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Locally Convex Inductive Limits of Normed Algebras.

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Introduction.

In the paper « Inductive limit of normed algebras » (see ref. [7]) S. Warner studied the case of an algebra A carrying the finest locally m-convex (in the sense of Michael) topology which makes continuous the inclusion maps of a family of subalgebras $\{A_i | i \in I\}$, where every A_i is endowed with a structure of normed algebra.

Since mathematical analysis is much more concerned with locally convex inductive limits than with locally m-convex ones, we propone a study of the above argument, replacing m-convex with convex. We point out that we are not able to exhibit one case in which the locally convex inductive topology differs from the locally m-convex one: in other words, to the extent of our knowledge, the following question is open:

« If τ is the locally convex inductive limit topology of a family of normed algebras, is then τ locally m-convex? »

At any rate, also in the case that the answer to the preceding question is yes, this works leads to some original results, by using techniques of Waelbroeck's b-algebras established after Warner's paper.

The subject is very similar to Waelbroeck's b-algebras and to pseudo-Banach algebras of Allan-Dales-Mc Clure (Studia Math. 40 (1971) pp. 55-69), but differs from these since we do not require any sort of

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completeness and mainly since our interest is topological, introducing bornologies only as a mean.

The content is in detail the following: in § 1 we show that the main elementary properties of normed algebras are preserved under the operation of locally convex inductive limit, except for the continuity of the inverse mapping $x \mapsto x^{-1}$ (where defined): the problem whether also this last property is preserved is intimately connected (in commutative case, equivalent) with the above question (see prop. 10).

In § 2 we furnish some hypothesis which assure the positive answer to the above question in commutative case: namely if A is a countable limit, or a Fréchet space, or a metrizable space carrying the Allan boundedness (see def. 2).

At last, in § 3, we restrict ourselves to the special case of topological algebras endowed with a Fréchet or Fréchet-Montel topology.

We end with examples and counterexamples.

Notations and terminology.

In this paper every algebra is over the complex field $\boldsymbol{C}$, and it is endowed with a unit that is marked e when this does not generate confusion: every subalgebra is implicitly supposed to contain the unit of the algebra.

We say that A is a *topological algebra* iff it is an algebra endowed with a structure of locally convex space (not necessarily Hausdorff) respect to which the product is separately continuous. We say that A is a *(semi)normed algebra* iff it is an algebra endowed with a (semi)norm ν which is also *submultiplicative*, that is $\nu(xy) \leqslant \nu(x) \cdot \nu(y)$ for every $x, y \in A$; in this case it is possible to find a topologically equivalent (semi)norm ν' in such a way that $\nu'(e) = 1$; we suppose implicitly that this last relation is verified when we introduce a submultiplicative (semi)norm, so that in every (semi)normed algebra the norm of the unit is equal to 1. Clearly a (semi)normed algebra is a topological algebra. If ν is the Minkowsky functional of an absolutely convex set V (namely: $\nu(x) \equiv \inf \{t > 0 | x \in tV\}$ for every x in the linear span of V) then ν is submultiplicative iff $V \cdot V \subseteq V$: in other words iff V is *idempotent*.

We say that A is a *LMCA* (locally m-convex algebra) iff it is an algebra endowed with a topology given by a family of submultiplicative seminorms (equivalently, a locally convex topology which has a fundamental system of idempotent 0-neighbourhoods). Clearly a LMCA is a topological algebra. Elementary properties of LMCA's have been investigated in [4] and we shall use them without explicit recalls.

Sometimes we use the notation (A, τ) (resp. (A, ν)) to mean the algebra A endowed with a topology τ (resp.: a seminorm ν).

Now let A be a topological algebra. We say that A has the *Q-property* iff there is a 0-neighbourhood V such that for every $x \in V$ there exists $(e-x)^{-1}$ in A (equivalently, iff the set of invertible elements of A is open). We say that A is a *continuous inverse algebra* iff A has the Q-property and the mapping $x \mapsto x^{-1}$ is continuous on the domain set. We denote M_A the set of continuous multiplicative linear functionals on A (not vanishing everywhere) endowed with the usual pointwise convergence topology. The *Fourier-Gelfand transform* is an application $\hat{}$ of A into $C(M_A)$ (the space of all complex-valued continuous functions on M_A) defined by $\hat{x}(f) = f(x)$ for every $x \in A$ and $f \in M_A$; we usually consider $C(M_A)$ with the topology of uniform convergence on compacta.

$\sigma_A(x)$ will denote the spectrum of x in A, namely $\sigma_A(x) \equiv \equiv \{\lambda \in \boldsymbol{C} | (\lambda e - x)^{-1}$ does not exists in $A\}$ and $r_A(x)$ will denote the spectral radius of x in A, namely $r_A(x) \equiv \sup\{|\lambda| \,|\, \lambda \in \sigma_A(x)\}$. Let A be a subalgebra of A: we say that A' is *algebraically dense* in A iff every element of A' which is not invertible in A' has no inverse in A too. R_A will denote the radical of A, namely $R_A \equiv \cap \{I \subseteq A | I$ maximal ideal$\}$: we say that A is *semisimple* iff $R_A = \{0\}$.

§ 1. In this paper we are interested about such a situation:

(*) A is an algebra, $\{A_i | i \in I\}$ is a family of subalgebras of A s.t. $A = \cup \{A_i | i \in I\}$. For every $i \in I$, A_i is a normed algebra, in such a way that the family $\{A_i | i \in I\}$ is *directed by continuous inclusions*: namely for every $i, j \in I$ there exists $l \in I$ s.t. $A_i \subseteq A_l$, $A_j \subseteq A_l$ and the inclusion maps are continuous.

If τ is a locally convex topology on A which makes continuous the inclusion map $\varphi_i \colon A_i \to A$ for every $i \in I$, then every 0-neighbourhood for τ absorbs the unit ball of A_i, which we denote S_i, for every $i \in I$. Then among all the topologies as above a finest one τ_l exists (not necessarily Hausdorff): a fundamental system of 0-neighbourhood for τ_l is the family of absolutely convex subsets of A absorbing S_i, for every $i \in I$. It is easy to check that every linear map

$\psi: A \to E$ (E locally convex space) is τ_l-continuous iff the restriction map $\psi \circ \varphi_i$ is continuous on the normed algebra A_i, for every $i \in I$. As a special case for every $x \in A$ the mappings of A into itself $y \mapsto xy$ and $y \mapsto xy$ are τ_l-continuous (fix $i \in I$: there exists $j \in I$ s.t. $x \in A_j$ and $A_i \subseteq A_j$: as $y \mapsto yx$ and $y \mapsto xy$ are continuous mappings of A_j into itself, they result continuous mappings of A_i into A by composition), so that A endowed with the topology τ_l is a topological algebra.

We adopt the following

TERMINOLOGY. *If A, $\{A_i | i \in I\}$ and τ_l are as above, we say that $\{A_i | i \in I\}$ is a L-family for A, and we call τ_l the LCIL (locally convex convex inductive limit) topology relative to the family $\{A_i | i \in I\}$.*

The properties of the pair (A, τ_l) are the object of our study, which however is not bounded to the set of topological algebras *defined* from a situation of type $(*)$, because in some cases, as shown in the latter part of this section, we may say that a given topological algebra A carries the LCIL topology relative to a L-family built *a posteriori*.

In analogy with the notation LMCA, we give the following

DEFINITION 1. *Let A be a topological algebra. We say that A is a BMCA iff it is possible to find a L-family for A, which A carries the LCIL topology relative to. Moreover when we say that A is a Banach-BMCA (resp.: $\aleph_0$-BMCA) we want to mean that it is possible by means of a L-family of Banach algebras (resp.: by means of a countable L-family).*

For examples see ex. I, II.

We have pointed out that in situation $(*)$ the LCIL topology need not be Hausdorff. In this case one is used to take the associated Hausdorff space: the following proposition then. assures that in this way one does not go out the terms of this paper.

PROPOSITION 1. *Let A be a BMCA (resp.: Banach-BMCA, $\aleph_0$-BMCA). For every J two sided ideal of A the topological quotient A/J is, with the obvious product, a BMCA (resp.: Banach-BMCA, $\aleph_0$-BMCA). As a particular case the Hausdorff space associated to A (namely the topological quotient of A for the ideal $J =$ closure of $\{0\}$) is, with the obvious product, a BMCA (resp.: Banach-BMCA, $\aleph_0$-BMCA).*

PROOF. Let $\{A_i|i\in I\}$ be a L-family for A wh'ch A carries the LCIL topology relative to; denote π the natural mapping of A into A/J, and consider $\pi(A_i)$ endowed with the quotient norm respect to A_i and π, for every $i\in I$: then it is easy to check that $\{\pi(A_i)|i\in I\}$ is a L-family for A/J and that the relative LCIL topology coincides with the quotient topology of A/J ▪

The proposition we prove now is very useful in the sequel: it is also an interesting counterpart of the fact that every LMCA is isomorphic to a dense subalgebra of an inverse limit of Banach algebras. We remind that a sequence $(x_n)_{n\in\boldsymbol{N}}$ in E (E topological vector space) is said to have limit $y\in E$ in the sense of Mackey (resp.: is said to be a Cauchy sequence in the sense of Mackey) iff there exists a bounded subset C and a sequence of real numbers $(\varepsilon_n)_{n\in\boldsymbol{N}}$ s.t. $\lim\varepsilon_n=0$ and $x_n-y\in\varepsilon_n C$, for every $n\in\boldsymbol{N}$ (resp.: $x_n-x_m\in\varepsilon_n C$ for every $m,n\in\boldsymbol{N}$, $m\geqslant n$); clearly every sequence that converges in the sense of Mackey is a Cauchy sequence in the sense of Mackey.

