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The interpreted type-free modal calculus MC^{∞}

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The Interpreted Type-Free Modal Calculus MC^{∞}.

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PART 3

Ordinals and cardinals in MC^{∞}

CHAPTER 5

ON ORDINALS AND THE AXIOM OF CHOICE

SOMMARIO - Si presenta un analogo modale della teoria dei numeri ordinali e cardinali svolta in [4], construendolo entro il calcolo MC^{∞} ove si trattano gli individui. Tale presentazione può farsi brevemente riguardo alla teoria pura dei suddetti numeri, essendo praticamente sufficiente dimostrare alcuni teoremi fondamentali essenzialmente modali all'inizio, e dare alcuni suggerimenti; la stessa presentazione è più laboriosa riguardo alle applicazioni della suddetta teoria ad insiemi (o proprietà) qualunque, e in particolare riguardo alla teoria degli universi. Per es. è importante considerare sia la relazione intensionale di equipotenza che quella estensionale. Lo stesso dicasi dei ranghi intensionale ed estensionale di un insieme.

SUMMARY - A modal analogue of the theory of ordinals and cardinals presented in [4] is constructed within MC^{∞}, where individuals are dealt with. This construction can be presented quikly, as far as the « pure number theory » is concerned, in that it is practically sufficient to prove certain

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basic and essentially modal theorems at the outset, and to give certain directions of a general use. More work is required for the applications of the theory above to arbitrary sets (or properties), and in particular to universes. For instance both the intensional relation of equipotence and the extensional one are important as well as the intensional and extensional ranks of sets.

32. Introduction to Part 3 (*).

A modal analogue of the theory of ordinals and cardinals in Chapters 2-4 of [IST] which is « introduction to set theory » of J. D. Monk, i.e. [4], is constructed in Part 2—cf. [2]—on the basis of the modal logical calculus MC^{∞} [Parts 1,2]—cf. [2]—which unlike [IST] takes individuals into account. This theory involves the axiom of choice with its main equivalents, and the main theorems on universes.

We first prove some basic essentially modal properties of ordinals [n. 33]. Then we show how the whole pure theory of ordinals in [IST] and in particular the theorems on transfinite induction, transitive closures, and natural numbers can be carried over to MC^{∞} easily, by means of rather standard changes [nn. 34-36]. The same holds with recursions of a general kind [n. 37] in spite of their not belonging to the pure number theory, provided only intensional properties of the attributes being considered be taken into account. Some corollaries concerning extensional attributes have been added [n. 37].

Hints suffice as far as ordinal arithmetics [n. 38], equivalence theorems for the axiom of choice [n. 40], and the pure theory of cardinals numbers [n. 41] in MC^{∞} are concerned. On the contrary the application of this theory in MC^{∞} to arbitrary sets has essentially modal features. In particular in MC^{∞} any element has both a intensional and an extensional power [n. 41]. This duplication of entities obviously affects some theorems.

Since the extensionalization $u^{(e)}$ of any non-empty set u is a proper class, it is useful to define the *rank-preserving extensionalization* $\mathscr{A}^{(rpe)}$ of any object $\mathscr{A}$ according to D39.6, in that it is a set when $\mathscr{A}$ is an element. This allows us to define the *limited-extension class* $\mathscr{A}^{(\mathscr{E})}$ of $\mathscr{A}$,

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(*) This pubblication was worked out in the sphere of activity of the groups for mathematical research of the Consiglio Nazionale delle Ricerche in the academic years 1971-72 and 1973-74.

which is the class of the rank-preserving [hence limited] extensions of the elements of $\mathscr{A}$. Unlike the intrinsic extension class $\mathscr{A}^{(E)}$ [D29.3] $\mathscr{A}^{(\mathscr{E})}$ is satisfactory also when $\mathscr{A}$ is a proper class.

Let us add that $\mathscr{A}^{(\mathscr{E})}$ is weakly separated ($WSep$) [D29.2] but, except for special choices of $\mathscr{A}$, it is not modally separated [D12.4]. This contributes to the fact that $\mathscr{A}^{(\mathscr{E})}$ and the theorems on $WSep$ [n. 29] have an essential role in the definition of the extensional rank $\varrho^E \mathscr{A}$ of $\mathscr{A}$ and in the proof of the main properties of ϱ^E [n. 39].

The changes of a modal nature to be performed on the theory of universes in [IST, Sec. 23] to construct a modal analogue of its in MC^{∞} are rather relevant in connection with the definition of the notion of universe, and some theorems on it [n. 42]. Therefore the proofs of some of these theorems have been explicitly written, at least in part.

33. Basic properties of ordinals.

The syntactical analogue $SA(12.2)_2$ of $(12.2)_2$—cf. ft. (6) in Part 2—and those of the other semantical theorem in n. 12 hold in MC^{∞}. As a preliminary we prove the following theorem whose part (a) is a strong analogue of [GIMC, Theor. 41.1 (II)] and whose part (b) is the extension of $SA(12.2)_2$, hence of [GIMC, Theor. 41.1 (VI)], to predicates.

THEOR. 33.1. (I) *If a predicate, say Pred, can be modally constant* [Conv. 12.1 (b)] *then it must be so*: $\vdash Pred \in^{\cup} MConst \supset^{\cap} Pred \in^{\cap} MConst$, *in particular* $A \in^{\cup} MConst \supset^{\cap} A \in^{\cap} MConst$.

(II) *If a predicate, Pred, is modally constant together with the entities for which it holds, then it is absolute*: $\vdash Pred \in MConst \wedge Pred \subseteq MConst \supset$ $\supset Pred \in Abs$.

PROOF. The matrix $Pred \in MConst$ is (a) $\forall_{\mathscr{A}}(\mathscr{A} \in^{\cup} Pred \supset \mathscr{A} \in^{\cap}$ $\in^{\cap} Pred)$, so that it is modally closed. Then part (I), i.e. $\diamond (a) \supset^{\cap} N(a)$, holds.

To prove part (II) assume (a), (b) $Pred \subseteq MConst$, (c) $\mathscr{A} =^{\cup} \mathscr{B}$, and ($d$) $\mathscr{A}, \mathscr{B} \in Pred$. By ($b$) and ($d$) $\mathscr{A}, \mathscr{B} \in MConst$. Hence $a \in \mathscr{A}$ yields $a \in^{\cap} \mathscr{A}$. Then $a \in^{\cup} \mathscr{B}$ by (c) and $a \in \mathscr{B}$ by (b). We easily conclude that $\mathscr{A} \subseteq \mathscr{B}$ holds. In the same way we prove $\mathscr{B} \subseteq \mathscr{A}$. Hence $\mathscr{A} = \mathscr{B}$. By the modal rule C we have (e) $\mathscr{A} =^{\cap} \mathscr{B}$. We easily conclude that (a), (b) $\vdash \forall_{\mathscr{A},\mathscr{B}}[(c) \wedge (d) \supset (e)]$, i.e. ($a$), ($b$) $\vdash Pred \in MSep$ [D12.4]. Then by D12.6 and the deduction theorem part (II) follows.

The notions of $\in$-*transitivity* ($\in$-*Trns*) and *ordinal classes* or briefly *ordinals* (*Ord*) are defined in ML^∞ by D5.4 and D14.5 respectively. By Convention 3.1 « Ord » expresses in ML^∞ the class of the ordinal sets, which are also called *ordinal numbers*—cf. [IST].

From D14.5 we deduce the third of the basic theorems—cf. Conventions 3.1,2 and 12.1*b*

$$(33.1)\quad \begin{cases} \vdash Ord \in MConst\,, & \vdash \mathrm{Ord} \in MConst\,, \\ \vdash Ord \subseteq MConst\,, & \vdash \mathrm{Ord} \subseteq \mathrm{MConst}\,, \\ \vdash \mathscr{A} \in Ord \supset \mathscr{A} \in Cl\,, & \vdash a \in \mathscr{A} \in Ord \supset a \in Ord\,. \end{cases}$$

Thence $(33.1)_{4,5}$ follow by the inclusion $\mathrm{Ord} \subseteq Ord$ and by D12.5 respectively.

