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Remarks on Holomorphic Vector Fields on Non-Compact Manifolds.

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Introduction.

Let M be a Kähler manifold and let Z and ω be respectively a holomorphic vector field and a holomorphic linear differential form on M. If M is compact, then the function $\omega(Z)$ is constant on M.

This fact yields some useful informations on the structure of the Lie algebra $h(M)$ of holomorphic vector fields on M, on the Lie algebra $i(M)$ of infinitesimal isometries on M and on the vanishing of certain cohomology groups on M [3].

The purpose of this note is that of extending some of the above results to the non-compact case. If M is a complete Kähler manifold more specific hypothesis are required for $\omega(Z)$ to be constant. We discuss in § 2 the case where $\omega(Z)$ is square summable on M, and the Ricci curvature is positive outside a compact of M, thus extending to the non-compact case some results of K. Yano [7].

Section 3 contains some results concerning the relationship between the zero set of Z and the vanishing of some cohomology group of M. Recent results of A. Lichnérowicz [5] and A. Howard [2] are extended to the non compact case.

In § 1, we discuss briefly some problems concerning $i(M)$ in the case when M is a complete Riemannian manifold.

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1. In this section, M will be a paracompact, connected, oriented manifold of dimension n, endowed with a positive definite, complete riemannian metric g of class C^∞. We shall denote by C^r (resp. $\mathfrak{D}^r$) the space of real C^∞ r-forms (resp: C^∞ r-forms with compact support); $*$: $C^r \to C^{n-r}$ is the canoncal real operator, associated with the riemannian metric such that $**\varphi = (-1)^{r(n-1)}\varphi$, for any $\varphi \in C^r$. Then, for $x \in M$ and $\varphi, \psi \in C^r$: $(\varphi \wedge * \psi)_x = A_x(\varphi, \psi)\, dm(x)$, where dm is the volume element defined by the riemannian metric and $A_x(\varphi, \psi)$ is the scalar product defined by the riemannian metric g at x. Let $\mathfrak{L}_r^2$ be the Hilbert space, which is the completion of $\mathfrak{D}^r$ with respect to the norm $\|\varphi\|^2 = (\varphi, \varphi) = \int_M \varphi \wedge * \varphi = \int_M A(\varphi, \varphi)\, dm$; d: $C^r \to C^{r+1}$ denotes the exterior differentiation operator and δ: $C^r \to C^{r-1}$—defined by

$$\delta\varphi = (-1)^{r-1} * d * \varphi \,,$$

for any $\varphi \in C^r$—is its formal adjoint.

In [6], it is shown that, if W_r' denotes the completion of $\mathfrak{D}^r$ with respect to the norm $\eta(\varphi)^2 = \|\varphi\|^2 + \|d\varphi\|^2 + \|\delta\varphi\|^2$, then $W_r' = \{\varphi \in \mathfrak{L}_r^2$: $d\varphi \in \mathfrak{L}_2^{r+1}, \delta\varphi \in \mathfrak{L}_2^{r-1}\}$. The Laplace-Beltrami (1) operator $\Delta: C^r \to C^r$, defined by $\Delta = d\delta + \delta d$, is essentially selfadjoint and its selfajoint extension, denoted by $\mathbf{\Delta}$, has domain: $W_r'' = \{\varphi \in W_r'$: $\delta d\varphi \in \mathfrak{L}_r^2$, $d\delta\varphi \in \mathfrak{L}_r^2\}$.

For $\varphi \in C^1$, we denote by $R\varphi$ the differential 1-form defined locally by $(R\varphi)_i\, dx^i = (R_i^J \varphi_J)\, dx^i$, where R_{iJ} are the local components of the Ricci tensor. We denote by ∇, the covariant derivation with respect to the riemannian connection defined by g. Assume that the Ricci tensor R satisfies the condition: (α) $A_x(R\varphi, \varphi) \geqslant 0$ for any $\varphi \in \mathfrak{D}^1$, outside a compact K of M. Then the following facts hold, for any $\varphi \in W_1'$:

(i) $\|\nabla\varphi\| < \infty$, where $\|\nabla\varphi\|^2 = \int A_x(\nabla\varphi, \nabla\varphi)\, dm(x)$

(ii) $\|\nabla\varphi\|^2 + (R\varphi, \varphi) = \|d\varphi\|^2 + \|\delta\varphi\|^2$; moreover if $\varphi \in W_1''$

$$\|\nabla\varphi\|^2 + (R\varphi, \varphi) = (\mathbf{\Delta}\varphi, \varphi) \,.$$

(iii) if $\varphi \in \mathfrak{L}_1^2$ and $\Delta\varphi = 0$, then $|\varphi|$ is bounded.

