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An Existence Theorem for Certain Solutions of Algebraic Differential Equations in Sectors.

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1. Introduction.

In this paper we consider first-order differential equations,

$$\Omega(z, y, y') = \sum_{k,j\geq 0} f_{kj}(z)\, y^k (y')^j = 0 \,, \tag{1}$$

where Ω is a polynomial in y and y', whose coefficients $f_{kj}(z)$ are complex functions, defined and analytic in a sectorial region which is approximately of the form,

$$a < \arg\big(z - \beta \exp[i(a+b)/2]\big) < b \,, \tag{2}$$

(for some $\beta \geq 0$), and where each non-zero $f_{kj}(z)$ has an asymptotic expansion in terms of real powers of z, as $z \to \infty$ over a filter base (denoted $F(a, b)$) which consists essentially of the sectors (2) as $\beta \to +\infty$. (We are using here the stronger concepts of « asympotically equivalent » ($\sim$) and « smaller rate of growth » ($\ll$) which were introduced by W. Strodt in [5; § 13]. For the reader's convenience, these concepts are reviewed in § 2 below, and we point out that the class of equations treated here contains, as a special case, the class of equations having polynomial coefficients.) The equations (1) were

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among those equations which were treated in [2], [5] and [7], where existence theorems were proved for solutions which are asymptotically equivalent over $F(a, b)$ to logarithmic monomials (i.e. functions of the form,

$$M(z) = Kz^{\alpha_0}(\log z)^{\alpha_1}(\log\log z)^{\alpha_2} \ldots (\log_q z)^{\alpha_q}, \tag{3}$$

for real α_j and complex $K \neq 0$). If (1) has a solution which is $\sim M$ over $F(a, b)$, then M must be a *critical* monomial [2; §§ 4, 5] of Ω (i.e. M is a *point of instability* of Ω in the sense that for some function $g \sim M$, $\Omega(z, g, g')$ is not $\sim \Omega(z, M, M')$ over $F(a, b)$). An algorithm for determining all critical monomials of Ω was developed in [2; §§ 21, 22] and the powerful Strodt-Wright theorem [7; p. 221] states, in part, that for every critical monomial M of Ω, there exists a solution $\sim M$ over a suitable $F(a_1, b_1)$. (For the special class treated in this paper, the algorithm shows that $\alpha_j = 0$ for $j \geqslant 2$).

In [1], existence theorems were proved for solutions of (1) which are of larger rate of growth than all logarithmic monomials over $F(a, b)$ and also for solutions which are of smaller rate of growth than all monomials. (These solutions are of the form $\exp\left[\int W\right]$, where W is $\sim$ to a logarithmic monomial of the form Kz^{α_0}).

In [3], a converse result was proved which shows that for the class of equations treated here, any solution $h(z)$ of (1), which is meromorphic in an element of $F(a, b)$ and which is « comparable » with all logarithmic monomials M over $F(a, b)$ (in the sense that for any M, one of the relations $h \ll M$, $M \ll h$ or $h \sim cM$ for some $c \neq 0$, is valid over $F(a, b)$), must either be asymptotically equivalent to a logarithmic monomial or of the form $\exp \int W$, where $W/Kz^{\alpha_0} \to 1$ over $F(a, b)$, for real α_0 and complex $K \neq 0$.