PROPOSITION 2. *Let A be a Hausdorff BMCA. Then A may be identified to a subalgebra of a Banach-BMCA A' s.t. every $y\in A'$ is the limit in the sense of Mackey of a sequence contained in A.*

PROOF. Suppose that A carries the LCIL topology relative to a L-family $\{A_i|i\in I\}$: denote S_i the unit ball of A_i, for every $i\in I$. We may suppose that S_i is closed in A, for every $i\in I$ (if it were not, denote $\overline{S}_i$ the closure of S_i in A and $A_{\overline{S}i}$ the span of $\overline{S}_i$ in A: $\overline{S}_i$ is an absolutely convex idempotent bounded subset of A, so if we norm $A_{\overline{S}i}$ with the Minkowsky functional of $\overline{S}_i$ we get a normed algebra. It is easy to check that $\{A_{\overline{S}_i}|i\in I\}$ is a L-family for A, that A carries the relative LCIL topology, that the unit ball of $A_{\overline{S}_i}$ is just $\overline{S}_i$).

We naturally identify A to a linear subspace of its completion $\tilde{A}$: also for every $i\in I$ we identify (only algebraically) $\tilde{A}_i$, the completion of A_i, to a linear subspace of $\tilde{A}$. This last identification is possible as the (unique) continuous extension $\tilde{\varphi}_i$ to completions of the inclusion map $\varphi_i\colon A_i\to A$ is injective: suppose in fact that $(x_n)_{n\in\boldsymbol{N}}$ is a Cauchy sequence in A_i s.t. $\lim x_n=0$ in A, then there exists a sequence of real numbers $(\varepsilon_n)_{n\in\boldsymbol{N}}$ s.t. $\lim\varepsilon_n=0$ and $x_n-x_m\in\varepsilon_n S_i$ for every $m\geqslant n$: passing to the limit in m in the topology of A we get $x_n\in\varepsilon_n S_i$ (remember S_i is closed in A), that is $\lim x_n=0$ in A_i.

For every $i\in I$, $\tilde{A}_i$ is a Banach algebra (with the obvious product) and $\{\tilde{A}_i|i\in I\}$ is a L-family for the linear subspace $A'=$

$\equiv \cup \{\tilde{A}_i | i \in I\}$. So we may endow A' with the obvious product and consider A a Banach-BMCA with the LCIL topology relative to $\{\tilde{A}_i | i \in I\}$.

The inclusion map of A' into $\tilde{A}$ is continuous, as the restriction map on $\tilde{A}_i$ is just the continuous mapping $\hat{\varphi}_i$, for every $i \in I$: as a consequence the topology induced by A' on A is finer than the induced one by $\tilde{A}$, which turns out to be the given topology of A. On the other side the inclusion map of A_i into A' is continuous by composition, for every $i \in I$. Then A is algebraically and topologically a subalgebra of A'.

If $y \in A'$ there exists $i \in I$ s.t. $y \in \tilde{A}_i$; as the natural map of A_i into $\tilde{A}_i$ has dense image, for every $i \in I$, we deduce that there exists a sequence in A_i which has y as its limit in $\tilde{A}_i$: it is easy to see that the convergence is also in the sense of Mackey in A' ∎

COROLLARY. *Let A be a Hausdorff BMCA: if every Cauchy sequence in the sense of Mackey converges, then A is a Banach-BMCA.*

Now we formule the general theorems on BMCA's.

THEOREM 1. *Let A be a BMCA. Following properties are true:*

a) *M_A is equicontinuous on A, compact in the pointwise convergence topology (surely non empty if A is commutative); the the Fourier-Gelfand transform is continuous*

b) *for every $x \in A$: $\sigma_A(x) \neq \emptyset$*

c) *if A is Hausdorff: for every $x \in A$ the mapping R_x of $\mathbf{C} - \sigma_A(x)$ into A defined by $R_x(\lambda) = (\lambda e - x)^{-1}$ is continuous, and $\lim R_x(\lambda) = 0$ as $\lambda \to \infty$, $\lambda \notin \sigma_A(x)$. Moreover R_x is holomorphic on the interior of $\mathbf{C} - \sigma_A(x)$ and*

$$(\lambda e - x)^{-1} = \sum_{0\,k}^{\infty} x^k \lambda^{-k-1}$$

holds for $|\lambda| > r_A(x)$

d) *A may not be a field, unless it is isomorphic to the complex field $\mathbf{C}$.*

PROOF. Let A carry the LCIL topology relative to a L-family $\{A_i | i \in I\}$: we denote S_i the unit ball of A_i, for every $i \in I$. We prove *a)*: for every $f \in M_A$, $i \in I$, $x \in S_i$ we have $|f(x)| \leqslant 1$. In fact if *ab absurdo* $|f_0(x_0)| > 1$ for some $x_0 \in S_{i_0}$ then f_0 is unbounded on A_{i_0} as $x_0^n \in S_{i_0}$ for every $n \in \boldsymbol{N}$ while $\lim f_0(x_0^n) = \lim [f_0(x_0)]^n = \infty$. We may conclude that ${}^0(M_A) \equiv \{x \in A \mid |f(x)| \leqslant 1$ for every $f \in M_A\} \supseteq \cup \{S_i \mid i \in I\}$, so that ${}^0(M_A)$ absorbs S_i for every $i \in I$, and so it is a 0-neighbourhood in A because it is absolutely convex: in other words M_A is equicontinuous on A. Continuity of Fourier-Gelfand transform follows by considering that that the set $\{x \in A \mid |\hat{x}(f)| \leqslant 1$ for every $f \in M_A\}$ is nothing but ${}^0(M_A)$.

For every $i \in I$, M_{A_i} is compact, non empty if A is commutative: this is true also for M_A, as it is isomorphic to the inverse limit of the family $\{M_{A_i} \mid i \in I\}$ respect to the transposed applications (see [1], (6.4), for instance).

b) By virtue of prop. 1 and prop. 2 we may restrict ourselves to the special case that A_i is Banach, for every $i \in I$.

Fixed $x \in A$, we denote $I_x = \{i \in I \mid x \in A_i\}$ and $\sigma_i(x) = \{\lambda \in \boldsymbol{C} \mid (\lambda e - x)^{-1}$ does not exists in $A_i\}$, for every $i \in I_x$. We say that $\sigma_A(x) = \cap \{\sigma_i(x) \mid i \in I_x\}$: it is clear that $\sigma_A(x) \subseteq \sigma_i(x)$ for every $i \in I_x$, on the other side if $\lambda \notin \sigma_A(x)$ then, as $A = \cup \{A_i \mid i \in I_x\}$, there exists $j \in I_x$ s.t. $(\lambda e - x)^{-1} \in A_j$, so that $\lambda \notin \sigma_i(x)$. As we have supposed that A_i is a Banach algebra for every $i \in I$, we see that $\{\sigma_i(x) \mid i \in I_x\}$ is a $(\supseteq)$-directed family of non empty compact sets, so $\sigma_A(x) \neq \emptyset$.

c) As property *c*) is inherited by subalgebras, by virtue of prop. 2 we may restrict ourselves to the special case that A_i is Banach, for every $i \in I$.

For every $x \in A$ let I_x be as above, and let us define for every $i \in I_x$ the mapping $R_{x,i}$ of $\boldsymbol{C} - \sigma_i(x)$ into A_i by $R_{x,i}(\lambda) = (\lambda e - x)^{-1}$: as in a Banach algebra the mapping $x \mapsto x^{-1}$ is continuous on the domain set, we have by composition that $R_{x,i}$ is continuous and we get from the identity $R_{x,i}(\lambda) = \lambda^{-1}(e - \lambda^{-1}x)^{-1}$ that $\lim R_{x,i}(\lambda) = 0$ for $\lambda \mapsto \infty$, $\lambda \notin \sigma_i(x)$.

As $R_{x|\boldsymbol{C}-\sigma_i(x)} = R_{x,i}$ (modulo inclusion mapping), $\boldsymbol{C} - \sigma_i(x)$ being open, for every $i \in I_x$ and $\cup \{\boldsymbol{C} - \sigma_i(x) \mid i \in I_x\} = \boldsymbol{C} - \cap \{\sigma_i(x) \mid i \in I_x\} = \boldsymbol{C} - \sigma_A(x)$, we see that R_x turns out to be continuous on the domain set. It is also easy to check that $\lim R_x(\lambda) = 0$ for $\lambda \to \infty$, $\lambda \notin \sigma_A(x)$.