To prove $(33.1)_6$ we start with (*a*) $b \in a \in \mathscr{A} \in Ord$. Then $\mathscr{A} \in \in\text{-}Trns$ by D14.5. Hence $b \in \mathscr{A}$ [D4.5]. Since $\mathscr{A} \in Ord$ by (*a*), this yields b, $a \in MConst \cap \in\text{-}Trns$ by D14.5. In addition, by the arbitrariness of b $(\in a)$ $b \in \mathscr{A}$ yields $a \subseteq MConst \cap \in\text{-}Trns$. Then $a \in Ord$ [D14.5]. On the basis of the deduction theorem $(33.1)_6$ holds.

Since $\vdash \mathrm{Ord} = Ord \cap El$, $(33.1)_2$ follows from $(33.1)_1$ by (18.8). Hence we have to prove only $(33.1)_1$. To this end we start with (*b*) $\mathscr{A} \in^{\cup} Ord$, which by D14.5 yields $\diamond p_i$ $(i = 1, \ldots, 4)$, where

$$p_1 \equiv_D \mathscr{A} \in MConst\,, \qquad p_2 \equiv_D \mathscr{A} \subseteq MConst\,,$$

$$p_3 \equiv_D \mathscr{A} \in \in\text{-}Trns\,, \qquad p_4 \equiv_D \mathscr{A} \subseteq \in\text{-}Trns\,.$$

From Theor. 33.1 (I) and $\diamond p_1$ we have Np_1:

Now we add the assumption (*c*) $b \in a \in \mathscr{A}$ which by Np_1 yields $a \in^{\cap} \mathscr{A}$, so that (*d*) $a \in MConst$ holds by $\diamond p_2$ and Theor. 33.1 (I). By (*d*) and (*c*) we have $b \in^{\cap} a$. By the arbitrariness of b and a we conclude that (*b*) yields p_2. Since $\vdash (b) \equiv N(b)$, (*b*) yields Np_2.

Now assume (*b*) and (*c*) again. The already proved steps $b \in^{\cap} a$, $a \in^{\cap} \mathscr{A}$, and $\diamond p_3$ yield $b \in^{\cup} A$ by D5.4. Thence we have $b \in^{\cap} \mathscr{A}$ by p_1; hence $b \in \mathscr{A}$. By the arbitrariness of a and b we conclude that (*b*) $[\equiv N(b)]$ yields p_3, hence Np_3.

In addition we can obviously conclude that $\forall_{\mathscr{A}}[(b) \supset^{\cap} Np_3]$.

Now we start with (*b*) and $a \in \mathscr{A}$. Furthermore we know that (*b*) yields $N(p_1 \wedge p_2 \wedge p_3)$; hence $a \in^{\cap} \mathscr{A}$ by p_1. This, (*b*), and $(33.1)_6$ yield $a \in^{\cup} Ord$. Since $\vdash \forall_{\mathscr{A}}[(b) \supset^{\cap} Np_3]$, this yields $a \in\text{-}Trns$. By the arbitrariness of a we conclude that (*b*) $[\equiv N(b)]$ yields p_4, hence Np_4.

We also conclude that (*b*) $\vdash N(p_1 \wedge p_2 \wedge p_3 \wedge p_4)$, so that $(33.1)_1$ holds by D14.5 and the deduction theorem. q.e.d.

By Theor. 33.1 (II), $(33.1)_{1,3}$ [$(33.1)_{2,4}$] yield the first [third] of the theorems

$$(33.2) \qquad \begin{cases} \vdash \mathit{Ord} \in \mathit{Abs}\,, & \vdash \mathit{Ord} \subseteq \mathit{Abs}\,, \\ \vdash \mathrm{Ord} \in \mathit{Abs}\,, & \vdash \mathrm{Ord} \subseteq \mathrm{Abs}\,. \end{cases}$$

To prove $(33.2)_2$ we start with (*a*) $\mathscr{A} \in \mathit{Ord}$, whence $\mathscr{A} \in \mathit{MConst} \wedge \wedge \mathscr{A} \subseteq \mathit{MConst}$ by D14.5. This and Theor. 33.1 (II) yield $\mathscr{A} \in \mathit{Abs}$. By the arbitrariness of $\mathscr{A}$ we easily conclude that $(33.2)_2$ holds. Thence $(33.2)_4$ follows for $\vdash \mathrm{Ord} \subseteq \mathit{Ord}$. q.e.d.

Theorems (33.1,2) are basic to extend the pure number theory in [IST, Chaps 2,3] to MC^{∞}. More precisely the theorems of this theory and their proofs are carried over to MC^{∞} by the convention that all variables should be restricted to absolute classes and by the change « $\{\}$ » $\to$ « $\{\}^{(i)}$ » (changes such as « $=$ » $\to$ « $=^{\frown}$ » or « $\subseteq$ » $\to$ « $\subseteq^{\frown}$ » are always possible but not always necessary). However by the analogue (10.1-3) for ML^{∞} of Convention 3.3 on restricted variables, we can practically take the conventions above into account by adding hypotheses of the form $A \in \mathit{Abs}$ in the theorems of [IST, Chaps 2,3]. These hypotheses can often be weakened to $A \in \mathrm{MConst}$, or they can be completely disregarded. This occurs e.g. in case $A \subseteq \mathit{Ord}$ or $A \in \mathit{Ord}$ respectively are implied by the other assumptions of the theorem being considered—cf. (33.5) below. We shall give some examples of analogues for MC^{∞} of some theorems in [IST, Chaps. 2,3] and their proofs. We shall substantially follow [IST pp. 69]—cf. fn. 5 inPart 2, n. 19.

$$(33.3) \quad \vdash 0 \in \mathrm{Ord} \quad (9.2) \qquad \vdash x \in \mathrm{Ord} \supset \mathfrak{S}x \in \mathrm{Ord} \quad [(13.14)_3]\,.$$

PROOF. By D2.4 ($0 =_D \Lambda$) and $(19.8)_1$, 0 is a set. Furthermore 0 is $\in$-transitive and modally constant by vacuous implication; and the same can be said of every member of 0 by vacuous implication. Thus $(33.3)_1$ holds.

To prove $(33.3)_2$ assume that (*a*) $x \in \mathrm{Ord}$. By $(13.14)_3$ $\mathfrak{S}x =^{\frown} x \cup \cup \{x\}^{(i)}$. By $(19.4)_1$ for $a =^{\frown} b =^{\frown} x$ and $(19.16)_2$ $\mathfrak{S}x \in \mathit{St}$. Furthermore by (*a*) and $(33.1)_4$ $x \in \mathit{MConst}$, so that $\mathfrak{S}x \in \mathit{MConst}$. Let $y \in \in z \in \mathfrak{S}x$ hold. Then either $z =^{\frown} x$ hence $y \in x$, or $z \in x$. Since x is

$\in$-transitive by (a) and D14.5, in the second case ($z \in x$) we have $y \in x$ again. Then $y \in \mathfrak{S}x$ in either case. Hence $\mathfrak{S}x$ is $\in$-transitive. We conclude that (b) $\mathfrak{S}x \in MConst \cap \in$-Trns.

Now let $y \in \mathfrak{S}x$ hold. Then $y \in x$ or $y =^{\cap} x$; by (a) and D14.5 y is both modally constant and $\in$-transitive. Hence $\mathfrak{S}x \subseteq \mathrm{MConst} \cap$ $\cap \in$-Trns. This and (b) yield $(33.3)_2$. q.e.d.