Of course, equations (1) can possess solutions which violate the above comparability condition such as solutions which are $\sim$ to functions of the form cz^{α}, where α is a nonreal complex number and $c \neq 0$. (It is easy to see that for such α, z^{α} and $z^{\mathrm{Re}(\alpha)}$ are not comparable over any $F(a, b)$). Thusfar, no general existence theorem for such solutions of nonlinear equations (1) has been proved, and in this paper, we prove such a result. Because of the noncomparability property of such functions cz^{α} with logarithmic monomials, the concept of « point of instability of Ω » cannot in general be used as the starting [7oint in the search for such solutions because as is pointed out in p; p. 253], it is possible for a function cz^{α}, with α nonreal, to be a point of instability of Ω, and yet there be no solution of (1) which is $\sim cz^{\alpha}$ over any $F(a, b)$. However, in the case of those Ω which are homogeneous as polynomials in y and y', it is true that for nonreal α the equation $\Omega = 0$ possesses a solution $\sim cz^{\alpha}$ over $F(a, b)$, if cz^{α} is a point of instability of Ω. (See §§ 4, 6 below). It is this fact which is the basis for our method for nonhomogeneous Ω. Roughly speaking, we show that under certain conditions if cz^{α} (with α nonreal) is a point of instability of a homogeneous part of Ω, and if this homogeneous part is in some sense « dominant » for $z^{\mathrm{Re}(\alpha)}$, then the whole equation $\Omega = 0$ possesses a solution which is $\sim cz^{\alpha}$ over $F(a, b)$. We remark that it is very easy to determine those functions cz^{α} which are points of instability of a homogeneous part of Ω (see § 6), and that in any given example, it is very easy to test whether the hypothesis is satisfied for the particular cz^{α} involved.

The proof of the main result consists of using the exact solution $\sim cz^{\alpha}$ of the homogeneous part of the equation involved (see § 4), to eventually transform the whole equation $\Omega = 0$ into a quasi-linear equation of the type treated by Strodt in [5; § 117], where a method of linear successive approximations was successful.

2. Preliminaries.

(*a*) [5; § 94]. Let $-\pi \leqslant a < b \leqslant \pi$. For each nonnegative real-valued function g on $(0, (b-a)/2)$, let $V(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \arg(z - h(\delta)) < b - \delta$ where $h(\delta) = g(\delta)\exp[i(a+b)/2]$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$, and is a filter base of simply-connected regions which converges to ∞ by [5; §§ 93, 95]. By $\log z$, we will mean the principal branch of the logarithm in $|\arg z| < \pi$. By induction, it is easy to see that the function $\log_{j+1} z = \log(\log_j z)$ is defined and analytic in some element of $F(a, b)$. If α is a complex number, then as usual, z^{α} will denote $\exp[\alpha \log z]$.

(*b*) [5; §§ 13, 17]. If f is analytic in an element of $F(a, b)$, then $f \to 0$ over $F(a, b)$ means that for any $\varepsilon > 0$, there is an element of $F(a, b)$ on which $|f(z)| < \varepsilon$. The statement $f \ll 1$ over $F(a, b)$ means that in addition to $f \to 0$, for all positive integers j and k we have $\theta_k^j f \to 0$, where $\theta_k f = (z \log z \ldots \log_{k-1} z) f'$, and where θ_k^j is the j-th iterate of the operator θ_k. Then $f \ll g$, $f \gg g$, $f \sim g$ and $f \approx g$ over $F(a, b)$ mean respectively, $f/g \ll 1$, $g/f \ll 1$, $f - g \ll g$ and finally $f \sim cg$

for some constant $c \neq 0$. The crucial property [5; § 28] of the relation « $\ll$ » is that if $f \ll 1$ over $F(a, b)$, then $\theta_j f \ll 1$ over $F(a, b)$ for all $j > 0$. From this, it easily follows that if $f \sim cz^{\alpha}$, where c and α are nonzero complex numbers, then $f' \sim \alpha c z^{\alpha-1}$. It is easily verified that for any complex number α and any $\varepsilon > 0$, we have $z^{\mathrm{Re}(\alpha)-\varepsilon} \ll \ll z^{\alpha} \ll z^{\mathrm{Re}(\alpha)+\varepsilon}$ over $F(a, b)$.

(*c*) [6; p. 244]. *A logarithmic field of rank zero* over $F(a, b)$ is a set L of functions, each defined and meromorphic in an element of $F(a, b)$, with the following properties: (i) L is a field (where, as usual, we identify two elements of L if they agree on an element of $F(a, b)$); (ii) L contains all functions of the form cz^{β}, for real β and complex $c \neq 0$, and (iii) for every element f in L except zero, there exists a function cz^{β}, with β real and $c \neq 0$, such that $f \sim cz^{\beta}$ over $F(a, b)$. (The set of all rational combinations of the functions cz^{β} is the simplest example of such a field, and this field clearly contains the field of rational functions.)