The last part of the assert follows by using lemma 3, postponed to theorem 2.

d) If A is a field the elements of A that are not of the form λe $(\lambda \in \boldsymbol{C})$ have empty spectrum: thesis follows from *b*) ∎

THEOREM 2. *Let A be a BMCA.*

Consider the following properties:

a) *A has the Q-property*

a′) *there is a* 0*-neighbourhood* V *s.t. for every* $x \in V$: $(e-x)^{-1}$ *exists*, $(e-x)^{-1} = \sum_{0}^{\infty} x^k$ *(in the sense of the topology of* A*) and* $\lim_{k\to\infty} x^k = 0$

b) *if* $y \in A$ *and* $(x_\alpha)_{\alpha\in\Lambda}$ *is a Cauchy net in* A *s.t.* $\lim x_\alpha y = \lim y x_\alpha = e$, *then* y *is an invertible element.*

c) A *carries the LCIL topology relative to a L-family of normed algebras endowed with the Q-property*

d) r_A *is a submultiplicative continuous seminorm on* A

e) *the maximals ideals of* A *are the kernels of the elements of* M_A

f) *(Wiener property) for every* $x \in A$:
x *is an invertible element iff* $f(x) \neq 0$ *for every* $f \in M_A$

g) *for every* $x \in A$: $r_A(x) = \sup\{|f(x)| \,|\, f \in M_A\}$

h) *there is a continuous homomorphism* ψ *of* A *into a commutative and semisimple Banach algebra* E, *with the following properties:* ψ *preserves the unit,* $\psi(A)$ *is topologically and algebraically dense in* E, *ker* $\psi = R_A$; *every maximal ideal of* A *is obtained as inverse image of one (and only one) maximal ideal of* E; ψ *preserves the spectrum of every element.*

If A *is Hausdorff then* $a) \Leftrightarrow a') \Rightarrow b) \Rightarrow c)$.
If A *is commutative then* $c) \Rightarrow d) \Rightarrow e) \Rightarrow f) \Rightarrow g) \Rightarrow h) \Rightarrow a)$.

Consequently all the properties listed above are equivalent for a Hausdorff commutative BMCA, and in particular are true for a Hausdorff commutative Banach-BMCA.

PROOF. – First suppose A is a Hausdorff BMCA.

The equivalence $a) \Leftrightarrow a')$ follows from theorem 1, *c*) and lemma 3 postponed to this theorem.

$a) \Rightarrow b)$: let W be a neighbourhood of e which consists of invertible elements: then there exists $\alpha \in \Lambda$ s.t. $x_\alpha y \in W$, so that $x_\alpha y$ is invertible and as a consequence y has a left inverse: analogously we find that y has a right inverse, then y is invertible.

$b) \Rightarrow c)$: let A carry the LCIL topology relative to a L-family $\{A_i | i \in I\}$: denote S_i the unit ball of A_i, for every $i \in I$. Define $\{\tilde{A}_i | i \in I\}$ as in prop. 2 and consider $A_i^* = \tilde{A}_i \cap A$ endowed with the induced topology by $\tilde{A}_i$, so that A_i^* is a normed algebra whose unit ball is $S_i^* = \tilde{S}_i \cap A$, where $\tilde{S}_i$ stands as usual for the unit ball of $\tilde{A}_i$, for every $i \in I$.

Fix $i \in I$: if $y \in S_i^*$ then the sequence $x_n = \sum_{0\, k}^{\infty} y^k$ is a Cauchy sequence in A_i^* and then in A, and $\lim (e - y) x_n = \lim x_n (e - y) = = \lim e - y^{n+1} = e$ in A_i^* and then in A. Under hypothesis $b)$, $e - y$ is invertible in A: as $\tilde{A}_i$ is a Banach algebra and $y \in \tilde{S}_i$, $e - y$ is invertible also in $\tilde{A}_i$, so that $e - y$ is invertible in A_i^*. This shows that A_i^* has the Q-property.

It is easy to check that $\{A_i^* | i \in I\}$ is a L-family for A; in proving prop. 2 we have shown that the LCIL topology relative to $\{\tilde{A}_i | i \in I\}$ induces on A exactly the LCIL topology relative to $\{A_i | i \in I\}$: as $A_i \subseteq A_i^* \subseteq \tilde{A}_i$ with continuous inclusions for every $i \in I$, it is easy to check that the LCIL topology on A relative to $\{A_i^* | i \in I\}$ also coincides with that one.

Now suppose that A is a commutative BMCA:

$c) \Rightarrow d)$: before all we observe that if A is a commutative normed algebra having the Q-property, A satisfies $d)$: in fact $\tilde{A}$, the completion of A, is a commutative Banach algebra (with the obvious product) and so $r_{\tilde{A}}$($=$ spectral radius on $\tilde{A}$) is a submultiplicative seminorm on A, but $r_{\tilde{A}|A} = r_A$ because of the implication $a) \Rightarrow b)$ which shows that a BMCA (in particular a normed algebra) having the Q-property is algebraically dense in its completion.

In the general case let A carry the LCIL topology relative to a L-family $\{A_i | i \in I\}$ s.t. A_i has the Q-property for every $i \in I$. Fixed $x \in A$, we have, with notations used in proving $b)$ of theorem 1, $\sigma_A(x) = \cap \{\sigma_i(x) | i \in I_x\}$: if we denote $r_i(x)$ the spectral radius on A_i for every $i \in I_x$ and observe that $\{\sigma_i(x) | i \in I_x\}$ is a ($\supseteq$)-directed family of compact sets (compactness follows from Q-property), we conclude that $r_A(x) = \inf \{r_i(x) | i \in I_x\}$.

This assures that $r_A(x) < +\infty$, for every $x \in A$. Moreover we can see that r_A is a seminorm: in fact given $x, y \in A$ for every $\varepsilon > 0$ we can find $i, j \in I$ s.t. $r_i(x) < r_A(x) + \varepsilon$, $r_j(y) < r_A(y) + \varepsilon$: then considering $h \in I$ s.t. $A_i \subseteq A_h$, $A_j \subseteq A_h$ we get $r_h(x) \leqslant r_i(x) < r_A(x) + \varepsilon$ and $r_h(y) \leqslant r_j(y) < r_A(y) + \varepsilon$ so that $r_A(x + y) \leqslant r_h(x + y) \leqslant r_h(x) + r_h(y) < < r_A(x) + r_A(y) + 2\varepsilon$. In an analogous way submultiplicativity is proved.

The continuity of the seminorm r_A follows from the continuity of the restriction to A_i, for every $i \in I$: in fact $(r_A)_{|A_i} \leqslant r_i$ and r_i is a continuous seminorm on A_i, as we have supposed A_i satisfying Q-property, for every $i \in I$.

$d) \Rightarrow e)$: let (A, r) be the algebra A endowed with the seminorm r_A: (A, r) is a semi-normed algebra endowed with the Q-property. The usual arguments in Banach algebras theory also apply to (A, r) and show that every maximal ideal of A is the kernel of a linear multi plicative functional (not vanishing everywhere) continuous on (A, r), and then on A. The converse is obvious.

$e) \Rightarrow f)$, $f) \Rightarrow g)$: arguments of Banach algebras theory apply.

Let us show $h)$ holds under the assumption of the equivalent properties $d), e), f), g)$: let (A, r) be as above: (A, r) is a seminormed algebra endowed with the Q-property. Let A' be the associate Hausdorff space, that is the topological quotient A/I, where $I = \{x \in A | r_A(x) = 0\}$, endowed with the norm r' defined by $r'(x + I) = r_A(x)$; A' is a normed algebra satisfying Q-property. Now let $\tilde{A}$ be the completion of A', endowed with the natural norm $\tilde{r}$: $\tilde{A}$ is, with the obvious product, a commutative Banach algebra. Let us naturally identify A' to a dense subalgebra of $\tilde{A}$, and denote ψ the natural mapping of A into $\tilde{A}$ (namely ψ is the composition of the canonical projection of A onto A' with the inclusion map of A' into $\tilde{A}$).

We see clearly that ψ is a continuous homomorphism, preserves unit and $\psi(A) = A'$ is dense topologically and algebraically in $\tilde{A}$ (remember implication $a) \Rightarrow b)$). Moreover $\ker \psi = I = \{x \in A | r_A(x) = 0)\} = \{x \in A | f(x) = 0 \text{ for every } f \in M_A\} = \cap \{I \subseteq A | I \text{ maximal ideal}\} \equiv R_A$ (the third equality follows from $g)$, the fourth from $e)$). M_A is an equicontinuous set on (A, r) as shown by $g)$, so that it is bijective to $M_{A'}$ and then to $M_{\tilde{A}}$ via ψ' ($\equiv$ transpose of ψ): so property $e)$, standing for $\tilde{A}$ too, gives easily that every maximal ideal of A is obtained (in only one way) as inverse image respect to ψ of a maximal ideal of $\tilde{A}$. Property $f)$ says $\sigma_A(x) = \{f(x) | f \in M_A\}$ for every $x \in A$: the analogous in true for $\tilde{A}$, and the corrispondence $M_A \overset{\psi'}{\leftrightarrow} M_A$ makes the equality $\sigma_A(x) = \sigma_{\tilde{A}}(\psi(x))$ hold for every $x \in A$.