The essentially modal theorem

$$(33.4) \qquad \vdash A \in \mathrm{MConst} \wedge A \subseteq \mathrm{Ord} \supset (\bigcup A, \bigcap A \in \mathrm{MConst}) \wedge \wedge (\bigcup A \cup \bigcap A \subseteq \mathrm{MConst})$$

is easy to prove and serves as a lemma for the following one.

$$(33.5) \qquad \vdash A \in \mathrm{MConst} \wedge A \subseteq \mathrm{Ord} \supset \bigcup A \in Ord\,. \qquad (9.4)$$

PROOF. We can prove that $\bigcup A$ is $\in$-transitive together with its elements exactly as in [IST, p. 70]. Now assume (a) $A \in$ MConst and (b) $A \subseteq$ Ord. Then by (33.4) we have (c) $\bigcup A \in$ MConst.

Add the assumption $u \in \bigcup A$, hence $u \in x \in A$ for some x. Then $u \in x \in$ Ord by (b), so that $u \in$ MConst by D14.5. Then $\bigcup A \subseteq$ MConst. This, (c), and the obvious $\in$-transitivity properties of $\bigcup A$ yield $\bigcup A \in$ Ord. q.e.d.

34. Analogues for MC^∞ of some theorems in the extensional theory of ordinals.

We can easily prove on the basis of [IST], the following theorems (among which $(34.1)_{4,5}$ are related to the Burali-Forti paradox)—cf. fn. 5 in n. 19:

$$(34.1) \qquad \begin{cases} \vdash (a, b) \notin \mathrm{Ord} \quad (9.5) & \vdash A \in Ord \supset A \subseteq \mathrm{Ord} \quad (9.6) \\ \vdash \mathrm{Ord} \in Ord \quad (9.7)\,, & \vdash \mathrm{Ord} \notin St \quad (9.8) \end{cases}$$

We make some remarks on the proofs (in MC^∞) of the basic theorems

$$(34.2) \qquad \begin{cases} \vdash x, y \in \mathrm{Ord} \supset x =^{\cap} y \vee x \in y \vee y \in x \quad (9.9)\,, \\ \vdash 0 \neq A \subseteq \mathrm{Ord} \wedge A \in MConst \supset \bigcap A \in \mathrm{Ord} \wedge \bigcap A \in A \quad (9.10)\,. \end{cases}$$

To prove $(34.2)_1$ we set, following substantially [IST, p. 71], $A =_D =_D \{x | x \in \mathrm{Ord} \wedge \exists_y (y \in \mathrm{Ord} \wedge \sim x =^{\cap} y \wedge x \notin y \wedge y \notin x)\}$; and we want to show that $A = 0$. Assume on the contrary that $A \neq 0$. By A17.7 choose $a \in A$ such that $a^{\cup} \cap A = 0$, hence $a \cap A = 0$. Since $a \in A$, the class $B =_D \{y | y \in \mathrm{Ord} \wedge \sim a =^{\cap} y \wedge a \notin y \wedge y \notin a\}$ is non-empty. Then, following exactly [IST], we prove that $b \cap B = 0$ for some $b \in B$, that $\sim a =^{\cap} b$, and that $a \subseteq b$ and $b \subseteq a$. Furthermore, since $a \in B$ and $b \in B$, $a, b \in \mathrm{Ord}$; hence $a, b \in Cl$. Then $a = b$. In addition from $a, b \in \in \mathrm{Ord}$ we deduce $a, b \in MConst$. Then $a =^{\cap} b$, which contrasts to the previous result $\sim a =^{\cap} b$. Hence we have $(34.2)_1$.

To prove $(34.2)_2$ we assume (a) $0 \neq A \subseteq \mathrm{Ord}$ and (b) $A \in MConst$. Then, following [IST, p. 72] exactly, we deduce from (a) that (c) $\bigcap A \in \in \text{-}Trns \wedge \bigcap A \subseteq \in\text{-}Trns$ holds.

From (a) and $(33.1)_4$ we deduce $A \subseteq MConst$, which by (b) and (33.4) yields (d) $\bigcap A \in MConst$. Furthermore, if $x \in \bigcap A$, then $x \in y$ for some $y \in A$. Thence we also have $y \in Ord$ by (a), so that by $(34.1)_2$ $x \in \mathrm{Ord}$. Hence by $(33.1)_4$ $x \in \mathrm{MConst}$. We conclude that $\bigcap A \subseteq \subseteq \mathrm{MConst}$. By D14.5 this, (d), and (c) yield (e) $\bigcap A \in Ord$. Now one can prove that $\bigcap A \in A$ exactly as in [IST, p. 72]. Then $\bigcap A \in El$ and by (e), $(34.2)_2$ holds. q.e.d.

With even less changes can we turn the proofs of Theors. 9.11-13 in [IST] into proofs of the theorems (34.3,5) below:

(34.3) $$A, B \in Ord \vdash \text{(i) to (vi)},$$

where

(i) is $A \in \mathrm{Ord} \vee A =^{\cap} \mathrm{Ord}$,

(ii) is $A \in B \equiv A \subset B$,

(iii) is $A \in B \supset \mathfrak{S} A =^{\cap} B \vee \mathfrak{S} A \in B$

(iv) is $C \subseteq A \supset \bigcup C =^{\cap} A \vee \bigcup C \in A$,

(v) is $A \in \mathrm{Ord} \supset \bigcup \mathfrak{S} A =^{\cap} A$,

(vi) $A =^{\cap} \mathfrak{S} \bigcup A \vee A =^{\cap} \bigcup A$.

Let us now define the preceding relations $\leqslant$ and $<$ for ordinal numbers and the *R-least element* $(R\text{-}LeastEl(A))$ and *R-least upper* *bound* (R-l.u.b.(A)) of the class A for any partial ordering R—cf. D31.1.

D34.1 $\leqslant =_D \{(x, y) | x, y \in \mathrm{Ord} \wedge (x \in y \vee x =^\frown y)\}$ (9.12),

D34.2 $< =_D \{(x, y) | x, y \in \mathrm{Ord} \wedge x \in y\}$,

D34.3 $a \in R\text{-}Least\ El(A) \equiv_D R \in P\mathrm{Ord} \wedge a \in A \wedge \forall_b (b \in A \supset aRb)$ (8.6, iv),

D34.4 $a \in R\text{-l.u.b.}(A) \equiv_D \forall_b (b \in A \supset bRa) [\forall_b (b \in A \supset bRc) \supset aRc]$.

CONVENTION 34.1. *Following* [IST, p. 73] *we use lower case Greek letters* $\alpha, \beta, \gamma, \ldots$ *to denote ordinal numbers* (Ord) *unless otherwise indicated.*

By $(33.2)_{4,2}$ and D12.4,6

(34.4) $\vdash \alpha = \beta \equiv \alpha =^\frown \beta$, $\vdash A, B \in Ord \supset (A = B \equiv A =^\frown B)$.

Lastly let us write the straightforward analogue of [IST, Theor. 9.13] for MC^∞.

(34.5) $\vdash \leqslant \in W\mathrm{Ord}$, $\vdash \mathrm{Fld}(\leqslant) =^\frown \mathrm{Ord}$,

$\vdash 0 \neq A \subseteq \mathrm{Ord} \wedge A \in MConst \supset \bigcap A =^\frown \leqslant\text{-LeastEl}(A)$,

$\vdash 0 =^\frown \leqslant\text{-LeastEl(Ord)}$, $\vdash \forall_\alpha \alpha =^\frown \{\beta | \beta < \alpha\}$,

$\vdash \beta < \mathfrak{S}\alpha \equiv \beta \leqslant \alpha$, $\vdash (N) \sim \exists_\beta \alpha < \beta < \mathfrak{S}\alpha$,

$\vdash \alpha \leqslant \beta \equiv \alpha \subseteq \beta$, $\vdash A \subseteq \mathrm{Ord} \supset \bigcup A =^\frown \leqslant\text{-l.u.b.}(A)$,

$\vdash \mathfrak{S}\alpha \leqslant \beta \equiv \alpha < \beta \equiv \mathfrak{S}\alpha < \mathfrak{S}\beta$, $\vdash \mathfrak{S}\alpha =^\frown \mathfrak{S}\beta \equiv \alpha =^\frown \beta$.