(*d*) let the equation (1) have coefficients in a logarithmic field L of rank zero. We say Ω is *nontrivial* if some coefficient is not identically zero. If $f_{kj} \not\equiv 0$, then

$$f_{kj}(z) = c_{kj} z^{a_{kj}} (1 + E_{kj}) , \tag{4}$$

where a_{kj} is real, c_{kj} is a nonzero complex number, and E_{kj} is an element of L which is $\ll 1$ over $F(a, b)$. We denote by $\delta_0(f_{kj})$ the number a_{kj}. (If $f_{kj} \equiv 0$, we set $\delta_0(f_{kj}) = -\infty$). Let $A = (k + j : f_{kj} \not\equiv 0)$. If $q \in A$, we denote by $\Omega^{(q)}$ the homogeneous part of Ω of degree q in the indeterminates y and y'. For a real number β, we set,

$$\Omega^{(q)}[*, \beta] = \beta q + \max\{\delta_0(f_{kj}) - j : k + j = q\} . \tag{5}$$

Finally [2; § 17], the *critical equation* of $\Omega^{(q)}$ is the equation $G_q(x) = 0$, where

$$G_q(x) = \sum \{c_{kj} x^j : (k, j) \in J_q\} , \tag{6}$$

where

$$J_q = \{(k, j) : k + j = q \quad \text{and} \quad \delta_0(f_{kj}) - j = \Omega^{(q)}[*, 0]\} . \tag{7}$$

3. We now state our main result.

The proof will be given in § 5.

THEOREM. Let $\Omega(z, y, y') = \sum f_{kj}(z) y^k (y')^j$ be a nontrivial polynomial in y and y' whose coefficients belong to a logarithmic field of rank zero over $F(a, b)$. Let $A = \{k + j : f_{kj} \neq 0\}$, and for $q \in A$, let $G_q(x) = 0$ be the critical equation of $\Omega^{(q)}$. Let α be a nonzero complex number for which there exists an element p in A such that,

(*a*) α is a simple root of the equation $G_p(x) = 0$, and

(*b*) $\Omega^{(q)}[*, \operatorname{Re}\alpha] < \Omega^{(p)}[*, \operatorname{Re}\alpha]$ for all $q \in A - \{p\}$.

Assume further that if $A \neq \{p\}$, then there exists an element t in $A - \{p\}$ such that,

(*c*) $G_t(\alpha) \neq 0$, and

(*d*) $\Omega^{(q)}[*, \operatorname{Re}\alpha] < \Omega^{(t)}[*, \operatorname{Re}\alpha]$ for all $q \in A - \{p, t\}$.

Then, for any complex number $c \neq 0$, the equation $\Omega = 0$, possesses a solution y_c, which is analytic in an element of $F(a, b)$ and satisfies $y_c \sim cz^\alpha$ over $F(a, b)$.

4. Lemma.

Let $\Omega(z, y, y') = \sum f_{kj}(z) y^k (y')^j$ be a nontrivial polynomial in y and y' whose coefficients belong to a logarithmic field of rank zero over $F(a,b)$, and assume that Ω is homogeneous as a polynomial in y and y' (i.e. for some p, $\Omega = \Omega^{(p)}$). Then, if α is a nonzero complex root of the critical equation $G_p(x) = 0$ of Ω, and c is a nonzero complex number, then the equation $\Omega = 0$ possesses a solution g, which is analytic in an element of $F(a, b)$, and satisfies $g \sim cz^\alpha$ over $F(a, b)$.

PROOF. Since $\Omega = \Omega^{(p)}$, if we divide the equation $\Omega = 0$ by y^p, and set $v = y'/y$, we obtain

$$H(v) = \sum_{k+j=p} f_{kj}(z) v^j = 0 . \tag{8}$$

Let $\beta = \Omega^{(p)}[*, 0]$, let J_p be as in (7) and let σ be a complex number. Using the representation (4) for the coefficients, and noting that $a_{kj} - j = \beta$ if $(k, j) \in J_p$, while $a_{kj} - j < \beta$ if $(k, j) \notin J_p$, it follows easily from (6) and (8) that,

$$(9) \qquad H(\sigma z^{-1}) = z^{\beta}\big(G_p(\sigma) + E\big) \text{ where } E \ll 1 \text{ over } F(a, b)\,.$$