It remains to show semisimplicity of A. From the last relation we get $\sigma_A(x) = \sigma_{A'}(\psi(x))$ and then $r_A(x) = r_{A'}(\psi(x))$ for every $x \in A$: from definition $r'(\psi(x)) = r_A(x)$ for every $x \in A$, so that $r' = r_{A'}$ on A'. As we noted, A' is algebraically dense in $\tilde{A}$, so that $(r_{\tilde{A}})_{|A'} = r_{A'}$: compared with the preceding relation it gives that $(r_{\tilde{A}})_{|A'} = r'$.

Since $\tilde{r}_{|A'} = r'$, we get $\tilde{r}_{|A'} = (r_{\tilde{A}})_{|A'}$: this means $\tilde{r} = r_{\tilde{A}}$ on $\tilde{A}$ because A' is topologically dense in A and both $\tilde{r}$ and $r_{\tilde{A}}$ are continuous on $\tilde{A}$. It is then immediate that $\tilde{A}$ is semisimple.

$h) \Rightarrow a)$ is obvious. ∎

REMARK. We have not required in the definition that a BMCA must be Hausdorff (nevertheless prop. 1 shows it is not dangerous for applications) in order that th. 1 and th. 2 keep their whole algebraic contents. Namely if A is an algebra which admits a L-family (cases may be provided considering lemma 5) then A must satisfy properties $b), d)$ of th. 1; if A is a commutative algebra which admits a L-family of normed algebras endowed with the Q-property, then A satisfies the elementary algebraic properties of Banach algebras: this affinity is pointed out in $h)$ of th. 2.

We now prove a lemma we need in the proofs of theorem 1 and 2, and which we still use in the sequel.

LEMMA 3. *Let A be a Hausdorff topological algebra s.t. for every $x \in A$ the mapping R_x: $\boldsymbol{C} - \sigma_A(x) \to A$ defined by $R_x(\lambda) = (\lambda e - x)^{-1}$ is continuous, and* $\lim R_x(\lambda) = 0$ *for $\lambda \to \infty$, $\lambda \notin \sigma_A(x)$. Then R_x is holomorphic on the interior of $\boldsymbol{C} - \sigma_A(x)$ and $(\lambda e - x)^{-1} = \sum_{0k}^{\infty} x^k \lambda^{-k-1}$ holds for $|\lambda| > r_A(x)$.*

A has the Q-property iff there exists a 0-neighbourhood V s.t. for every $x \in V$: $(e - x)^{-1}$ exists and $(e - x)^{-1} = \sum_{0k}^{\infty} x^k$, $\lim_{k \to \infty} x^k = 0$.

PROOF. From the identity $(\lambda - \lambda')^{-1}[R_x(\lambda') - R_x(\lambda)] = R_x(\lambda') \cdot R_x(\lambda)$ (for $\lambda, \lambda' \notin \sigma_A(x), \lambda \neq \lambda'$) and the continuity of R_x it follows that there exists the limit of the incremental ratio of R_x at every λ interior to $\boldsymbol{C} - \sigma_A(x)$: so R_x is holomorphic on the interior of $\boldsymbol{C} - \sigma_A(x)$ and then R_x has on the set $\{\lambda \in \boldsymbol{C} \,|\, |\lambda| > r_A(x)\}$ (not necessarily non empty) a development of type $R_x(\lambda) = \sum_{-\infty}^{+\infty} a_n \lambda^n$. The hypothesis $\lim R_x(\lambda) = 0$ for $\lambda \to \infty$, $\lambda \notin \sigma_A(x)$ says that $a_n = 0$ for $n \geqslant 0$, and at last we compute the desired formule.

Now suppose that A has the Q-property: then $V = \{x \in A \,|\, r_A(x) < 1\}$ is a 0-neighbourhood and for every $x \in V$ we write the preceding formule in $\lambda = 1$. At last from the identity $x^k = e - (e - x)\left(\sum_{0j}^{k-1} x^j\right)$ we deduce also $\lim_{k \to \infty} x^k = 0$ ∎

REMARK. By use of lemma 3 one immediately solves a problem posed in [7] (p. 215, n. 4): if A is a commutative metrizable LMCA endowed with the Q-property and for every $x \in A$ there exists $\varrho > 0$

s.t. $\lim_{k\to\infty}(\varrho x)^k = 0$, then has A a 0-neighbourhood V s.t. $\lim_{k\to\infty} x^k = 0$ for every $x \in V$?

In fact one observes that in a LMCA the mapping $x \mapsto x^{-1}$ is continuous on the domain set so that hypothesis of lemma 3 are easily checked.

Now let us see in which cases we may say that a given topological algebra is a BMCA: at first we give a

NOTATION. *Let A be a topological algebra: then $\mathfrak{B}_A$ will denote the class of all closed bounded idempotent absolutely convex sets of A, containing the unit of the algebra.*

The attention upon $\mathfrak{B}_A$ has been brought for the first time in [1]: we recall in two lemmas some properties of $\mathfrak{B}_A$:

LEMMA 4. *Let A be a topological algebra, C a bounded idempotent subset of A: then there exists $B \in \mathfrak{B}_A$ s.t. $C \subseteq B$.*

PROOF. The set $C \cup \{e\}$ is a bounded idempotent set. The absolutely convex hull of a bounded idempotent set is a bounded idempotent set; the same for the operation of topological closure ∎

NOTATION. *Let A be a topological algebra and $B \in \mathfrak{B}_A$: then A_B will denote the linear span of B in A, seminormed with the Minkowsky functional ν_B of B (namely, $\nu_B(x) = \inf\{t > 0 | x \in tB\}$ for every $x \in A_B$).*

LEMMA 5. *Let A be a Hausdorff topological algebra. For every $B \in \mathfrak{B}_A$, A_B is a normed algebra. $A = \cup\{A_B | B \in \mathfrak{B}_A\}$ iff for every $x \in A$ there exists $\varrho > 0$ s.t. $\lim_{k\to\infty}(\varrho x)^k = 0$. The following properties imply each other in the order:*

a) $\mathfrak{B}_A$ is directed by inclusion

b) the family $\{A_B | B \in \mathfrak{B}_A\}$ is directed by continuous inclusions

c) for every $B', B'' \in \mathfrak{B}_A$ the set $B' \cdot B''$ is bounded.

If A is commutative properties a), b), c) are equivalent and are implied by joint continuity of the product of A.

(the proof is not difficult and it is left to the reader)

Now we are able to give the following

PROPOSITION 6. *Let A be a Hausdorff topological algebra and let $\mathcal{K}$ be a subfamily of $\mathcal{B}_A$ endowed with one of the following properties:*

a) every bounded set of A is absorbed by an element of $\mathcal{K}$

b) the family $\{A_B|B\in\mathcal{K}\}$ is directed by continuous inclusions and if $(x_n)_{n\in N}$ is a sequence in A then $\lim x_n=0$ in A iff there exists $B\in\mathcal{K}$ s.t. $x_n\in A_B$ for every $n\in N$ and $\lim x_n=0$ in A_B

Then $\{A_B|B\in\mathcal{K}\}$ is a L-family for A; the relative LCIL topology coincides with the given topology of A iff this latter one is bornological.

If A' is a subalgebra of A and $\mathcal{K}'=\{B\cap A|B\in\mathcal{K}\}$, then $\{(A')_C|C\in\mathcal{K}'\}$ is a L-family for A': the relative LCIL topology coincides with the induced one by A iff this latter one is bornological.

PROOF. First let us show that $A=\cup\{A_B|B\in\mathcal{K}\}$. Fix $x\in A$: $\{x\}$ is a bounded set so under hypothesis a) x is absorbed by an element of $\mathcal{K}$, under hypothesis b) one uses the trick of considering the sequence $(n^{-1}x)_{n\in N}$.

$\{A_B|B\in\mathcal{K}\}$ is directed by continuous inclusions also under hypothesis a): for every $B_1, B_2\in\mathcal{K}$, $B_1\cup B_2$ is a bounded set and so there exists $D\in\mathcal{K}$ absorbing $B_1\cup B_2$: then $A_{B_i}\subseteq A_D$ $(i=1, 2)$ with continuous inclusions.

So in each case $\{A:|B\in\mathcal{K}\}$ is a L-family for A.

Every BMCA carries a bornological topology (as locally convex inductive limit of bornological topologies). On the other side the given topology τ of A is less fine than the LCIL topology τ_l relative to the family $\{A_B|B\in\mathcal{K}\}$; moreover suppose that τ is bornological. Absume hypothesis a): an absolutely convex set absorbing the elements of $\mathcal{K}$ absorbs all (τ)-bounded sets too, so by definitions a 0-neighbourhood for τ_l is a 0-neighbourhood for τ too. Absuming hypothesis b), we check the continuity of the identity map $i: (A, \tau)\to(A, \tau_l)$ by means of sequences (see [3], p. 203, cor.): in fact if $\lim x_n=0$ for τ there exists $B\in\mathcal{K}$ s.t. $x_n\in A_B$ for every $n\in N$ and $\lim x_n=0$ in A_B, then $\lim x_n=0$ in every topology on A which makes continuous the inclusion map of A_B into A for every $B\in\mathcal{K}$, in particular $\lim x_n=0$ for τ_l.