Assume that R satisfies the stronger condition: (β) there exists

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(1) These and further results on the behavior of $\mathbf{\Delta}$ will be found in a forthcoming paper of the author.

$\gamma > 0$, such that: $A_x(R\varphi, \varphi) \geqslant \gamma A_x(\varphi, \varphi)$, for any $\varphi \in \mathfrak{D}^1$, outside a compact set K of M. Then $\mathfrak{L}_1^2 = H_1 \oplus d(\delta W_1'') \oplus \delta(dW_1'')$, where $H_1 = \{\text{kernel } \mathbf{\Delta}\}$ has a finite dimension.

For any vector field $X = \sum_i \zeta^i(\partial/\partial x^i)$ on M, ζ shall denote the 1-form $\sum_i \zeta_i dx^i$, corresponding to X, under the duality defined by the metric g, i.e. $\eta(X) = A(\zeta, \eta)$, for any $\eta \in C^1$.

THEOREM 1.1. If condition (α) holds and ζ, $R\zeta$ are in $\mathfrak{L}_1^2$, then conditions 1) $\Delta\zeta = 2R\zeta$, 2) $\delta\zeta = 0$, imply that X is an infinitesimal isometry (i.e. X generates a local 1-parameter group of local isometries).

PROOF. Since $\zeta \in W_1''$ and $\mathbf{\Delta}\zeta = 2R\zeta$, then $\|\nabla\zeta\| < \infty$ and $(\mathbf{\Delta}\zeta, \zeta) = \|\nabla\zeta\|^2 + (R\zeta, \zeta)$; so

$$(a)\quad (R\zeta, \zeta) - \|\nabla\zeta\|^2 = 0\,.$$

Moreover, it follows from a straight-forward computation that

$$(b)\qquad -\operatorname{div}(A_X X + (\operatorname{div} X)X) = \sum_{i,j}(R_{ij}\zeta^i\zeta^j + \nabla_j\zeta^i\cdot\nabla_i\zeta^j - \nabla_i\zeta^i\cdot\nabla_j\zeta^j),$$

where $A_X = L_X - \nabla_X$, and L_x is the Lie derivation with respect to X.

Let η be the 1-form corresponding to $A_X X$, then $\int_M |\eta|\cdot dm < \infty$ since $\eta^i = -\sum_j (\nabla_j\zeta^i)\zeta^j$, $\int_M |\delta\eta|\, dm < \infty$ by (b) since $\operatorname{div} X = 0$. So from Gaffney Lemma (cf. e.g. [6], p. 51)

$$(c)\qquad (R\zeta, \zeta) - \|\nabla\zeta\|^2 = \tfrac{1}{2}\int_M \operatorname{trace}\left((A_X + {}^tA_X)^2\right) dm\,.$$

Since trace $((A_X + {}^tA_X)^2)$ is the square of the lenght of the symmetric tensor $A_X + {}^tA_X$, and $\frac{1}{2}\int_M \operatorname{trace}((A_X + {}^tA_X)^2)\, dm = 0$ by (a) and (c), then $A_X + {}^tA_X = 0$, which is equivalent to say that X is an infinitesimal isometry ([3], pag. 43).

REMARK 1.1. It is well known [3], that if X is an infinitesimal isometry, then 1) $\Delta\zeta = 2R\zeta$ and 2) $\delta\zeta = 0$.

REMARK 2.1. K. Yano has shown (in [7]) that, if V is a compact oriented riemannian manifold, every infinitesimal affine transformation is an infinitesimal isometry; J. Hano (in [1]) gave the following extension: if V is complete, every infinitesimal affine transformation with bounded lenght is an infinitesimal isometry. From theorem 1.1, it follows that:

COROLLARY 2.1. If condition (α) is satisfied, every infinitesimal affine transformation such that ζ and $R\zeta$ are in $\mathfrak{L}_1^2$, is an infinitesimal isometry.