35. Transfinite induction. Recursion.

As a lemma for extending [IST, Sects. 10-13] to MC^∞ quickly, we remark that—cf. D25.1 and D11.10

(35.1) $\vdash \mathscr{F} \in A \to B \wedge A, B \in Ord \supset \mathscr{F} \in AFn \wedge$

$\wedge (\mathscr{F}, \breve{\mathscr{F}} \in Fnc \equiv \mathscr{F}, \breve{\mathscr{F}} \in Fn)$, where—cf. the remark following D25.1—$\mathscr{F} \in A \to B$, unlike $\mathscr{F} \in {}^{A}B$, is compatible with the condition that $\mathscr{F}$ should be a proper class.

THEOR. 35.1. (*First principle of transfinite induction*)

$$(35.2) \quad \vdash A \in Ord\, \forall_{\alpha}[\alpha \in A \wedge \forall_{\beta}(\beta < \alpha \supset \beta \in B) \supset \alpha \in B] \supset A \subseteq B \quad (10.1).$$

Note the absence of an assumption such as $B \in MConst$. In spite of this (35.2) is proved the same way as [IST, Theor. 10.1]. Likewise the proof of statement 1 in [IST, p. 75] is substantially the one of the theorem

$$(35.3) \quad \vdash F \in A \to \mathrm{Ord} \wedge A \in \mathrm{Ord} \wedge \forall_{\alpha\beta}(\alpha, \beta \in A \wedge \alpha < \beta \supset F'\alpha < F'\beta) \supset$$
$$\supset \forall_{\alpha}(\alpha \in A \supset \alpha \leqslant F'\alpha) \quad [D4.2].$$

We now define *limit ordinal* (*LimOrd*):

$$\text{D35.1} \qquad \alpha \in LimOrd \equiv_{D} \sim \alpha =^{\cap} 0 \wedge \sim \exists_{\beta}(\alpha =^{\cap} \mathfrak{S}\beta)\ .$$

Hence α is a *successor ordinal*, i.e. $\exists_{\beta}(\alpha =^{\cap} \mathfrak{S}\beta)$, iff $\alpha \notin LimOrd$. The proofs of theorems (35.4-6) below are substantially those of Theors. 10.3,4 and statement 2 in [IST, p. 76].

$$(35.4) \quad \begin{cases} \vdash \alpha \notin LimOrd \equiv \bigcup \alpha < \alpha\,, \qquad \vdash \alpha \in \mathrm{LimOrd} \equiv \bigcup \alpha = \alpha \neq 0\,, \\ \vdash \alpha \in LimOrd \equiv \alpha \neq 0 \wedge \forall_{\beta}[\beta < \alpha \supset \exists_{\gamma}(\beta < \gamma < \alpha)]\,. \end{cases}$$

THEOR. 35.2. (*second principle of transfinite induction*)

$$(35.5) \qquad \vdash A \in Ord \wedge (\mathrm{i}) \wedge (\mathrm{ii}) \wedge (\mathrm{iii}) \supset A \subseteq B\,,$$

where

(i) is $0 \in B$,

(ii) is $\forall_{\alpha}[\alpha \in A \cap B \wedge \mathfrak{S}\alpha \in A \supset \mathfrak{S}\alpha \in B]$, and

(iii) is $\forall_{\alpha}[\alpha \in LimOrd \cap A \wedge \forall_{\beta}(\beta < \alpha \supset \beta \in B) \supset \alpha \in B]$.

THEOR. 35.3. *We have*

$$(35.6) \quad \vdash F \in Fn \wedge \mathrm{Dmn}\, F = \mathrm{Ord} \wedge F'\,0 =^{\cap} 0 \wedge \forall_{\alpha}(p_{\alpha} \wedge p'_{\alpha}) \supset \forall_{\alpha}(q_{\alpha} \wedge r_{\alpha} \wedge s_{\alpha})$$

where

$$(35.7)\quad \begin{cases} p_\alpha \equiv_D F' \mathfrak{S}\alpha =^\cap S^{mc} F'\alpha \\ \qquad\qquad [\text{D29.1 or } S^{mc}F'\alpha =_D (SF'\alpha) \cap \text{MConst}], \\ p'_\alpha \equiv_D \alpha \in \text{LimOrd} \supset F'\alpha =^\cap \bigcup_{\beta<\alpha} F'\beta\,, \qquad q_\alpha \equiv_D F'\alpha \in MConst, \\ r_\alpha \equiv_D F'\alpha \subseteq \text{MConst}\,, \qquad s_\alpha \equiv_D \forall_x (x \in F'\alpha \supset x \subseteq^\cap F'\alpha) \end{cases}$$

PROOF. Following substantially [IST p. 77] we deduce $\forall_\alpha(q_\alpha \wedge r_\alpha \wedge s_\alpha)$ « by induction on α » on the basis of (35.5). Assume the antecedent of (35.6) and set

$$(35.8)\qquad B \equiv_D \{\alpha | q_\alpha \wedge r_\alpha \wedge s_\alpha\}\,.$$

Since $F'0 \equiv^\cap 0$, q_0, r_0, and s_0 hold vacuously. Thus $0 \in B$, i.e. (i) in Theor. 35.2 holds.

Assume that $\alpha \in B$ (to show that $\mathfrak{S}\alpha \in B$). Then we get q_α, r_α, and s_α. By an assumption we have p_α, Furthermore q_α yields $S^{mc}F'\alpha \in MConst$ by $(29.3)_3$, and $S^{mc}F'\alpha \subseteq \text{MConst}$ by D29.1. Hence $F'\mathfrak{S}x \in MConst$ and $F'\mathfrak{S}\alpha \subseteq \text{MConst}$. Then $q_{\mathfrak{S}\alpha} \wedge r_{\mathfrak{S}\alpha}$ holds.

Now, following [IST, p. 77], we assume that $x \in F'\mathfrak{S}\alpha$ and $y \in x$ with a wiew to proving $s_{\mathfrak{S}\alpha}$. By p_α, $x \in S^{mc}F'\alpha$, i.e. (a) $x \subseteq F'\alpha$, and (b) $x \in MConst$ [D29.1]. From (a) we have $y \in F'\alpha$. By r_α and s_α it follows that $y \in MConst$ and $y \subseteq^\cap F'\alpha$ respectively, so that $y \in S^{mc}F'\alpha =^\cap$ $=^\cap F'\mathfrak{S}\alpha$ (by p_α). Since $y (\in x)$ is arbitrary, (b) and $q_{\mathfrak{S}\alpha}$ yield $x \subseteq^\cap F'\mathfrak{S}\alpha$. We conclude that we have $s_{\mathfrak{S}\alpha}$. Since we have also proved $q_{\mathfrak{S}\alpha} \wedge r_{\mathfrak{S}\alpha}$, we conclude that $\mathfrak{S}\alpha \in B$. That is (ii) in Theor. 35.2 holds.

Now we assume (c) $\alpha \in \text{LimOrd}$ and (d) $\forall_\beta(\beta < \alpha \supset \beta \in B)$, hence $q_\beta \wedge r_\beta \wedge s_\beta$ for $\beta < \alpha$. Furthermore p'_α holds. Then so do q_α and r_α obboviously. Furthermore let $x \in F'\alpha$. By p'_α we have $x \in F'\beta$ for some $\beta < \alpha$. Then $\beta \in B$ by (d); hence, by s_β, $x \subseteq^\cap F'\beta$. Then $x \subseteq^\cap F'\alpha$ by p'_α. We have deduced s_α. We conclude that, besides (i) and (ii), (iii) in Theor. 35.2 holds. Then by (35.5) for $A =^\cap \text{Ord}$, we have $\text{Ord} \subseteq B$. Hence $\forall_\alpha(\alpha \in B)$, i.e. $\forall_\alpha(q_\alpha \wedge r_\alpha \wedge s_\alpha)$. Thus (35.6) has been proved. q.e.d.