Proof of last part consists in checking that $\mathcal{K}'\subseteq\mathcal{B}_{A'}$, and that properties a), b) of $\mathcal{K}$ are inherited by $\mathcal{K}'$ ∎

REMARK. A subalgebra of a BMCA in general need not be a BMCA: however proposition 6 gives particular cases in which this is true.

Let us provide cases in which *a*) and *b*) of prop. 6 are verified. First we give the following

DEFINITION 2. *Let A be a topological algebra: we say that A carries the Allan boundedness iff every bounded set of A is absorbed by an element of $\mathcal{B}_A$.*

This property will be obtained in a corollary of

PROPOSITION 7. *Let A be a commutative Hausdorff continuous inverse algebra. Then every compact absolutely convex subset of A is absorbed by an element of $\mathcal{B}_A$.*

PROOF. Let C be an absolutely convex compact subset of A: C is bounded and so is absorbed by the 0-neighbourhood $V = \{x \in A | r_A(x) < 1\}$, then there exists $\varrho > 0$ s.t. $x \mapsto (e - x)^{-1}$ is defined and continuous on $\varrho C \subseteq V$. Then the set $\{(e - x)^{-1} | x \in \varrho C\}$ is compact and then is a bounded set, and so its closed absolutely convex hull D is a bounded set. Under the given hypothesis the formule

$$x^n = \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^n (\lambda e - x)^{-1} \, d\lambda$$

stands for every $x \in V$, for every $n \in \boldsymbol{N}$ (from the holomorphic functional calculus in continuous inverse algebras introduced by L. WAELBROECK; however it may be deduced by substituting $(\lambda e - x)^{-1} = \sum_{0\,k}^{\infty} x^k \lambda^{-k-1}$ (see lemma 3) in the right-handed member and then interchanging $\int$ with $\sum$). So we get that $x \in \varrho C$ implies $x^n \in D$. It is possible then to deduce (see [6], p. 122) the existence of an idempotent set T s.t. $(2 \exp 1)^{-1} \cdot \varrho C \subseteq T \subseteq D$: T is then a bounded set and one ends the proof using lemma 4. ∎

COROLLARY. *Let A be as in proposition 7 and suppose that every bounded set of A is relatively compact in A. Then A carries the Allan boundedness.*

PROOF. The closed absolutely convex hull of a bounded set is a compact absolutely convex set ∎

A lot of BMCA's carry the Allan boundedness:

PROPOSITION 8. *For a Hausdorff topological algebra A following properties are equivalent*:

a) A is a $\aleph_0$-BMCA

b) *the topology of A is bornological and there exists a countable subfamily $\mathcal{H}$ of $\mathcal{B}_A$* s.t. *every bounded set of A is absorbed by an element of $\mathcal{H}$.*

PROOF. *a*) $\Rightarrow$ *b*): every BMCA has a bornological topology. Let $\{A_i | i \in I\}$ be a L-family for A, which A carries the LCIL topology relative to, and card $I = \aleph_0$. A bounded set D of A is contained in the closure of the sum of a finite number of bounded subsets of the elements of the L-family (see [3], p. 312): from definition of L-family this means that exists $i \in I$, $k \in \boldsymbol{N}$ s.t. $D \subseteq k\bar{S}_i$, where $\bar{S}_i$ stands for the closure in A of the unit ball S_i of A_i: now $\bar{S}_i \in \mathcal{B}_A$ because S_i is idempotent and so its closure.

b) $\Rightarrow$ *a*): see prop. 6 ∎

At last let us provide a case in which property *b*) of prop. 6 is verified.

PROPOSITION 9. *Let A be a commutative metrizable LMCA. Suppose that there exists a 0-neighbourhood V s.t.* $\lim\limits_{k\to\infty} x^k = 0$ *for every $x \in V$. Then $\{A_B | B \in \mathcal{B}_A\}$ is directed by continuous inclusions, if $(x_n)_{n\in\boldsymbol{N}}$ is a sequence in A them* $\lim x_n = x_0$ *in A iff there exists $B \in \mathcal{B}_A$ s.t. $x_n \in A_B$ for every $n \in \{0, 1, \ldots\}$ and* $\lim x_n = x_0$ *in A_B.*

PROOF. Product is jointly continuous in a LMCA, then former part of the thesis follows by using lemma 5.

The proof of the latter part is substantially taken from the proof of th. 7 of [7]. Clearly if $\lim x_n = x_0$ in A_B then it stands also in A. On the other side suppose first that $(z_n)_{n\in\boldsymbol{N}}$ is a sequence contained in V, with $\lim z_n = 0$: we claim that D, the smallest idempotent set containing $\{z_n | n \in \boldsymbol{N}\} \cup \{e\}$, is bounded. In fact if we fix a 0-neighbourhood W s.t. $W \cdot W \subseteq W \subseteq V$, there exists $\bar{n} \in \boldsymbol{N}$ s.t. $z_n \in W$ for every $n > \bar{n}$: now the generic element of D is of the form $z = z_1^{m_1} \cdot z_2^{m_2} \ldots z_{\bar{n}}^{m_{\bar{n}}} \prod\limits_{n<\bar{n}} z_n^{m_n}$ where $(m_n)_{n\in\boldsymbol{N}}$ is a sequence of definitively 0 non negative integers. The term $z_i^{m_i}$ belongs to the bounded set $\{z_i^m | m = 0, 1 \ldots\}$: moreover $\prod\limits_{n<\bar{n}} z_n^{m_n} \in W \subseteq V$ as W is idempotent: then if $k \in \boldsymbol{N}$ is such that $\{z_i^m | m = 0, 1 \ldots\} \subseteq kW$ $(i = 1, 2, \ldots \bar{n})$, we have $z \in (kW)^{\bar{n}} \cdot W \subseteq k^{\bar{n}} W$. As k and $\bar{n}$ do not depend upon the choice of z in D we get D is bounded.

Now let $(x_n)_{n\in N}$ be a sequence in A s.t. $\lim x_n = x_0$ in A: as A is metrizable there exists a sequence of real numbers $(\varepsilon_n)_{n\in N}$ s.t. $\lim \varepsilon_n = 0$ and $\lim \varepsilon_n^{-1}(x_n - x_0) = 0$ in A. Put $z_n = \varepsilon_n^{-1}(x_n - x_0)$, take $n' \in N$ s.t. $z_n \in V$ for every $n > n'$ and apply the preceding result to $(z_n)_{n>n'}$: then C, the smallest idempotent set containing $\{z_n | n > n'\} \cup \{e\}$, is a bounded set. As V absorbs points, there exists $\varrho > 0$ s.t. $\varrho x_i \in V$ $(i = 0, 1 \dots n')$, as a consequence the set $C_i = \{(\varrho x_i)^m | m = 0, 1 \dots\}$ is bounded $(i = 0, 1, \dots n')$. Then the set $C \cdot \prod_{0 \leqslant i \leqslant n'} C_i$ contains C and C_i $(i = 0, 1, \dots n')$, it is bounded as finite product of bounded sets (by joint continuity of product in A) and it is idempotent by commutativity: then it is contained in some $B \in \mathfrak{B}_A$ by lemma 4.

In this way we get $x_i \in A_B$ $(i = 0, 1 \dots n')$ and $x_n - x_0 = \varepsilon_n z_n \in \varepsilon_n C \subseteq \varepsilon_n B$ for every $n > n'$, hence $\{x_n | n = 0, 1 \dots\} \subseteq A_B$ and $\lim x_n = x_0$ in A_B ∎

§ **2.** BMCA's satisfy all elementary properties of normed algebras except for the continuity of the mapping $x \mapsto x^{-1}$ where it is defined: for BMCA's this property is intimately connected with local m-convexity, in the sense of the following

PROPOSITION 10. *Let us consider the following statements:*

a) *every Banach-BMCA is a LMCA*

b) *every BMCA is a LMCA*

c) *in every BMCA the mapping $x \mapsto x^{-1}$ is continuous on the domain set*

d) *in every Banach-BMCA the mapping $x \mapsto x^{-1}$ is continuous on the domain set.*

Then a) $\Leftrightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d). Restricting to commutative case all enunciates turn out to be equivalent.

PROOF. $b) \Rightarrow a)$ is obvious. $a) \Rightarrow b)$: the associated Hausdorff BMCA is a subalgebra of a LMCA (by virtue of prop. 2) and then is a LMCA. $b) \Rightarrow c)$ as in every LMCA the mapping $x \mapsto x^{-1}$ is continuous on the domain set; $c) \Rightarrow d)$ is obvious.

In commutative case $d) \Rightarrow a)$: in fact let A be the associated Hausdorff Banach-BMCA, then the implication $c) \Rightarrow a)$ of th. 2 works, that is A satisfies Q-property. If in A the mapping $x \mapsto x^{-1}$ is supposed to be continuous on the domain set, A is a LMCA for a result of [5]. ∎

Among the problems obtained by putting in interrogative form the preceding statements, the most natural is the following:

(**) IS EVERY BMCA A LMCA?

We do not know the answer to this question: the aim of this section is to furnish adjunctive hypothesis assuring a positive answer.