Let σ_0 be a complex number which is not a root of G_p. Then since $G^p(\alpha) = 0$, we have,

$$(10) \qquad H(\alpha z^{-1}) \ll z^{\beta} \approx H\big((\sigma_0/\alpha)\,\alpha z^{-1}\big)\,.$$

Thus by [2; § 5(*a*)], αz^{-1} is a point of instability of H, so in the terminology of [6; § 5], the instability multiplicity of αz^{-1} for H is at least 1, that is,

$$(11) \qquad \operatorname{inst}(\alpha z^{-1}, H) \geqslant 1\,.$$

(This shows immediately that H is of degree $\geqslant 1$). Since H has coefficients in a logarithmic field of rank zero, it follows from [6; Theorem II, p. 244] (by applying this result to, in the terminology of [6; p. 246], the logarithmic quadruple $(F, E_0 * (F), R, S_0)$, where $F = F(a, b)$ and R is the set of real numbers), that there exists a logarithmic field of rank zero over $F(a, b)$ in which H factors completely into linear factors. Hence there exist distinct functions $\varphi_1, \ldots, \varphi_q$ all lying in a logarithmic field of rank zero over $F(a, b)$, and all nonidentically zero, such that for some (k, j) we have

$$(12) \qquad H(v) = f_{kj}\, v^{m_0}(v - \varphi_1)^{m_1} \ldots (v - \varphi_q)^{m_q}\,,$$

where $m_1, \ldots, m_q$ are positive integers. Now if $q = 0$ or none of the functions $\varphi_1, \ldots, \varphi_q$ are $\sim \alpha z^{-1}$, then by building up H one factor at a time, it would follow from repeated applications of [6; §§ 24, 25], that $\operatorname{inst}(\alpha z^{-1}, H) = 0$, contradicting (11). Thus $q \geqslant 1$ and for some j, $1 \leqslant j \leqslant q$, we have,

$$(13) \qquad \varphi_j \sim \alpha z^{-1} \quad \text{and} \quad H(\varphi_j) \equiv 0 \quad \big(\text{by } (12)\big)\,.$$

Since φ_j lies in a logarithmic field of rank zero, we have,

$$(14) \qquad \varphi_j = \alpha z^{-1}(1 + w_0)\,,$$

where $w_0 \ll 1$ and w_0 also lies in the field. If $w_0 \equiv 0$, then $H(\alpha z^{-1}) \equiv 0$, so clearly cz^α would be a solution of $\Omega = 0$ and the proof would be complete. If $w_0 \not\equiv 0$, then by property (iii) for a field of rank zero, (and noting that $w_0 \ll 1$), we have, over $F(a, b)$,

$$w_0 \sim Kz^\lambda, \quad \text{for some } \lambda < 0 \text{ and } K \neq 0 . \tag{15}$$

Recalling that the elements of $F(a, b)$ are simply-connected, and choosing a point r in the domain of w_0, let

$$U(z) = \int_r^z \alpha\zeta^{-1} w_0(\zeta)\, d\zeta . \tag{16}$$

In view of (15), it follows from [4; Lemma ζ (b), p. 272], that there is a complex number K_1 such that

$$K_1 + U \sim (\alpha K/\lambda)\, z^\lambda \quad \text{over } F(a, b) . \tag{17}$$

Now set,

$$U_1 = K_1 + U \quad \text{and} \quad W = \exp U_1 . \tag{18}$$

Then if we set $g = cz^\alpha W$, if follows easily from (14) that $g'/g = \varphi_j$, and hence since $H(\varphi_j) \equiv 0$ (by (13)), we see that g is a solution of $\Omega = 0$. Hence the proof will be complete if we show $g \sim cz^\alpha$, or equivalently that $W \sim 1$ over $F(a, b)$. To this end, let $V = W - 1$. Since $\lambda < 0$, clearly $U_1 \to 0$ and hence $W \to 1$ over $F(a, b)$. Thus $V \to 0$ over $F(a, b)$. Now let k be a positive integer, and refer to the operators θ_k^j in § 2(b). Clearly $\theta_k V = W\theta_k U_1$, since $W' = WU_1'$. By induction on j, it is easy to see that for $j \geqslant 1$ we have,