PROOF. Let X be an infinitesimal affine transformation, then $\Delta\zeta = 2R\zeta$ and $\delta\zeta$ is a constant function ([3], pag. 44-45). Since $\zeta \in W_1''$, we have $\delta\zeta \in \mathfrak{L}^2$. If Vol $M = \infty$, then $\delta\zeta = 0$. If Vol $M < \infty$, then $\delta\zeta \in \mathfrak{L}^1$ and $\zeta \in \mathfrak{L}^1$, so from Gaffney Lemma $\int_M \delta\zeta\, dm = 0$, which implies $\delta\zeta = 0$.

REMARK 3.1. If the Ricci tensor is negative definite, then M has no infinitesimal isometry such that $\zeta \in \mathfrak{L}^2$ and $R\zeta \in \mathfrak{L}^2$. Indeed $\Delta\zeta = 2R\zeta$ implies $\zeta \in W_1''$ and $0 < 2(R\zeta, \zeta) = (\Delta\zeta, \zeta) = \|d\zeta\|^2 \geqslant 0$. In particular, if M is an Einstein manifold with $c < 0$, M has no infinitesimal isometry in $\mathfrak{L}^2$.

2. Throughout this and the following section M will always be a paracompact, connected, complex manifold of dimension n, endowed with a complete Kähler metric. We shall denote by $C^{p,q}$ (resp. $\mathfrak{D}^{p,q}$) the space of $C^\infty(p, q)$-forms (resp. $C^\infty(p, q)$-forms with compact support); the canonical isomorphism $*\colon C^{p,q} \to C^{n-q,n-p}$, defined by the star operator of the Riemannian structure underlying the hermitian structure of M, allows us to introduce in $\mathfrak{D}^{p,q}$ the scalar product $\int_M \varphi \wedge * \bar{\psi}$. That enables us to define, as in the Riemannian case, the spaces $\mathfrak{L}^2_{(p,q)}$ and $W'_{(p,q)}$. The complex Laplace operator $\square\colon C^{p,q} \to C^{p,q}$, defined by $\square = \theta\bar\partial + \bar\partial\theta$, where θ, is the formal adjoint of $\bar\partial$, is essentially selfadjoint, and the domain of its selfadjoint extension is $W''_{(p,q)} = \{\varphi \in W'_{(p,q)}\colon \bar\partial\theta\varphi \in \mathfrak{L}^2_{(p,q)}, \theta\bar\partial\varphi \in \mathfrak{L}^2_{(p,q)}\}$. Let $R_{\alpha\bar\beta}$ be the Ricci tensor of the Kähler metric $g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta$, $R_{\alpha\bar\beta} = -(\partial^2 \log g)/(\partial z^\alpha\, \partial\bar z^\beta)$.

THEOREM 1.2. Let the riemannian structure of M satisfy condition (α) of § 1. Let $Z = \sum \zeta^\alpha(\partial/\partial z^\alpha)$ be a complex vector field of type (1, 0) and let $\zeta = \sum \zeta_{\bar\alpha}\, d\bar z^{\alpha}(\zeta_{\bar\alpha} = g_{\bar\alpha\beta}\zeta^\beta)$ be the corresponding (0, 1)-form. Then, if $\zeta \in W''_{(0,1)}$ and $(\square\zeta, \zeta) = (R''\zeta, \zeta)$, where $R''\zeta = \sum R_{\alpha\beta} \cdot \zeta^\alpha\, d\bar z^\beta$, Z is holomorphic.

PROOF. Since $\|\nabla\zeta\| < \infty$ and $(\square\zeta, \zeta) = \|\nabla''\zeta\|^2 + (R''\zeta, \zeta)$ (∇'' is defined for any tensor field K by the properties $\nabla K = \nabla' K + \nabla'' K$ $\nabla''_W K = 0$ $\nabla'_{\bar W} K = 0$ for all vectors W of type $(1, 0)$), then $\nabla''\zeta = 0$, i.e. $\nabla'' Z = 0$ which implies that Z is holomorphic ([3] pag. 93).

REMARK 1.2. It is a well known fact ([3] pag. 93) that, if Z is holomorphic, then $\square\zeta = R''\zeta$.

Let Z be, as before, a complex vector field on M. Then $Z = X - iJX$, where X is a vector field on the underlying differentiable manifold, and J defines the complex structure of M. Let ζ be the corresponding $(1, 0)$ form to Z.