36. On natural numbers and transitive closures.

CONVENTION. *Following* [IST, p. 78] *we shall use the letters h, l, m, n, as variables restricted to natural numbers, unless otherwise indicated.*

Thus

(36.1) $$(\forall m)\, p \equiv (\forall m)(m \in \omega \supset p)$$

where

$$\omega =_D \bigcap \{A \mid 0 \in A \wedge \forall_x (x \in A \supset \mathfrak{S}x \in A)\}\,.$$

We can state and prove in MC^∞ [IST, Ths. 11.2-6] substantially as they are in [IST]. They say that *ω is a set and the smallest limit ordinal* and they include the complete and ordinary induction principle and the following consequence of the regularity axiom

(36.2) $$\sim \exists_f [f \in {}^{\omega}V \wedge (\forall n)\, f' \mathfrak{S}n \in f' n] \qquad \text{[D27.1]}.$$

Remembering D27.1 we define the *transitive closure* $\boldsymbol{T}R$ of the relation (contained in) R—cf. [IST, Def. 11.7]:

D36.1 $$\boldsymbol{T}R =_D \big\{(a, b) \mid a, b \in \mathrm{Fld}\, R \wedge (\exists m, f)(m \neq 0 \wedge f \in {}^{\mathfrak{S}m}(\mathrm{Fld}\, R) \wedge \wedge f' 0 =^{\frown} a \wedge f' m =^{\frown} b \wedge (\forall n)[n < m \supset (f' n, f' \mathfrak{S}n) \in R])\big\}.$$

We can state and prove in MC^∞ [IST, Ths 11.8-10] substantially as they are in [IST]. These theorems can be presented as follows:

TEOHR. 36.1. *If* $R \subseteq V^2$, *then* (i) $R \subseteq \boldsymbol{T}R$, (ii) $\mathrm{Fld}\, R = \mathrm{Fld}(\boldsymbol{T}R)$, (ii) $\boldsymbol{T}R \in Trns$, (iv) $S \in Trns \wedge R \subseteq S \supset \boldsymbol{T}R \subseteq S$, *and* (v) $R \in Trns \supset R = \boldsymbol{T}R$—cf. D30.3.

THEOR. 36.2. *If* $A \subseteq Eqv$, $B = \bigcup_{R \in A} \mathrm{Fld}\, R$, *and* $B \in St$, *then* $B = \mathrm{Fld}(\boldsymbol{T} \bigcup A)$ *and* $\boldsymbol{T} \bigcup A = \bigcap \{S \mid \bigcup A \subseteq S \wedge S \in Eqv\}$.

THEOR. 36.3.
$\vdash (R, A \in St) \wedge R \subseteq V^2 \supset (\boldsymbol{T}R)'' A = \bigcap \{C \mid A \subseteq C \wedge R'' C \subseteq C\}$ [D4.3].

37. Recursions of a general kind.

Since Sec. 12 in [IST] concerns functions with fields formed by ordinals, in particular *normal* and *half normal* sequences of ordinals, it belongs to pure mathematics and can practically be regarded as embodied into MC^∞ (assuming Fn as the analogue for MC^∞ of the notion of function). Now we consider recursions—cf. [IST, Theor. 13.1]:

THEOR. 37.1 (*General recursion principle*). *Let* R *be a modally constant and asymmetric well founded relation—so that* $R \cap \boldsymbol{I} =^{\cap} 0$, *cf. ft.* 11 *in Part* 2—*such that for all* $a \in \mathrm{Fld}\, R$, $\{b|bRa\}$ *is a set, and let* $F \in (\mathrm{Fld}\, R) \times V \to V$ [D25.1]. *Then there is a unique* $G \in Fn$ *such that* $\mathrm{Dmn}\, G = \mathrm{Fld}\, R$, *and for all* $a \in \mathrm{Fld}\, R$,

$$Ga =^{\cap} F'(a, \{b|bRa\} 1 |G) . \tag{37.1}$$

Remark, first, that the assumption $R \in MConst$ makes the proof of the theorem very similar with the one of [IST, Theor. 13.1], in that it implies, for every a, that $\{b|bRa\} \in MConst$, so that $\{b|bRa\} 1 G \in \in Fn$ [D11.14]. Second, given the function F above and $S \subseteq V^2$, there exist a unique $R \in MConst$ such that $R = S$. Hence the extensions of S and F determine the function G above. Now it is evident that Theor. 37.1 keeps holding after crossing out the assumption $R \in MConst$ and after substituting the class (set) $\{b|bRa\}$ in (37.1) with the modally constant class u that equals $\{b|bRa\}$. This shows that the aforementioned assumption is not restrictive.

PROOF of Theor. 37.1. Following substantially [IST, p. 88] we set

$$\begin{cases} M =_D \{h|h \in Fn \wedge \mathrm{Dmn}\, h \subseteq \mathrm{Fld}\, R \wedge \forall_\alpha [a \in \mathrm{Dmn}\, h \supset \\ \supset \{b|bRa\} \subseteq \mathrm{Dmn}\, h \wedge h'\, a =^{\cap} F(a, \{b|bRa\} 1 h)]\} . \end{cases} \tag{37.2}$$

Hence $M \subseteq Fn$ and $M \in MConst$ because $R \in MConst$. Now, setting $G =_D \bigcup M$, it is easy to deduce $G \in Fn$ from (37.2) and (28.4). We easily deduce $\mathrm{Dmn}\, G \subseteq \mathrm{Fld}\, R$ also. To establish the converse inclusion and the uniqueness property mentioned in the theorem, it suffices to follow [IST, pp. 88, 89] with these provisions. Remark that $M \in \in MConst$ and assume that the subclass N mentioned in [IST] also is modally constant. More precisely our analogue for the steps (3) and (4) in [IST, p. 88] is

$$\Lambda \neq N \in S^{mc} M \supset \bigcap N \in M \wedge (N \in \mathrm{El} \supset \bigcup N \in M) \quad [\text{D29.1}]. \tag{37.3}$$

q.e.d.

COROLLARY *to Theor.* 37.1. *Add to the hypotheses of Theor.* 37.1 *that* R [F] *should be extensionally invariant with respect to its last* [*first*] *argument* (*in the* Γ-*case being considered*), *more precisely* [Conv. 10.1]

$$(\ )\, bRa \wedge a = a_1 \supset bRa_1 , \qquad (\ )\, a = a_1 \supset F(a, f) = F(a_1, f) \quad [\text{D12.1}]; \tag{37.4}$$

and disregard the assumption $R \in MConst$. *Then there is a unique extensionally invariant function* $F(\in Fnc)$ [D13.4] *such that for all* $a \in$ Fld

$$(37.1') \quad G'a =^\frown F'(a, u_a 1 G) \quad where \quad u_a = \{b|bRa\} \quad \text{and} \quad u_a \in MConst.$$

PROOF. By Theor. 37.1 and the subsequent remark there is a unique $G \in Fn$ such that (37.1′) holds for all $a \in \text{Fld}\, R$. To prove that this G is extensionally invariant, assume $a, a_1 \in \text{Fld}\, R$. Then by $(37.4)_1$ the sets $\{b|bRa\}$ and $\{b|bRa_1\}$ coincide; hence $u_a =^\frown u_{a_1}$ by $(37.1')_{2,3}$: Then $u_a 1 G =^\frown u_{a_1} 1 G$. Furthermore $(37.1')_1$ holds for all $a \in \text{Fld}\, R$. Then by $(37.4)_2$ $G'a = G'a_1$. We conclude that $G \in Fnc$. q.e.d.

The theorems 37.2-8 below are the analogues for MC^∞ of [IST, Ths. 13.2-8]. They differ from the latter only by some of our usual changes and by the addition of some assumptions of the form $A \in MConst$ The proofs of the theorems below are obtained from those of the corresponding theorems like the one of Theor. 37.1.