PROPOSITION 11. *Let A be an algebra and $\{A_i | i \in I\}$ a L-family for A totally ordered by inclusions, whose norms are less or equal than* 1. Then *the relative LCIL topology makes A be a LMCA.*

PROOF. Let S_i be the unit ball of A_i, for every $i \in I$. The family of absolutely convex hulls of sets of the form $\cup \{\varrho_i S_i | i \in I\}$ $(0 < \varrho_i \leqslant 1$ for every $i \in I)$ is a fundamental system of 0-neighbourhoods for the LCIL topology: it is easy to check that $\cup \{\varrho_i S_i | i \in I\}$ is idempotent, the thesis follows from the fact that the absolutely convex hull of an idempotent set is idempotent ∎

One may reach through another way a result of [2], namely that under the hypothesis of prop. 11 the mapping $x \mapsto x^{-1}$ is continuous on the domain set respect to the relative LCIL topology: it is enough to observe that this property is satisfied in every LMCA.

We obtain a significative result with

PROPOSITION 12. *Every commutative $\aleph_0$-BMCA is a LMCA.*

PROOF. Let A be an algebra carrying the LCIL topology relative to a L-family $\{A_i | i \in I\}$, card $I = \aleph_0$: let S_i be the unit ball of A_i, for every $i \in I$.

There exists a bijection $\sigma: \boldsymbol{N} \to I$, hence an application $\tilde{\sigma}$ of $\boldsymbol{N}$ into the family of subsets of I defined by $\tilde{\sigma}(n) = \{\sigma(m) | m \leqslant n\}$.

Fix $n \in \boldsymbol{N}$: define A_n as the linear span in A of the set $\cup \{A_i | i \in \tilde{\sigma}(n)\}$ and define S_n as the absolutely convex hull of the family $\mathcal{S}_n = \{S_{i_1} \cdot S_{i_2} \dots S_{i_k} | i_1, i_2 \dots i_k \in \tilde{\sigma}(n)\}$: the elements of $\mathcal{S}_n$ are in finite number by commutativity and are bounded sets as $\{A_i | i \in I\}$ is directed by continuous inclusions: then S_n is a bounded set of A. We assign as norm to A_n the Minkowsky functional of S_n: as A_n is algebra and S_n is idempotent we get that A_n is a normed algebra.

Then, by virtue of the preceding proposition, the finest locally convex topology on A which makes continuous the inclusion maps of A_n into A for every $n \in \boldsymbol{N}$ is locally m-convex; as S_n is a bounded subset of A this topology is finer than the given one of A: the converse is also true as a topology on A that makes continuous the inclusion map of A_n into A for every $n \in \boldsymbol{N}$ makes continuous the inclusion map of A_i for every $i \in I$ (by surjectivity of σ). ∎

In order to answer question (∗∗) we have furnished adjunctive hypothesis on the side of the L-family: now we give hypothesis on the topology of A, supposed to be BMCA.

PROPOSITION 13. *Let A be a commutative BMCA, endowed with a Fréchet topology: then A is a LMCA.*

The proof follows immediately from the

LEMMA 14. *Let A be a commutative topological algebra with a Fréchet topology, endowed with the following property: «for every $x \in A$ there exists $\varrho > 0$ s.t. $\lim_{k\to\infty} (\varrho x)^k = 0$».*

Then A is a LMCA.

PROOF. For a commutative topological algebra with a Fréchet topology the property of being LMCA is equivalent to the following one (see [9], p. 47): «for every $x \in A$ and every entire function $\varphi(\lambda) = \sum_{0\,k}^{\infty} a_k \lambda^k$, the series $\sum_{0\,k}^{\infty} a_k x^k$ converges in A».

Now fix $x \in A$ and $\varphi = \sum_{0\,k}^{\infty} a_k \lambda^k$ entire function, take $\varrho > 0$ s.t. $\lim_{k\to\infty} (\varrho x)^k = 0$ and denote D the absolutely convex hull of the set $\{(\varrho x)^k \cdot |k = 0, 1, \ldots\}$; put $y_n = \sum_{0\,k}^{n} a_k x^k$, then

$$y_n - y_m = \sum_{n+1}^{m}{}_k\, a_k x^k = \sum_{n+1}^{m}{}_k\, a_k \varrho^{-k} (\varrho x)^k \in \left(\sum_{n+1}^{m}{}_k\, |a_k| \varrho^{-k} \right) \cdot D$$

for every $m > n$. As the series $\sum_{0\,k}^{\infty} a_k \varrho^{-k}$ converges absolutely (to the number $\varphi(\varrho^{-1})$) and D is bounded, we get that $(y_n)_{n\in N}$ is a Cauchy sequence in A, so converges in A ∎

Let us try to extend the result of prop. 13, leaving the hypothesis of completeness

PROPOSITION 15. *Let A be a commutative metrizable BMCA. The following condition: «the family $\{A_B|B \in \mathfrak{B}_A\}$ is directed by continuous inclusions and the set of the elements of every Cauchy sequence in A is absorbed by an element of $\mathfrak{B}_A$» assures that A is a LMCA.*

The given condition is also necessary.

PROOF. First the product is jointly continuous in A. In fact if $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are sequences in A s.t. $\lim x_n = \lim y_n = 0$, then by metrizability there exists a sequence of real numbers $(\varepsilon_n)_{n\in\mathbf{N}}$ s.t. $\lim \varepsilon_n = 0$ and $\lim \varepsilon_n^{-1} x_n = \lim \varepsilon_n^{-1} y_n = 0$. Then there exists $B', B'' \in \mathfrak{B}_A$ and $k', k'' \in \mathbf{N}$ s.t. $\varepsilon_n^{-1} x_n \in k'B'$ and $\varepsilon_n^{-1} y_n \in k''B''$ for every $n \in \mathbf{N}$: as $\{A_B|B \in \mathfrak{B}_A\}$ is directed by continuous inclusions there exists $B \in \mathfrak{B}_A$, $k \in \mathbf{N}$ s.t. $\varepsilon_n^{-1} x_n, \varepsilon_n^{-1} y_n \in kB$ for every $n \in \mathbf{N}$. This says $\lim x_n = \lim y_n = 0$ in A_B, so that $\lim x_n y_n = 0$ in the normed algebra A_B and then in A.

We naturally identify A to a linear subspace of its completion $\tilde{A}$, and extend to $\tilde{A}$ in a (unique) continuous way the jointly continuous product of A: thus $\tilde{A}$ is a topological algebra with a Fréchet topology. We will show that $\tilde{A}$ satisfies the property considered in lemma 14: then we will get that $\tilde{A}$ is a LMCA, and so A as a subalgebra of $\tilde{A}$.

Fix $y \in \tilde{A}$: then there is a Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ in A s.t. $\lim x_n = y$ in $\tilde{A}$. From hypothesis there exists $B \in \mathfrak{B}_A$, $k \in \mathbf{N}$ s.t.. $x_n \in kB$ for every $n \in \mathbf{N}$: then $k^{-1}y \in \tilde{B}$, where $\tilde{B}$ stands for the closure of B in $\tilde{A}$: as the closure of an idempotent set is again idempotent we get that $\{(k^{-1}y)^n | n = 0, 1 \ldots\} \subseteq \tilde{B}$ is a bounded subset of $\tilde{A}$: then $\lim\limits_{n\to\infty} [(2k)^{-1}y]^n = 0$.

Conversely suppose that A is a LMCA: product is then jointly continuous, so $\{A_B|B \in A\}$ is directed by inclusions (see lemma 5). In order to get the latter part of the condition, consider the completion $\tilde{A}$ as above, with the product of A extended by means of joint continuity. $\tilde{A}$ is a commutative complete LMCA, then $\tilde{A}$ satisfies Wiener property, or equivalently $\sigma(x) = \{f(x) | f \in M_A\}$ for every $x \in A$.

M_A is equicontinuous on A (see th. 1), and so is $M_{\tilde{A}}$ on $\tilde{A}$: then $W = \{x \in \tilde{A} | r_{\tilde{A}}(x) \leq \frac{1}{2}\} = \{x \in \tilde{A} | |f(x)| \leq \frac{1}{2}$ for every $f \in M_{\tilde{A}}\}$ is a 0-neighbourhood in $\tilde{A}$ s.t. $e + W$ consists of invertible elements: that is, $\tilde{A}$ has the Q-property.

The mapping $x \mapsto x^{-1}$ is continuous on the domain set in every LMCA: then $\tilde{A}$ falls under the hypothesis of lemma 3, we get a 0-neighbourhood V in $\tilde{A}$ s.t. $\lim\limits_{k\to\infty} x^k = 0$ for every $x \in V$.

Now consider a Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ in A: get $y \in \tilde{A}$ s.t.

$\lim x_n = y$ in $\tilde{A}$, applying prop. 9 we can deduce the existence of a $C \in \mathfrak{B}_{\tilde{A}}$ s.t. $\lim x_n = y$ stands in $(\tilde{A})_C$: as a consequence the set $\{x_n | n \in \boldsymbol{N}\}$ is absorbed by C, then it is absorbed by $C \cap A$ which is an element of $\mathfrak{B}_A$ ∎

COROLLARY. *Let A be a commutative metrizable topological algebra, and let A carry the Allan boundedness. Then A is a LMCA.*

PROOF. The family $\{A_B | B \in \mathfrak{B}_A\}$ is a L-family for A and A is a BMCA (see prop. 6): moreover every Cauchy sequence is a bounded set, so it is absorbed by an element of $\mathfrak{B}_A$ ∎

One may observe that the hypothesis of prop. 12 is in a certain way complementar to those of prop. 13 and 15: it is well known in fact that a metrizable topology which is locally convex inductive limit of a countable family of normed spaces is normable.