$$\theta_k{}^j V = WQ_j(\theta_k U_1, \theta_k^2 U_1, \ldots, \theta_k^j U_1) , \tag{19}$$

where $Q_j(u_1, \ldots, u_j)$ is a polynomial in $u_1, \ldots, u_j$, with constant coefficients and $Q_j(0, \ldots, 0) = 0$. Since $\lambda < 0$, it follows from (17) that $U_1 \ll 1$. Thus $\theta_k^j U_1 \to 0$ over $F(a, b)$ for all positive k and j. Since $W \to 1$, it follows from (19) that $\theta_k^j V \to 0$ for all positive k and j and hence $V \ll 1$ over $F(a, b)$. Thus $W \sim 1$ and the proof of the lemma is now complete.

5. Proof of the main result (§ 3).

We can assume at the outset that $A \neq \{p\}$, for if $A = \{p\}$, then $\Omega = \Omega^{(p)}$ and the result follows from the previous lemma.

Let the nonzero coefficients of Ω have the representation (4) and let G_q and J_q be as in (6) and (7). By hypothesis, we have,

$$(20) \qquad G_p(\alpha) = 0 \quad \text{and} \quad G_p'(\alpha) \neq 0 .$$

(Note that $p \geqslant 1$ for if $p = 0$, $G_p(x)$ would have no roots.) For convenience, let

$$(21) \qquad b_q = \Omega^{(q)}[*, 0] \quad \text{for } q \in A ,$$

and let c be any nonzero constant. By the previous lemma, there exists an analytic function g in an element of $F(a, b)$ such that over $F(a, b)$,

$$(22) \qquad g \sim cz^{\alpha} \quad \text{and} \quad \Omega^{(p)}(z, g(z), g'(z)) \equiv 0 .$$

Since $\alpha \neq 0$, it follows from § 2(b) that we have the representations,

$$(23) \quad g = cz^{\alpha}(1 + E_1) \quad \text{and} \quad g' = \alpha c z^{\alpha-1}(1 + E_2) \quad \text{where } E_j \ll 1.$$

We now consider the equation,

$$(24) \qquad \Omega(z, y, y')/g^p z^{b_p} = 0 .$$

Under the change of dependent variable $y = gw$, (24) is transformed into the equation,

$$(25) \qquad \Lambda(z, w, w') = \sum d_{nm} w^n (w')^m = 0 ,$$

where for each $q \geqslant 0$, and $m \leqslant q$, we have

$$(26) \qquad d_{q-m,m} = \sum_{j=m}^{q} T_{qmj} ,$$

where

$$T_{qmj} = g^{-p} z^{-b_p} f_{q-j,j} \binom{j}{m} g^{q-j+m} (g')^{j-m} . \tag{27}$$

From the second relation in (22), it follows that

$$d_{p0} \equiv 0 . \tag{28}$$

We now consider $d_{p-1,1}$. In view of the representations (4) and (23), and the definitions of b_p, J_p and $G_p(x)$, it follows from a straightforward calculation of (26) for the case $q = p$ and $m = 1$, that over $F(a, b)$,

$$d_{p-1,1} \sim \lambda z , \qquad \text{where } \lambda = G_p'(\alpha) \neq 0 \text{ by } (20) . \tag{29}$$