THEOR. 37.2 (*iteration principle*). *Let* $a \in A \in MConst \cap St$ *and* $f \in A \to A$ [D25.1]. *Then there is a unique function* $g \in {}^\omega A$ [D27.1] *such that* $g'0 =^\frown a$ *and* $g'\mathfrak{S}m =^\frown f'g'm$ *for all* $m \in \omega$.

THEOR. 37.3 (*General recursion principle for ordinals*). *Assume that* $A \in Ord$ *and* $F \in V \to V$. *Then there is a unique function* $G \in A \to V$ *such that, for every* $\alpha \in A$, *we have* $G'\alpha =^\frown F(\alpha 1 G)$.

THEOR. 37.4 (*Usual recursion principle for ordinals*). *Assume that* $0 < A \in Ord$, $a \in B \in MConst$, $F \in B \to B$, *and* $G \in C \to B$ *where* $C =_D$ $=_D \{f|\exists_\alpha(\alpha \in A \wedge f \in {}^\alpha B)\}$. *Then there is a unique function* $H \in A \to B$ *such that*

(i) $H'0 =^\frown a$,

(ii) $H'\mathfrak{S}\alpha =^\frown F'H'\alpha$ *for every* α *for which* $\mathfrak{S}\alpha \in A$, *and*

(iii) $H'\beta =^\frown G(\beta 1 H)$ *for every limit ordinal* $\beta \in A$.

Remark that since $\omega \in$ Ord and ordinal numbers are absolute classes —cf. $(33.2)_4$—, the extensional invariance of the functions g, G, and H considered in the Corollary above holds trivially. Hence the analogues of this corollary for these theorems have no interest.

THEO. 37.5 (*General recursion principle with a parameter*). *Assume that* $R \cap I = 0$—cf. ft 11 *in Part* 2—, *that* $R \in WFnd$, *that for all* $a \in$ Fld R

we have $\{b|bRa\} \in St$, *and that* $F \in V \times \mathrm{Fld}\, R \times V \to V$. *Then there is a unique function* $G \in V \times \mathrm{Fld}\, R \to V$ *such that, for every* $a \in V$ *and* $b \in \mathrm{Fld}\, R$

$$G'(a, b) =^{\frown} F'(a, b, u_{a,b} 1 G) \tag{37.5}$$

where

$$u_{a,b} = \{(c, d)|c =^{\frown} a \wedge dRc\}, \qquad u_{a,b} \in MConst.$$

THEOR. 37.6 (*General recursion principle for ordinals, with a parameter*). *Assume that* $A \in Ord$ *and* $F \in (V \times A \times V) \to V$ [D25.1]. *Then there is a unique* $G \in (V \times A) \to V$ *such that, for all* a *and all* $\alpha \in A$

$$G'(a, \alpha) =^{\frown} F'(a, \alpha, \{(b, \beta)|b =^{\frown} a \wedge \beta < \alpha\} 1 G). \tag{37.6}$$

THEOR. 37.7 (*Usual recursion principle for ordinals with a parameter. Assume that* $A \in Ord$, $B \in MConst$, $F \in B \to B$, $G \in (B \times A \times B) \to B$, *and* $H \in (B \times A \times V) \to B$. *Then there is a unique* $K \in (B \times A) \to B$ *such that, for all* $a \in B$

(i) $K'(a, 0) =^{\frown} F'(a)$,

(ii) $K'(a, \mathfrak{S}\alpha) =^{\frown} G'(a, \alpha, K'(a, \alpha))$ *for every* α *for which* $\mathfrak{S}\alpha \in A$,

(iii) $K'(a, \beta) =^{\frown} H'(a, \beta, \{(b, \gamma)|b =^{\frown} a \wedge \gamma < \beta\} 1 K)$ *for all* $\beta \in A \cap \cap LimOrd$.

THEOR. 37.8 (*Primitive recursion*). *Assume that* $B \in MConst$, $f \in \in B \to B$, *and* $g \in (B \times \omega \times B) \to B$. *Then there is a unique* $h \in (B \times \omega) \to B$ *such that, for all* $x \in B$ *and all* $m \in \omega$

(i) $h'(a, 0) =^{\frown} f' a$,

(ii) $h'(a, \mathfrak{S}m) =^{\frown} g'[a, m, h'(a, m)]$.

The analogues of the Corollary to Theor. 37.1 hold for Theors. 37.5-8.

COROLLARY *to Theors.* 37.5-8. *Briefly, the function* G *in* (37.5) [(37.6)] *is extensionally invariant* ($G \in Fnc$) [D13.4] *in case* (37.5) *holds for all* $a \in V$ *and* $b \in \mathrm{Fld}\, R$, *the ternary function* F *is extensionally invariant,* *with respect to the first and second arguments,* i.e.

$$(37.7)\qquad (\ )\,[(a, b, c), (a_1, b_1, c) \in \mathrm{Dmn}\, F]\, a = a_1 \wedge b = b_1 \supset F(a, b, c) = \\ = F(a_1, b_1, c)\,,$$

and R is extensionally invariant with respect to its second argument—cf. $(37.4)_1$—[*in case* (37.6) *holds for all* $a \in V$ *and all* $\alpha \in A$, *and in addition F is extensionally invariant with respect to its first argument*].

Likewise the function K [h] considered in Theor. 37.7 [Theor. 37.8] *is extensionally invariant* ($K \in Fnc$) *in case F, G, and H [f and g] are extensionally invariant with respect to the first argument.*

38. Hints at ordinals arithmetics.

The fixed point theorem for normal functions [IST, Theor. 13.9] concerns pure mathematics and can be stated and proved in MC^{∞} the way it is in [IST].

THEOR. 38.1. *If* $R \in \mathrm{WOrd}$ (*hence* $R \in St$), *then there is a unique α and a unique function f such that f is an* (*intensional*) *isomorphism from* $(\lambda, \beta, \gamma)(\beta \leqslant \gamma < \alpha)$ *onto R*—cf. [IST, Theor. 13.10]:

$$(38.1)\qquad \vdash R \in \mathrm{WOrd} \supset (\exists_1 \alpha)(\exists_1 f)\ \mathrm{Ism}\,[f, (\lambda\beta, \gamma)(\beta \leqslant \gamma < \alpha), R][D31.12]\,.$$

PROOF. Add the assumption $R \in MConst$ and prove the theorem using the proof of [IST, Theor. 13.10] with the usual suitable changes. Then, since $\mathrm{Ism}(F, R, S)$ is extensional with respect to R, by $(31.3)_2$, the additional assumption above can be disregarded. q.e.d.

Following substantially [IST, Theor. 13.11], we introduce the *ordinal* $\tau(R)$ of the (intensional) well ordering R

$$\text{D38.1}\qquad \tau(R) =_D (\imath\alpha)\, \exists_f [R \in \mathrm{WOrd} \wedge \mathrm{Ism}(f, R, \{(\beta, \gamma) | \beta \leqslant \gamma < \alpha\})]\,.$$

Furthermore we state the following theorem to be proved substantially as [IST, Theor. 13.11].

$$(38.2)\qquad \vdash R, S \in \mathrm{WOrd} \supset [\exists_F\, \mathrm{Ism}(F, R, S) \equiv \tau(R) = \tau(S)]\,.$$

* * *

It is quite straight forward to embody into MC^∞ the purely mathematical part of ordinal arithmetic, and more precisely Sec. 14 in [IST] and the part of Sect. 15 up Theor. 15.15. Thus we can speak of the *ordinal sum* $\alpha \dotplus \beta$, *ordinal multiplication* $\alpha \cdot \beta$ of α and β, and *ordinal exponentiation* α^β. The operation $\dotplus$ is defined recursively [Theor. 37.7] by the conditions

$$(38.3) \qquad \begin{cases} \alpha \dotplus 0 =^\cap \alpha\,, \qquad \alpha \dotplus \mathfrak{S}\beta =^\cap \mathfrak{S}(\alpha \dotplus \beta) \\ \alpha \dotplus \gamma =^\cap \bigcup_{\delta<\gamma} (\alpha \dotplus \delta) \quad \text{if} \quad 0 \neq \gamma = \bigcup \gamma\,. \end{cases}$$