§ 3. In this section we restrict our interest to commutative topological algebras endowed with a Fréchet topology and to the class generated by these with the operation of locally convex inductive limit.

PROPOSITION 16. *Let A be a commutative topological algebra endowed with a Fréchet topology. Then following properties are equivalent:*

a) *for every $x \in A$ there exists $\varrho > 0$ s.t.* $\lim\limits_{k\to\infty} (\varrho x)^k = 0$

b) *A is a LMCA, and there exists a 0-neighbourhood V s.t.* $\lim\limits_{k\to\infty} x^k = 0$ *for every $x \in V$*

c) *$\{A_B | B \in \mathfrak{B}_A\}$ is directed by continuous inclusions. If $(x_n)_{n\in \boldsymbol{N}}$ is a sequence in A then $\lim x_n = x_0$ iff there exists $B \in \mathfrak{B}_A$ s.t. $x_n \in A_B$ $(n = 0, 1 \ldots)$ and $\lim x_n = x_0$*

d) *A is a Banach BMCA.*

e) *A has the Q-property*

f) *A is a continuous inverse algebra.*

These properties are inherited by closed subalgebras of A and topological quotient by a closed ideal (endowed with the obvious product).

PROOF *a)* ⇒ *b)*. A turns out to be a *LMCA* by using lemma 14, the second part of *b)* follows from prop. 10 of [7].

b) ⇒ *c)*: see prop. 9.

$c) \Rightarrow d)$: using prop. 6 we see that A carries the LCIL topology relative to the L-family $\{A_B | B \in \mathfrak{B}_A\}$, and it is easy to check that A_B is a Banach algebra, for every $B \in \mathfrak{B}_A$:

$d) \Rightarrow e)$: see implication $c) \Rightarrow a)$ of th. 2.

$e) \Rightarrow f)$: in a topological algebra endowed with a Fréchet topology and the Q-property, the mapping $x \mapsto x^{-1}$ is continuous on the domain set (see [6], p. 113).

$f) \Rightarrow a)$: hypothesis of lemma 3 are satisfied, so there exists a 0-neighbourhood V s.t. $\lim_{k \to \infty} x^k = 0$ for every $x \in A$; thesis follows by observing that V absorbs points.

If I is a closed subalgebra (resp.: closed ideal) of A then I (resp.: A/I) is a topological algebra endowed with a Fréchet topology, then it is enough to show that one property is inherited: $a)$, for instance ▪

PROPOSITION 17. *Let A be a commutative topological algebra endowed with a Fréchet topology. If A carries the Allan boundedness then A satisfies the equivalent properties of prop.* 16.

If A is Montel space the converse is also true.

PROOF. If A carries the Allan boundedness every point of A is absorbed by some $B \in \mathfrak{B}_A$: this assures property $a)$ of prop. 16.

Conversely, if A is Montel space, that is every bounded subset of A is relatively compact, then corollary of prop. 7 gives that A carries the Allan boundedness ▪

For examples see ex. III, IV, V.

Concluding this paper let us see what happens in the situation (*) of § 1, weakening opportunately the hypothesis on the family $\{A_i | i \in I\}$. If A_i is not necessarily a normed algebra, but more generally a topological algebra, we may still speak about the LCIL topology, namely the finest locally convex topology τ_l on A among those making continuous the inclusions of A_i into A, for every $i \in I$: this topology admits as a fundamental system of 0-neighbourhoods the family of all absolutely convex sets of A which absorb some 0-neighbourhood in A_i, for every $i \in I$.

The proof of the following proposition rests on the transitivity of the operation of locally convex inductive limit.

PROPOSITION 18. *Let A be a commutative algebra and $\{A_i | i \in I\}$ a family of subalgebras of A s.t. $A = \cup \{A_i | i \in I\}$; for every $i \in I$,*

A_i is a metrizable LMCA and there exists a 0-neighbourhood V_i in A_i s.t. $\lim_{k\to\infty} x^k = 0$ for every $x \in V_i$ (resp.: for every $i \in I$, A_i is a topological algebra with a Frechet topology satisfying one of the properties of prop. 16) in such a way that $\{A_i | i \in I\}$ is directed by continuous inclusions.

Then A endowed with the relative LCIL topology τ_l is a BMCA (resp.: a Banach-BMCA).

To ask wheter under hypothesis of prop. 18 (A, τ_l) turns out to be LMCA is equivalent to ask wheter every BMCA is a LMCA. Otherwise if A_i is a LMCA with Fréchet topology without any other hypothesis answer is no (see ex. XI).

We state without proof this more particular result, useful for applications (see ex. VI).

PROPOSITION 19. *Let A be a commutative algebra and $\{A_n | n \in \boldsymbol{N}\}$ a sequence of subalgebras of A s.t. $A = \cup\{A_n | n \in \boldsymbol{N}\}$: suppose that, for every $n \in \boldsymbol{N}$, A_n is a LMCA with a Fréchet-Montel topology, and A_n is a topological subspace of A_{n+1}. Then A, endowed with the relative LCIL topology is a (Hausdorff, complete) Banach-BMCA that carries the Allan boundedness; if $(x_n)_{n\in N}$ is a sequence in A then $\lim x_n = x_0$ in A iff there exists $B \in \mathfrak{B}_A$ s.t. $x_n \in A_B$ $(n = 0, 1 \ldots)$ and $\lim x_n = x_0$ in A_B.*

Examples and counterexamples.

We remind that if A is a topological algebra without unit one is used to consider the unitary algebra A^+ associated to A, that is $A^+ = \boldsymbol{C} \times A$ with the product topology and a product defined by $(\lambda, x) \cdot (\mu, y) = (\lambda\mu, \lambda y + \mu x + xy)$.

Examples of commutative topological algebras which are at one time Banach-BMCA and $\aleph_0$-BMCA.

I) Let $A = C_0(\boldsymbol{R})$ be the algebra of all complex-valued continuous functions on $\boldsymbol{R}$ vanishing outside a compact set, endowed with the LCIL topology relative to the L-family $\{A_k | k \in \boldsymbol{N}\}$ where $A_k = C([-k, k])$ is the Banach algebra of all complex valued continuous functions on $\boldsymbol{R}$ vanishing outside $[-k, k]$ endowed with the sup norm.

If we consider A^+, the unital algebra associated to A, it is easy to check that A^+ carries the LCIL topology relative to the L-family $\{(A_k)^+ | k \in \boldsymbol{C}\}$, where $(A_k)^+$ is the unital algebra associated to A_k:

II) Let $A = \mathcal{H}(K)$ be the algebra of all holomorphic functions on a neighbourhood of K, K compact subset of $\boldsymbol{C}$: we endow A with the $LCIL$ topology relative to the L-family $\{\overline{\mathcal{H}}(K_n)|n \in \boldsymbol{N}\}$ where $\{K_n|n \in \boldsymbol{N}\}$ is a fundamental sequence of compact neighbourhoods of K and, for every $n \in \boldsymbol{N}$, $\overline{\mathcal{H}}(K_n)$ is the Banach algebra of all complex-valued continuous functions on K_n, holomorphic on the interior of K_n, endowed with the sup norm.

Examples of commutative topological algebras with a Fréchet-Montel topology, which satisfy the equivalent properties of proposition 16 (*and then carry the Allan boundedness*).

III) Let $A = \boldsymbol{C}^{\boldsymbol{N}}$ the algebra of the sequences of complex numbers with the convolution product and the topology of the coordinate-wise convergence: A is a Fréchet-Montel space and satisfy the Q-property (the non invertible elements of A are those with 0 as first coordinate).

