Lambda \dotplus (\Lambda' \dotplus F') . \tag{7}$$

Since $\Lambda' \dotplus F'$ is a Λ-lattice, and hence is Λ-projective, there exists a Λ-lattice L such that $\Lambda' \dotplus F' \dotplus L$ is a free Λ-lattice. Adding L to both sides of (7), it follows that X is stably isomorphic to Λ as

Λ-lattices. Therefore $[X] = 0$ in $\mathrm{Cl}\,\Lambda$. This completes the proof that β is an isomorphism.

For later use, we quote without proof

(8) THEOREM (Jacobinski [1]). Let K be a global field, and let Λ be an arbitrary order. If A satisfies the Eichler condition relative to R, then $[X] = [Y]$ in $\mathrm{Cl}\,\Lambda$ if and only if $X \cong Y$.

2. Endomorphism rings.

For any Λ-lattice M, let $\mathrm{End}_\Lambda M$ denote its Λ-endomorphism ring. It is well known that if Λ' is a maximal R-order in A, and M' is any Λ'-lattice, then $\mathrm{End}_{\Lambda'} M'$ is also a maximal order. We now establish a corresponding result for hereditary orders.

(9) THEOREM. Let Λ be a hereditary order, and let M be any non-zero Λ-lattice. Then $\mathrm{End}_\Lambda M$ is a hereditary R-order in $\mathrm{End}_A KM$.

PROOF. It is clear that $\mathrm{End}_\Lambda M$ is an R-order in $\mathrm{End}_\Lambda KM$, any we must show that it is hereditary. Since M is Λ-projective, we mad choose a Λ-lattice M' such that $M \dotplus M' \cong \Lambda^{(n)}$ for some n. Set $\Gamma = \mathrm{End}_\Lambda \Lambda^{(n)}$; the projection map $e\colon \Lambda^{(n)} \to M$ is then an idempotent in the ring Γ. We have

$$\mathrm{End}_\Lambda M \cong e\Gamma e\ .$$

Now $\Lambda^{(n)}$ is a progenerator for the category of left Λ-modules, and hence Γ is Morita equivalent to Λ. But Morita equivalence preserves the property of being hereditary, since by Theorem 1, an order Λ is hereditary if and only if every submodule of a finitely generated projective Λ-module is Λ-projective. It follows that Γ is hereditary, and it remains to prove that also $e\Gamma e$ is hereditary.

Let L be any left ideal in $e\Gamma e$; then $L \subset e\Gamma e \subset \Gamma$, and we can form the left ideal ΓL in Γ. Now L is finitely generated as $e\Gamma e$-module, so there exists a left $e\Gamma e$-epimorphism (for some r)

$$\varphi\colon (e\Gamma e)^{(r)} \to L\ , \qquad \varphi(\alpha_1, \ldots, \alpha_r) = \sum_{i=1}^{r} \alpha_i l_i\ , \qquad \alpha_i \in e\Gamma e\ .$$

We may extend φ to a left Γ-epimorphism

$$\varphi' : \Gamma^{(r)} \to \Gamma L\,, \qquad \varphi'(\gamma_1, \dots, \gamma_r) = \sum_{i=1}^{r} \gamma_i l_i\,, \qquad \gamma_i \in \Gamma.$$

But ΓL is Γ-projective since Γ is hereditary, and hence there exists a Γ-homomorphism $\psi' : \Gamma L \to \Gamma^{(r)}$ such that $\varphi' \psi' = 1$ on ΓL. We may write

$$\psi'(y) = (\psi_1(y), \dots, \psi_r(y))\,, \qquad y \in \Gamma L.$$

Then

$$y = \varphi' \psi'(y) = \sum_{i=1}^{r} \psi_i(y)\, l_i\,, \qquad y \in \Gamma L.$$

Also we have

(10) $$\psi_i(\gamma y) = \gamma \psi_i(y)\,, \qquad \gamma \in \Gamma,\ y \in \Gamma L,$$

so in particular

(11) $$\psi_i(x) = e\psi_i(x)\,, \qquad x \in L.$$

Now define $\psi : L \to (e\Gamma e)^{(r)}$ by setting

$$\psi(x) = (\psi_1(x)\, e, \dots, \psi_r(x)\, e)\,, \qquad x \in L.$$

Each $\psi_i(x)\, e \in e\Gamma e$ by (11), and ψ is a left $e\Gamma e$-homomorphism by (10). We have

$$\varphi\psi(x) = \sum_{i=1}^{r} \psi_i(x)\, e \cdot l_i = \sum_{i=1}^{r} \psi_i(x)\, l_i = x\,, \qquad x \in L.$$

This shows that ψ splits the epimorphism φ, and proves that L is $e\Gamma e$-projective. Hence $e\Gamma e$ is hereditary, and the theorem is established.

3. Restricted genus.

Two Λ-lattices M, N are in the same *restricted genus* if $M \vee N$ and $\Lambda' M \cong \Lambda' N$ for some maximal order Λ' containing Λ. We shall apply our preceding results to prove the following result of Jacobinski [2]:

(12) THEOREM. Let Λ be a hereditary R-order in the separable K-algebra A, where K is a global field and where $\operatorname{End}_A KM$ satisfies the Eichler condition relative to R. If M, N are Λ-lattices in the same restricted genus, then $M \cong N$.

PROOF. Let Λ' be a maximal order containing Λ, such that $\Lambda' M \cong \Lambda' N$. Let us set

$$\Gamma = \operatorname{End}_\Lambda M \,, \qquad \Gamma' = \operatorname{End}_{\Lambda'} \Lambda' M \,, \qquad B = \operatorname{End}_A KM \,.$$

Then $\Gamma \subset \Gamma' \subset B$, and Γ' is a maximal order in the separable K-algebra B. The order Γ is hereditary, by Theorem 9.

Replacing N by αN for some nonzero $\alpha \in R$, we may assume hereafter that $N \subset M$. We shall view M as a left Λ-, right Γ-bimodule. Set

$$J = \operatorname{Hom}_\Lambda(M, N) \,.$$

Then J is a left ideal in Γ. We claim that $J \vee \Gamma$, and that $N = MJ$.

For each P, there is an isomorphism $M_P \cong N_P$, and hence we may write

$$N_P = M_P u_P \qquad \text{for some } u_P \in u(B_P).$$

Then

$$J_P = \operatorname{Hom}_{\Lambda_P}(M_P, N_P) = \Gamma_P u_P \,,$$

which shows that $J \vee \Gamma$. This also proves that $N = MJ$, since the equality holds locally at all P.

Now we observe that

$$\Gamma' J = \operatorname{Hom}_{\Lambda'}(\Lambda' M, \Lambda' N) \,,$$

since this holds locally at each P. But $\Lambda' N = \Lambda' M \cdot y$ for some $y \in u(B)$, since $\Lambda' M \cong \Lambda' N$. Therefore $\Gamma' J = \Gamma' y$, which shows that the element $[J] \in \operatorname{Cl} \Gamma$ maps onto the zero element of $\operatorname{Cl} \Gamma'$. Hence $[J] = 0$ by Theorem 2, and so J is stably isomorphic to Γ. Therefore $J = \Gamma z$ for some $z \in u(B)$ by Theorem 8, whence

$$N = MJ = M \cdot \Gamma z = Mz \cong M \,,$$

as desired. This completes the proof of the theorem.

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