Obviously

$$(38.4) \qquad \vdash \dotplus \notin El\,, \qquad \vdash \dotplus \in \mathrm{Ord}^{(2)} \to \mathrm{Ord} \qquad [D25.1]$$

It is natural to consider the ternary extensionalization $\dotplus^{(3e)}$ of $\dotplus$ [D12.2] and to accept the convention

$$(38.5) \quad \vdash a \dotplus^{(e)} b =_D \dotplus^{(e)/}(a, b) =^\cap (\imath c)\, \exists_{\alpha\beta} \quad (a = \alpha \wedge b = \beta \wedge c = \alpha \dotplus \beta)\,.$$

Hence in case $a, b \in \mathrm{Ord}^{(e)}$, we have $a \dotplus^{(e)} b \in \mathrm{Ord}^{(e)}$. In the remaining case $a \dotplus^{(e)} b = a^*$. (It is convenient to write « $\dotplus$ » again for « $\dotplus^{(e)}$ ».)

39. Intensional and extensional modal ranks in MC^∞, rank-preserving extensionalizations, and limited-extension classes.

We define in MC^∞ the (*intensional*) *rank* ϱa of any element a and the class $\boldsymbol{M}_\alpha$ of the elements whose ranks are $< \alpha$—cf. [IST, (15,16,19)]:

$$\text{DD39.1,2} \qquad \varrho a =_D \bigcap \{\alpha | \forall_b (b \overset{\cup}{\in} a \supset \varrho b < \alpha)\}\,, \qquad \boldsymbol{M}_\alpha =_D \{a | \varrho a < \alpha\}\,.$$

Thus ϱx is the least ordinal larger than every ϱb with $b \overset{\cup}{\in} x$. D39.1 is easily justified, using the general recursion theorem, in a way very similar to the justification of its analogue (5.7) for EC^∞. This justification is given by Theor. 5.2. Since by $(21.8)_2 \vdash \{b | b \overset{\cup}{\in} a\} = a^{\cup} \in St$, the proof of Theor. 5.2 can be extended to MC^∞ easily.

Here are the analogues for MC^{∞} of Theors. 15.17,18, and 20 in [IST]:

(39.1)
$$\begin{cases} \vdash a \overset{\smile}{\in} b \supset \varrho x < \varrho y\,, \quad \vdash a \overset{\frown}{\subseteq} b \supset \varrho a \leqslant \varrho b, \quad \vdash \varrho(a \cup b) = \varrho a \cup \varrho b \\ \vdash \varrho \bigcup a = \bigcup \{\varrho b | b \overset{\smile}{\in} a\} \leqslant \varrho a\,, \qquad \vdash \varrho\{a\}^{(\iota)} = \varrho a \dotplus 1\,, \\ \vdash \varrho \boldsymbol{S}^{\frown} a = \varrho a \dotplus 1 = \varrho \boldsymbol{S}^{mc} a\,, \qquad \vdash \varrho \alpha = \alpha\,. \end{cases}$$

(39.2)
$$\begin{cases} \vdash \boldsymbol{M}_\alpha \in St\,, \qquad \vdash \boldsymbol{M}_\alpha = (\mathrm{In} \cup \bigcup_{\beta<\alpha} \boldsymbol{S}\boldsymbol{M}_\beta)^{\frown}\,, \qquad \vdash \varrho \boldsymbol{M}_\alpha = \alpha\,, \\ \vdash x \in \boldsymbol{M}_\alpha \supset x \subset \boldsymbol{M}_\alpha\,, \qquad \vdash \alpha < \beta \supset^{\frown} \boldsymbol{M}_\alpha \subset \boldsymbol{M}_\beta \wedge \boldsymbol{M}_\alpha \in \boldsymbol{M}_\beta\,, \\ \vdash (\mathrm{In} \cup \boldsymbol{S}\boldsymbol{M}_\alpha)^{\frown} = \boldsymbol{M}_{\alpha\dotplus 1}\,, \qquad \vdash \alpha \in \mathrm{LimOrd} \supset \boldsymbol{M}_\alpha = (\bigcup_{\beta<\alpha} \boldsymbol{M}_\beta^{(e)})^{\frown}\,, \\ \vdash V = (\bigcup_{\alpha \in Ord} \boldsymbol{M}_\alpha^{(e)})^{\frown}\,. \end{cases}$$

Remark that theorems (5.10), $(5.12)_2$, and $(5.13)_{1,2}$ in EC^{∞} are turned into $(39.2)_{2,6,7,8}$ respectively by changes that are similar with one another but differ from the foregoing standard changes used to turn [IST] into a theory based on MC^{∞}. Let us add that the changes of the new kind are important for e.g.

$$\vdash \mathrm{In}^{\frown} \cup \boldsymbol{S}^{\frown} \boldsymbol{M}_\alpha \subset \boldsymbol{M}_{\alpha\dotplus 1}$$

—cf. $(39.2)_6$.

Of course we cannot replace $\overset{\frown}{\subseteq}$ with $\subseteq$ in $(39.1)_2$, or $\boldsymbol{S}^{\frown}$ with $\boldsymbol{S}$ in $(39.1)_6$, or $\boldsymbol{S}$ with $\boldsymbol{S}^{\frown}$ in $(39.2)_2$, or else $\boldsymbol{M}_\beta^{(e)}$ with $\boldsymbol{M}_\beta$ in $(39.2)_7$. The proofs of Theorems (39.1,2) are sufficiently clear from [IST, pp. 113-14].

Incidentally the same holds for theorem (39.3) below on the so called *R-type* $\tau_R a$ *of* a—cf. Def. 15.21 in [IST]—which is of interest in case R is an equivalence relation:

D39.3
$$\tau_R a =_D \{b | aRb \wedge \varrho b = \bigcap \{\varrho c | cRa\}\}\,,$$

(39.3)
$$\vdash R \in Eqv \wedge a, b \in \mathrm{Fld}\, R \supset \tau_R a \neq 0 \wedge [\tau_R a = \tau_R b \equiv aRb] \wedge \tau_R a \in St$$

Let us note the following extensionality property of τ_R:

(39.4)
$$\vdash R = S \supset \tau_R a = \tau_S a\,.$$

As a special case of R-types we obtain order types—cf. [IST, p. 114].

The *extensional* (*modal*) *rank* $\varrho^E a$ of any element a can be defined as $\tau_= a$, i.e. the least (intensional modal) rank of an object b that coincides with a:

D39.4 $$\varrho^E a =_D \bigcap \{\varrho b, | b = a\} .$$

Hence we obviously have the first two of the theorems

(39.5) $$\vdash \varrho^E a \leqslant \varrho a \,, \quad \vdash a = b \supset \varrho^E a = \varrho^E b \,, \quad \vdash a \in \mathrm{MConst} \supset \varrho^E a = \varrho a \,.$$

The second justifies our denomination of ϱ^E; the third also does if one remembers that in our semantical theory for ML^∞ the extension corresponding to the quasi intension ξ in a Γ-case γ, was identified with the L-determined QI equivalent to ξ in γ [n. 9].

To prove $(39.5)_3$ assume (a) $a \in \mathrm{MConst}$ and (b) $a = b$. By (a) $c \mathrel{\breve{\in}} a$ yields $c \in a$, hence $c \mathrel{\breve{\in}} b$ by (b). So $\varrho a \leqslant \varrho b$ by D39.1. We easily conclude that $\vdash (a) \supset \forall_b (a = b \supset \varrho a \leqslant \varrho b)$, hence $\vdash (a) \supset \varrho a = \bigcap \{\varrho b | b = a\}$. Then by D39.4, $(39.5)_3$ holds.