THEOREM 2.2. Suppose that M satisfies the hypothesis of theorem 1.2. If $\zeta \in \mathfrak{L}^2_{(0.1)}$ and $R''\zeta \in \mathfrak{L}^2_{(0.1)}$, X is an infinitesimal isometry if and only if Z is holomorphic and $\operatorname{div} X = 0$.

PROOF. The 1-form corresponding to X is $\frac{1}{2}(\zeta + \bar\zeta)$ By Remark 1.1 if X is an infinitesimal isometry, $\operatorname{div} X = 0$ and

$$\Delta(\zeta + \bar\zeta) = 2 \sum R_{ij}(\zeta + \bar\zeta)^J dx^i ,$$

then $\square\zeta = R''\zeta$, so $\zeta \in W''_{(0.1)}$ and $(\square\zeta, \zeta) = (R''\zeta, \zeta)$ and Z is holomorphic by theorem 1.2. If Z is holomorphic and $\operatorname{div} X = 0$, then by Remark 1.2 $\zeta \in W''_{(0.1)}$ and $\square\zeta = R''\zeta$; so $\frac{1}{2}(\zeta + \bar\zeta) \in W''_1$ and $\Delta(\zeta + \bar\zeta) = 2 \sum_{i,j} R_{ij}(\zeta + \bar\zeta)^J dx^i$.

The conclusion follows from theorem 1.1.

THEOREM 3.2. If condition (β) of § 1 holds, and if Z is a holomorphic vector such that $\zeta \in \mathfrak{L}^2_{(0.1)}$, then:

i) if $\operatorname{vol} M = \infty$, there exists a unique $f \in \mathfrak{L}^2_{(0.0)}$ such that $\zeta = \bar\partial f$.

ii) if $\operatorname{vol} M < \infty$, there exists $f \in \mathfrak{L}^2_{(0.0)}$, unique after the normalization $\int_M f \cdot dm = 0$, such that $\zeta = H\zeta + \bar\partial f$, where $\Delta H\zeta = 0$; moreover if zero Z, the zero set of Z, is $\neq \emptyset$, then $\zeta = \bar\delta f$.

PROOF. Since M is a Kähler manifold, condition (β) enables us to write $\zeta = H\zeta + \bar\partial(\theta\eta)$, $\theta\eta \in \mathfrak{L}^2_{(0.0)}$ and $\Delta H\zeta = 0$, moreover $|H\zeta|$ is bounded (§ 1). Therefore, $\overline{H\zeta}(Z)$ is a constant, since it is a holomorphic function in $\mathfrak{L}^2_{(0.0)}$. (This constant is zero if $\operatorname{vol} M = \infty$. It vanishes also when vol $M < \infty$, provided that Zero $Z \neq \emptyset$). But $\overline{H\zeta}(Z)$ $(z) = A(H\zeta, \zeta)$ $(z) = |H\zeta|^2$ $(z) + A(H\zeta, \bar\partial f)(z)$ then $\|H\zeta\|^2 = \int_M \overline{H\zeta}(Z)\, dm$.

As for the uniqueness of f, let g be in $\mathfrak{L}^2_{(0,0)}$ such that $\zeta = H\zeta + \bar\partial g$, then $\bar\partial(f - g) = 0$ so $f - g$ is constant in $\mathfrak{L}^2_{(0,0)}$.

THEOREM 4.2. If M is as in theorem 3.2 and vol $M < \infty$, then $\zeta = \bar\partial f$ if, and only if, $\alpha(Z) = 0$, for any holomorphic 1-form in $\mathfrak{L}^2_{(1,0)}$.

PROOF. Since α is harmonic, then $|\alpha|$ is bounded; so $\alpha(Z)$ is a holomorphic function in $\mathfrak{L}^2_{(0,0)}$. Then $\alpha(Z)(z) = k$ and

$$\int_M k\, dm = \int_M A(\bar\alpha, \bar\partial f)(z)\, dm = 0 ,$$

so $k = 0$. To prove the converse, just take $\alpha = H\bar\zeta$.

THEOREM 5.2. Under the hypothesis on M and Z of theorem 3.2 and with the above notation, X in an infinitesimal isometry if, and only if, the real part of f: $\operatorname{Re} f$, is a constant.