From the representations (4) and (23), we see that if $f_{p-j,j} \not\equiv 0$, then $T_{pmj} \approx z^{a_{p-j,j}-j-b_p+m}$. By definition of b_p, it follows that for each j, either T_{pmj} is $\approx z^m$ or is $\ll z^m$. Thus by (26), it easily follows that for each m,

$$d_{p-m,m} \text{ is either } \approx z^m \text{ or is } \ll z^m . \tag{30}$$

We now consider $d_{q-m,m}$ where $q \in A - \{p\}$. From (4) and (23), we see that if $f_{q-j,j} \not\equiv 0$, then

$$T_{qmj} \approx z^{\beta(q,j)+m} , \tag{31}$$

where,

$$\beta(q, j) = \alpha q + a_{q-j,j} - j - (\alpha p + b_p) . \tag{32}$$

Now from (b) of the hypothesis, there exists $\delta > 0$ such that $\Omega^{(q)}[*, \operatorname{Re} \alpha] < \Omega^{(p)}[*, \operatorname{Re} \alpha] - 2\delta$ for all $q \in A - \{p\}$. Hence,

$$\operatorname{Re} \big(\beta(q, j)\big) < -2\delta ,$$

and thus from (31) (and § 2(b)), we have $T_{qmj} \ll z^{m-\delta}$. Thus from (26),

$$d_{q-m,m} \ll z^{m-\delta} \qquad \text{for } q \in A - \{p\} . \tag{33}$$

Of course if $q \notin A$, then each $f_{q-j,j} \equiv 0$ so clearly each $d_{q-m,m} \equiv 0$ by (26).

Hence, with (33) we have,

$$d_{q-m\,m} \ll z^{m-\delta} \quad \text{if } q \neq p\,. \tag{34}$$

Under the change of dependent variable $w = 1 + v$, equation (25) is transformed into the equation,

$$\Phi(z, v, v') = \sum D_{kj} v^k (v')^j = 0\,, \tag{35}$$

where

$$D_{kj} = \sum_{n \geqslant k} \binom{n}{k} d_{nj}\,. \tag{36}$$

From (28) and (34), we see that,

$$D_{k0} \ll z^{-\delta} \quad \text{over } F(a, b) \text{ for } k \geqslant 0\,. \tag{37}$$

From (29) and (34), we see that since $\delta > 0$,

$$D_{01} \sim \lambda z \quad \text{over } F(a, b)\,, \tag{38}$$

while for $k \geqslant 1$,

$$D_{k1} \text{ is either } \approx z \text{ or } \ll z\,. \tag{39}$$

Finally, from (30) and (34), we see that for each k, j,

$$D_{kj} \text{ is either } \approx z^j \text{ or } \ll z^j\,. \tag{40}$$

We now investigate the coefficient D_{00} more closely. Clearly, $D_{00} = \Omega(z, g, g')/g^p z^{b_p}$: In view of (22) and the definition of the set A, we have

$$D_{00} = g^{-p} z^{-b_p} \sum \{\Omega^{(q)}(z, g, g') : q \in A - \{p\}\}\,. \tag{41}$$

In view of the representations (4) and (23), and the definitions of b_q, J_q and $G_q(x)$, it follows from a straightforward calculation, that for $q \in A$, we have

$$\Omega^{(q)}(z, g, g') = c^q z^{\alpha q + b_q} \big(G_q(\alpha) + E_q\big) \quad \text{where } E_q \ll 1\,. \tag{42}$$

Now by hypotheses (c) and (d), there exist $t \in A - \{p\}$ and $\delta_1 > 0$ such that $G_t(\alpha) \neq 0$, and

$$(43) \qquad \Omega^{(q)}[*, \operatorname{Re}\alpha] < \Omega^{(t)}[*, Re\ \alpha] - 2\delta_1 \qquad \text{for } q \in A - \{p, t\}\,.$$

Thus from (42),

$$(44) \qquad \Omega^{(t)}(z, g, g') \sim c^t \lambda_1 z^{\alpha t + b_t} \qquad \text{where } \lambda_1 = G_t(\alpha) \neq 0\,.$$

Furthermore, since $\operatorname{Re}(\alpha q + b_q - (\alpha t + b_t)) < -2\delta_1$ by (43) for $q \in A - \{p, t\}$, it follows from § 2(b) that $z^{\alpha q + b_q - (\alpha t + b_t)} \ll z^{-\delta} \ll 1$, and hence by (42) and (44), we have that for each $q \in A - \{p, t\}$, the function $\Omega^{(q)}(z, g, g')$ is $\ll$ the function $\Omega^{(t)}(z, g, g')$ over $F(a, b)$.