IV) Let A be the Schwartz space $\mathcal{S}$, that is the algebra (for the usual product) of all complex-valued continuous functions x on $\boldsymbol{R}$ with derivative of each order s.t. for every $r, n \in \{0, 1, \ldots\}$

$$\sup_{t \in \boldsymbol{R}} (1 + |t|^2)^{n/2} |D^r x(t)| < +\infty .$$

A is naturally endowed with the seminorms

$$\nu_n(x) = \sup_{\substack{t \in \boldsymbol{R} \\ 0 \leqslant r \leqslant n}} (1 + |t|^2)^{n/2} |D^r x(t)|$$

which make A a topological algebra endowed with a Frechét-Montel structure. The associated unital algebra A^+ is Fréchet-Montel too. In order to show that A^+ has the Q-property, it will be enough to show that there is a 0-neighbourhood V in A s.t. for every $x \in V$ there exists $y \in A$ s.t.

$$x + y + xy = 0 .$$

If we choose $V = \{x \in A | \nu_0(x) = \sup_{t \in \boldsymbol{R}} |x(t)| \leqslant \frac{1}{2}\}$, then $y \equiv -x(1 + x)^{-1}$ is defined, has derivative of each order and belongs to $\mathcal{S}$. In fact for $n = 0, 1 \ldots$

$$\sup_{t \in \boldsymbol{R}} (1 + |t|^2)^{n/2} \cdot |-x(t) \cdot (1 + x)^{-1}| \leqslant 2 \sup_{t \in \boldsymbol{R}} (1 + |t|^2)^{n/2} \cdot |x(t)| < +\infty ,$$

while for every $r \in \boldsymbol{N}$ the computation of

$$D^r[-x(1 + x)^{-1}] = -D^r[(1 + x)^{-1}] ,$$

shows that

$$\sup_{t \in \boldsymbol{R}} (1 + |t|^2)^{n/2} \cdot |D^r[-x(t) \cdot (1 + x(t))^{-1}]| < +\infty .$$

V) Let K be a compact subset of $\boldsymbol{R}$ and let $A = C^\infty(K)$ be the algebra of all complex-valued continuous functions on $\boldsymbol{R}$ with continuous derivative of each order vanishing outside K, endowed with the seminorms

$$\nu_p(x) = \sup_{t \in K} |D^p x(t)| \quad (P = 0, 1 \ldots)$$

A is a Fréchet-Montel space, then A^+ is also a Fréchet Montel space: it is easy to check that A^+ satisfies the Q-property.

Example to proposition 19.

VI) Let $A = C_0^\infty(\boldsymbol{R})$ be the algebra of all complex-valued functions on $\boldsymbol{R}$ with continuous derivative of each order vanishing outside a compact set, endowed with the LCIL topology relative to the family $\{A_k | k \in \boldsymbol{N}\}$ where $A_k = C^\infty([-k, k])$ is defined as in ex. V.

Consider A^+: it is easy to check that A^+ carries the LCIL topology relative to the sequence $\{(A_k)^+ | k \in \boldsymbol{N}\}$: $(A_k)^+$ is a Fréchet-Montel space and is a topological subspace of $(A_{k+1})^+$, as this happens to A_k and A_{k+1}.

Moreover it is interesting to note that A is a LMCA (for a result of [4], p. 64), so that A^+ too is a LMCA.

Counterexamples.

VII) Let T be a completely regular Hausdorff topological space, and $\mathcal{K}$ a family of compact subsets of T, s.t. $T = \cup \mathcal{K}$. Let $A = C(T)$ be the algebra of all complex-valued continuous function on T, endowed with the topology of uniform convergence on the elements of $\mathcal{K}$.

If we want A to be a BMCA, we must pretend a priori that M_A is compact (in the usual topology of pointwise convergence, see *a*) of th. 1): it is well known that M_A is naturally homeomorphic to T, so the above condition forces A to be a Banach algebra.

Then for us there is interest in A only if it is not a BMCA but nevertheless satisfies properties we are in acquintance with. To this regard we present two examples of commutative Hausdorff complete LMCA, carrying the Allan boundedness and satisfying the property « if $(x_n)_{n \in \boldsymbol{N}}$ is a sequence in A then $\lim x_n = x_0$ in A iff there exists $B \in \mathcal{B}_A$ s.t. $x_n \in A_B$ for $n = 0, 1 \ldots$ and $\lim x_n = x_0$ in A_B » which is neither a BMCA (not bornological in fact), nor satisfies Q-property, nor has a 0-neighbourhood V s.t. $\lim_{k\to\infty} x^k = 0$ for every $x \in V$:

a) $T =$ {ordinals less than the first uncountable ordinal} endowed with the order topology, $\mathcal{K} =$ {compact subsets of T}

b) $T = [0, 1]$ with the usual topology, $\mathcal{K} =$ {compact and countable subsets of T}.

VIII) Counterexample a) and b) of VII) may be strenghtened. Let $A = C^1([0, 1])$ be the algebra of all complex-valued continuous functions on $[0, 1]$ with continuous derivative also in extreme points, endowed with the seminorms $\nu_0(x) = \sup_{0 \leqslant t \leqslant 1} |x(t)|$, $\nu_k(x) = \sup_{t \in K} |Dx(t)|$ where K varies among the countable and compact subsets of $[0, 1]$.

Then A is a commutative sequentially complete Hausdorff LMCA, carrying the Allan boundedness, satisfying Q-property and property: « If $(x_n)_{n \in \boldsymbol{N}}$ is a sequence in A then $\lim x_n = x_0$ in A iff there exists $B \in \mathcal{B}_A$ s.t. $x_n \in A_B$ for $n = 0, 1 \ldots$ and $\lim x_n = x_0$ in A_B », endowed with a 0-neighbourhood V s.t. $\lim_{k\to\infty} x^k = 0$ for every $x \in V$; but A is not a BMCA (not bornological in fact).

The example is due to Warner ([7], ex. 15).

IX) Consider $A = C([0, 1])$ endowed with the weak topology relative to the sup norm. As the boundedness of A coincides with the boundedness relative to the sup norm, it follows that A carries the Allan boundedness: now A is neither a LMCA (from a result of [8], but it is not hard to prove it in a direct way), nor a BMCA (not bornological in fact), nor satisfies Q-property, nor has a 0-neighbourhood V s.t. $\lim_{k\to\infty} x^k = 0$ for every $x \in V$.

X) Let $A = \boldsymbol{C} \times C_0(\boldsymbol{R})$ be the algebra of all complex-valued continuous functions on $\boldsymbol{R}$ constant outside a compact set, endowed with the topology of uniform convergence on compacta.

A is a commutative metrizable LMCA satisfying property: « for every $x \in A$ there exists $\varrho > 0$ s.t. $\lim_{k\to\infty} (\varrho x)^k = 0$ », but it is not a BMCA, nor carries the Allan boundedness, nor satisfies Q-property.

(The same for $A = L^\infty([0, 1])$, endowed with the seminorms $\nu_p(f) = \left(\int_0^1 |f|^p \, dt\right)^{1/p} (p \in \boldsymbol{N})$)

XI) A counterexample exhibited in [7] (ex. 6) shows a metrizable LMCA (A, τ) and an increasing sequence $\{A_n | n \in \boldsymbol{N}\}$ of subalgebras endowed with the induced topology τ_n s.t. $A = \cup \{A_n | n \in \boldsymbol{N}\}$ and the relative LCIL topology τ_l is not locally m-convex.

Now we observe that τ_l is Hausdorff and that the (unique) continuous extensions $\tilde{\varphi}_n$ of the inclusion map $\varphi_n: (A_n, \tau_n) \to (A, \tau_l)$ to completions is injective (if ab absurdo it were not, then the (unique) continuous extension of the inclusion map of (A_n, τ_n) into (A, τ) should be not injective, as composition of $\tilde{\varphi}_n$ with the extension to completions of the identity map of (A, τ_l) into (A, τ)): so we can identify (only algebraically) $\tilde{A}_n$, the completion of (A_n, τ_n), to a linear subspace of $\tilde{A}$, the completion of (A, τ_l).

Now we denote $A' = \cup \{\tilde{A}_n | n \in \boldsymbol{N}\}$, observe that $\tilde{A}_n$ is a LMCA with the obvious product, endow A' with the obvious product and with τ', the LCIL topology relative to $\{\tilde{A}_n | n \in \boldsymbol{N}\}$. We observe that τ' induces on A (naturally identified to a subspace of $\tilde{A}$ and then of A') exactly τ_l: this fact may be made clear with the same arguments used in the proof of prop. 2. Then τ' may not be locally m-convex, as τ_l is not so (local m-convexity is inherited by subalgebras). So we have exhibited an algebra A' and an increasing sequence $\{\tilde{A}_n | n \in \boldsymbol{N}\}$ of subalgebras endowed with a structure of LMCA with Fréchet topology, $\tilde{A}_n$ topological subspace of $\tilde{A}_{n+1}$, s.t. the relative LCIL topology is not locally m-convex.

REFERENCES

[1] G. R. Allan, *A spectral theory for locally convex algebras,* Proc. London Math. Soc., (3) **15** (1965), pp. 399-421.

[2] M. Bonnard, *Trois results sur les limites inductives et projectives d'algébres localement convexes,* C. R. Acad. Sci. Paris, **260** (1965), pp. 2655-7.

[3] A. Grothendieck, *Espaces vectoriels topologiques,* 2-nd edition, São Paulo, 1958.

[4] E. Michael, *Locally multiplicatively-convex algebras,* Memoirs of the A.M.S., no. 11 (1952).

[5] P. Turpin, *Une remarque sur les algèbres à inverse continu,* C. R. Acad. Sci Paris, Serie A, **270** (1970), pp. 1686-8.

[6] L. Waelbroeck, *Topological vector spaces and algebras,* Springer Lecture Notes in Math., **230** (1971).

[7] S. WARNER, *Inductive limits of normed algebras*, Transactions of A.M.S., **82** (1956), pp. 190-216.

[8] S. WARNER, *Weakly topologized algebras*, Proceedings of A.M.S., **8** (1957), pp. 314-6.

[9] ZELAZKO W., *Metric generalizations of Banach algebras*, Rozprawy Matematyczne, vol. 47, PWN-Polish Scientific Publishers, Warsaw, 1965.

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