PROOF. By theorem 2.2, we need only show that $\operatorname{div} X = 0$ if, and only if, $\operatorname{Re} f =$ constant. Indeed $\delta(\zeta + \bar\zeta) = \delta(\bar\partial f + \partial\bar f) = \Delta(\operatorname{Re} f)$; so $\operatorname{div} X = 0 \Leftrightarrow \delta(\zeta + \bar\zeta) = 0 \Leftrightarrow \Delta(\operatorname{Re} f) = 0 \Leftrightarrow \operatorname{Re} f =$ constant, since $\operatorname{Re} f$ belongs to $\mathfrak{L}^2_{(0,0)}$.

3. Some applications.

In this section, M will be as at the beginning of § 2. Let $h(M)$ and $i(M)$ be respectively the space of holomorphic vector fields and of infinitesimal isometries on M.

THEOREM 1.3. If M is a Kähler manifold and $R_{ij} = 0$ then $h(M) \cap \mathfrak{L}^2$ coincides with $i(M) \cap \mathfrak{L}^2$, and consists of parallel vector fields (for the Riemannian connection).

Moreover, if vol $M = \infty$, they vanish.

PROOF. If $Z \in h(M) \cap \mathfrak{L}^2$, then $\Box\zeta = 0$ and $\bar\zeta(Z)(z) = |\zeta|^2(z)$ is a harmonic function in $\mathfrak{L}^2_{(0,0)}$, so it is constant. ($\zeta = 0$ if vol $M = \infty$)

and since $\Delta(|\zeta|^2) = 2A(\Delta\zeta, \zeta) - 2A(R\zeta, \zeta) - 2|\nabla\zeta|^2$, we have $\nabla\zeta = 0$. Moreover, since $\zeta = H\zeta$, then $\delta(\zeta + \bar{\zeta}) = 0$, which implies $Z \in i(M) \cap \mathfrak{L}^2$.

THEOREM 2.3. Let M be a Kähler manifold, which satisfies the condition (α), with $K = \Phi$. If Z is a non-trivial holomorphic vector field, such that the corresponding form ζ is in $\mathfrak{L}^2_{(0,1)}$, then any holomorphic n-form φ in $\mathfrak{L}^2_{(n,0)}$ vanishes if Zero $Z \neq \emptyset$.

PROOF. If φ is an n-holomorphic form in $\mathfrak{L}^2_{(n,0)}$, $|\varphi|$ must be constant, since $(\Delta\varphi, \varphi) = \|\nabla\varphi\|^2$. At this point, we can remark that if vol $M = \infty$, φ must vanish, without any further hypothesis on the holomorphic vector fields. If vol $M < \infty$, then $\zeta = H\zeta + \lim_n \bar{\partial} f_n$ in $\mathfrak{L}^2_{(0,1)}$ where f_n has compact support, and $H\zeta = 0$. Since condition (α) holds, $|H\zeta|$ is bounded and $\overline{H\zeta}(Z)$ is a holomorphic function which belongs to $\mathfrak{L}^2_{(0,0)}$. Then $\overline{H\zeta}(Z)$ is a constant, equal to zero if Zero $(Z) \neq \emptyset$. So $\zeta = \lim \bar{\partial} f_n$; since $d\varphi = 0$ and the $(n-1)$-holomorphic form $i_Z(\varphi)$ (where i_Z denotes the interior product with respect to Z) is in $\mathfrak{L}^2_{(n-1,0)}$; then $d \circ i_z(\varphi) = 0$ and $L_Z(\varphi) = d \circ i_z(\varphi) + i_z \circ d(\varphi) = 0$. Moreover, since $i_z(*\bar{\varphi}) = 0$ and $d(*\bar{\varphi}) = 0$: $L_Z(*\bar{\varphi}) = 0$. Hence $L_Z(\bar{f}_n |\varphi|^2 dm) = Z(\bar{f}_n)|\varphi|^2 dm$ for all n, and by Stokes's theorem:

$$0 = \int_M A(\bar{\partial} f_n, |\varphi|^2 \zeta \, dm = |\varphi|^2 \int_M A(\bar{\partial} f_n, \zeta)\, dm = |\varphi|^2 \langle \bar{\partial} f_n, \zeta\rangle \to |\varphi|^2 \|\zeta\|^2 .$$

REMARK 3.3. The condition (α) has been used in the above proof only to grant that $|H\zeta|$ is bounded. So theorem 3.3 still holds if condition (α) on the Ricci curvature of M, is replaced by the condition that the holomorphic vector field Z with Zero$(Z) \neq \emptyset$ has a bounded lenght.

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