Thus from (41), (44) and (23), we have,

$$(45) \qquad D_{00} \sim K_1 z^{\beta} \qquad \text{over } F(a, b)\,,$$

where

$$(46) \qquad \beta = \alpha t + b_t - (\alpha p + b_p) \qquad \text{and} \qquad K_1 = c^{t-p} \lambda_1\,.$$

By hypothesis (b), we have,

$$(47) \qquad \operatorname{Re}(\beta) < 0\,.$$

In view of (45) and (38), we have,

$$(48) \qquad D_{00}/D_{01} \sim (K_1/\lambda)\, z^{\beta - 1}\,.$$

We consider the equation,

$$(49) \qquad h' = -D_{00}/D_{01}\,.$$

Under the change of dependent variable,

$$(50) \qquad h = -(K_1/\lambda\beta)\, z^{\beta}(1 + \varphi)\,,$$

followed by multiplication by $(\lambda/K_1) z^{1-\beta}$, we have from (48) that equation (49) is transformed into an equation of the form,

$$(51) \qquad \varphi + (z/\beta)\,\varphi' = U\,, \qquad \text{where } U \ll 1 \text{ over } F(a, b)\,.$$

In view of (47), it follows from [4; Lemma $\delta(b)$, p. 271], that equation (51) possesses a solution φ_0, analytic in an element of $F(a, b)$, and satisfying $\varphi_0 \ll 1$ over $F(a, b)$. Thus by (50), the equation (49) possesses a solution h_0 such that

$$h_0 \sim -(K_1/\lambda\beta) z^{\beta} \quad \text{over } F(a, b) . \tag{52}$$

Under the change of dependent variable $v = h_0 + h_0 u$, equation (35) is transformed into the equation,

$$\Psi(z, u, u') = \sum t_{mn} u^m (u')^n = 0 , \tag{53}$$

where the coefficients are given by the formula,

$$t_{mn} = (h_0)^n \sum_{k \geqslant m} \binom{k}{m} \Gamma_{kn} , \tag{54}$$

where

$$\Gamma_{kn} = \sum_{i=0}^{k} \binom{k-i+n}{n} D_{i,k+n-i} (h_0)^i (h_0')^{k-i} . \tag{55}$$

Since h_0 solves equation (49), it follows from (48) and (52) that for $0 \leqslant i \leqslant k$,

$$h_0^i (h_0')^{k-i} \approx z^{k\beta-k+i} \quad \text{over } F(a, b) . \tag{56}$$

It thus follows from (40) and (55) that for each k, n,

$$\Gamma_{kn} \text{ is either } \approx z^{k\beta+n} \text{ or } \ll z^{k\beta+n} \text{ over } F(a, b) . \tag{57}$$

Now $\Gamma_{00} = D_{00}$ and $\Gamma_{10} = D_{01} h_0' + D_{10} h_0$. Since h_0 solves (49), it follows from (54) that

$$t_{00} = D_{10} h_0 + \sum_{k \geqslant 2} \Gamma_{k0} . \tag{58}$$

In view of (37) and (52), $D_{10} h_0 / z^{\beta} \ll 1$. Furthermore, if $k \geqslant 2$, then since $\mathrm{Re}\big((k-1)\beta\big) < 0$ (by (47)), it follows from (57) and § 2(b) that $\Gamma_{k0}/z^{\beta} \ll 1$. Hence from (58),

$$t_{00}/z^{\beta} \ll 1 \quad \text{over } F(a, b) . \tag{59}$$

Now $\Gamma_{01} = D_{01}$ so by (38),

$$\Gamma_{01} \sim \lambda z . \tag{60}$$

For $k \geqslant 1$, $\mathrm{Re}(k\beta + 1) < 1$ by (47), so by (57) and § 2(*b*), $\Gamma_{k1} \ll z$ for $k \geqslant 1$. Thus by (54) and (60), clearly, $t_{01} \sim h_0 \lambda z$. Thus by (52),

$$t_{01}/z^{\beta} \sim (-K_1/\beta) z \quad \text{over } F(a, b) . \tag{61}$$

We now consider $t_{10} = \sum_{k \geqslant 1} k\Gamma_{k0}$. Now $\Gamma_{10} = D_{01} h_0' + D_{10} h_0$. Since h_0 solves (49), it follows easily from (37), (38), (48) and (52), that, $D_{10} h_0 \ll \ll D_{01} h_0' \sim -K_1 z^{\beta}$. Thus,

$$\Gamma_{10} \sim -K_1 z^{\beta} . \tag{62}$$

Now for $k \geqslant 2$, $\mathrm{Re}\big((k-1)\beta\big) < 0$ (by (47)), so clearly from (57) and § 2(*b*), we have

$$\Gamma_{k0} \ll z^{\beta} \quad \text{for } k \geqslant 2 . \tag{63}$$

Thus with (62), we have $t_{10} \sim -K_1 z^{\beta}$, and hence

$$t_{10}/z^{\beta} \sim -K_1 \quad \text{over } F(a, b) . \tag{64}$$

From (63), it immediately follows that over $F(a, b)$,

$$t_{m0}/z^{\beta} \ll 1 \quad \text{for } m \geqslant 2 . \tag{65}$$

Finally, we consider the ratio t_{mn}/t_{01} for $n \geqslant 1$ and $m + n \geqslant 2$. We rewrite (54) in the form,

$$t_{mn}/t_{01} = \sum_{k \geqslant m} \binom{k}{m} (h_0)^n (\Gamma_{kn}/t_{01}) . \tag{66}$$

In view of (52), (57) and (61), the term in the summation corresponding to k is either $\approx$ or $\ll z^{\beta(k+n-1)+n-1}$. But $k \geqslant m$, so $k + n - 1 \geqslant m + n - 1 \geqslant 1$. Thus by (47), $\mathrm{Re}\big(\beta(k + n - 1)\big) < 0$, and hence it follows from § 2(*b*) that each term in the sum on the right of (66) is $\ll z^{n-1}$,

$$t_{mn}/t_{01} \ll z^{n-1} \quad \text{over } F(a, b) \text{ if } n \geqslant 1 \text{ and } m + n \geqslant 2 . \tag{67}$$

If follows from the asymptotic relations (59), (61), (64), (65) and (67) that the polynomial $\Psi(z, u, u')/(-K_1 z^{\beta})$ is *normal* over $F(a, b)$ in the sense of [5; § 83]). In view of (47), it follows from [5; §§ 117, 118] that the equation,

$$\Psi(z, u, u')/(-K_1 z^{\beta}) = 0 \,, \tag{68}$$

possesses a unique solution u_0, which is analytic in an element of $F(a, b)$ and satisfies,

$$u_0 \ll 1 \text{ over } F(a, b) \,. \tag{69}$$

From (35), it follows that $w_0 = 1 + h_0 + h_0 u_0$ satisfies equation (25). Thus the function $y_0 = g(1 + h_0 + h_0 u_0)$ is analytic in an element of $F(a, b)$ and is a solution of the original equation $\Omega(z, y, y') = 0$. In view of (69) and the fact that $h_0 \ll 1$ over $F(a, b)$ by (47) and (52) (and § 2(b)), we see that $y_0 \sim g$ over $F(a, b)$ and hence by (22), $y_0 \sim cz^{\alpha}$. This concludes the proof of the theorem.

6. Remark.

We point out here that for the differential polynomials Ω, and the functions cz^{α} which are treated in the main result, it was shown in the course of the proof that cz^{α} is a point of instability of a homogeneous Ω, if and only if α is a root of the critical equation of Ω. (If $\Omega = \Omega^{(q)}$ and if α is not a root of the critical equation $G_q(x) = 0$, then from the calculation in (42) it follows that for *any* function $g \sim cz^{\alpha}$, we have that $\Omega(z, g, g') \sim c^q G_q(\alpha) z^{\alpha q + b_q}$, so clearly cz^{α} is not a point of instability of Ω. Conversely, if α is a root of $G^q(x) = 0$, then by § 4, the equation $\Omega = 0$ possesses a solution $\sim cz^{\alpha}$ and hence clearly cz^{α} is a point of instability of Ω.)

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