Now we can define the *rank-preserving extensionalization* $x^{(rpe)}$ of any set x and the *limited-extension class* $\mathscr{A}^{(\mathscr{E})}$ of $\mathscr{A}$ [D29.3].

D39.5 $$\mathscr{A}^{(rpe)} =_D (\lambda b)\, \exists_a (a \in \mathscr{A} \wedge a = b \wedge \varrho b \leqslant \varrho^E a) \,,$$

D39.6 $$\mathscr{A}^{(\mathscr{E})} =_D [\mathscr{A}^{(rpe)}]^{(E)} \,.$$

It is easy to prove by D39.1 that

(39.6) $$\vdash \varrho x^{(rpe)} = \varrho x \,, \qquad \vdash x^{(rpe)} \in St \,.$$

Furthermore by D39.6 and D29.3

(39.7) $$\begin{cases} \vdash u \in \{a\}^{(\mathscr{E})} \equiv u \in \mathrm{MConst} \wedge u = \{b | b = a \wedge \varrho b \leqslant \varrho^E a\} \,, \\ \vdash [\{a\}^{(i)}]^{(\mathscr{E})} = \{a\}^{(\mathscr{E})} \,, \qquad \vdash \mathscr{A}^{(\mathscr{E})} = \{\{a\}^{(\mathscr{E})} | a \in \mathscr{A}\} \,, \\ \hfill \vdash \mathscr{A}^{(rpe)} = \bigcup \mathscr{A}^{(\mathscr{E})} \,. \end{cases}$$

By $(39.7)_1$ we can say that the non-empty set u such that $\{u\}^{(i)} = \{a\}^{(\mathscr{E})}$ is the *limited extension* of a. This and $(39.7)_3$ motivate the name chosen for $\mathscr{A}^{(\mathscr{E})}$.

D29.2 and $(39.7)_{1,4}$ yield the first of the theorems

$$(39.8) \qquad \vdash \mathscr{A}^{(\mathscr{E})} \in WSep\,, \qquad \vdash (\{2\}^{(i)})^{(\mathscr{E})} \notin MConst\,,$$
$$\vdash \exists_A (A \in \text{MConst} \wedge A^{(\mathscr{E})} \notin MConst)\,.$$

The second is obvious. It yields the third, which shows that the analogue for $u^{(\mathscr{E})}$ of theorem $(29.7)_3$ on $u^{(E)}$ is false. This is an advantage of $\mathscr{A}^{(E)}$ with respect to $\mathscr{A}^{(\mathscr{E})}$ in the case $\mathscr{A} \in St$. However in the remaining case only $\mathscr{A}^{(\mathscr{E})}$ is interesting -see below.

D14.3 yields easily the first of the theorems

$$(39.9) \qquad \begin{cases} \vdash A \approx^{(e)} A^{(\mathscr{E})}\,, & \vdash A^{(\mathscr{E})} \approx^{(e)} B^{(\mathscr{E})} \equiv A^{(\mathscr{E})} \approx B^{(\mathscr{E})} \equiv A \approx^{(e)} B\,, \\ \vdash u^{(\mathscr{E})} \approx^{(e)} u^{(E)}\,, & \vdash u^{(\mathscr{E})} \approx u^{(E)}\,. \end{cases}$$

The second follows from $(39.8)_1$, $(29.4)_3$, and $(39.9)_1$; the third from $(39.9)_1$ and $(29.8)_1$. It yields the fourth by $(39.8)_1$, $(29.7)_1$, and $(29.4)_{1,3}$.

For A and B sets, theorems $(39.9)_{1,2}$ are the analogues of theorems $(29.8)_{1,2}$ on the intrinsic extension class $\mathscr{A}^{(E)}$ [D29.3]. The former theorems hold also in case A and B are (arbitrary) proper classes, unlike their analogues for $\mathscr{A}^{(E)}$. Thus only the first of the classes $\mathscr{A}^{(\mathscr{E})}$ and $\mathscr{A}^{(E)}$ is satisfactory for an (arbitrary) proper class $\mathscr{A}$. This is perhaps the main motive for introducing the notion of limited-extension class.

CHAPTER 6

ON THE AXIOM OF CHOICE AND ORDINALS

40. Hints at the axiom of choice and its equivalents.

To carry over to MC^∞ Chapter 3 in [IST] on the axiom of choice is very straightforward, so that brief hints suffice. The (intensional) relational axiom of choice A17.10 implies the axiom of choice below —cf. [IST, p. 116]—which is not equivalent to it—cf. D25.1.

(40.1) (axiom of choice)

$$\vdash \forall_{\mathscr{A}} (\mathscr{A} \in A \supset 0 \neq \mathscr{A} \in St) \supset \exists_F [F \in A \to \bigcup A \wedge \forall_{\mathscr{A}} (\mathscr{A} \in A \supset F'\mathscr{A} \in \mathscr{A})]\,.$$

The proof of (40.1) is practically the one of [IST, Theor. 16.1]. The analogue holds for the proof of the equivalence of (40.1) with any of the assertions (40.2-4) (in MC^∞) below, written as examples of such equivalents.

(40.2) (*Counting principle*)

$$\vdash \exists_F(F, \breve{F} \in Fn \wedge \mathrm{Dmn}\, F = x \wedge \mathrm{Rng}\, F \in \mathrm{Ord}),$$

(40.3) (*Well ordering principle*)

$$\vdash \exists_R(R \in \mathrm{WOrd} \wedge \mathrm{Fld}\, R = x) \qquad [\mathrm{D31.9}]$$

(40.4) (*Trichotomy principle*)

$$\vdash \exists_{\mathscr{A},\mathscr{B}}[\mathscr{A} \subseteq A \wedge \mathscr{B} \subseteq B \wedge (A \approx \mathscr{B} \vee \mathscr{A} \approx B)] \qquad [\mathrm{D14.1}].$$

The dychotomic form of (40.4) has simplicity reasons. We need not write explicitly the versions in MC^∞ of the multiplicative principle, Zermelo's principle, Zorn's lemma, the maximality principle, Kuratowski's principle, and the mapping principle—cf. [IST, Sec. 16]. Furthermore it is useless to write explicitly the analogues for MC^∞ of the applications of the axiom of choice in [IST, Sec. 17].

41. Basic considerations on cardinal numbers in $\mathbf{MC^\infty}$.

As we already said, the version for MC^∞ of [IST, Chapter 4] is practically equal to the original one, as far as the pure theory of cardinals (i.e. cardinal numbers) is concerned, and is nearly so in connection with many applications of this theory. In some applications of this theory some notions and theorems in [IST] have an intensional analogue in MC^∞ as well as an extensional or total one—cf. e.g. DD14.2,3 $(41.6)_{1,2}$, and (41.7,8) below.

We now define *cardinals* (Card), the *intensional power* or *cardinality* $|x|$ of any set x, and the *extensional power* $|x|^E$ of x:

D41.1 $$\mathrm{Card} =_D \{\alpha | \sim \exists_\beta(\beta < \alpha \wedge \beta \approx \alpha)\} \qquad [\mathrm{D14.1}],$$

D41.2 $$|\mathscr{A}| =_D (\imath\beta)(\beta \in \mathrm{Card}^{(e)} \wedge \beta \approx \mathscr{A}) \quad [\mathrm{D14.1}],$$

D41.3 $$|\mathscr{A}|^E =_D |\mathscr{A}^{(\mathscr{E})}| \quad [\mathrm{D39.6}].$$

CONVENTION 41.1. *Following* [IST, pp. 129] *we use lower case Greek* [*German*] *letters as variables restricted to ordinals* [*cardinals*], *unless otherwise indicated.*

The analogues for MC^{∞} of Theors. 18.2,3 and some parts of Theor. 18.5 in [IST] are

$$(41.1)\qquad \vdash \mathfrak{m} \neq \mathfrak{n} \equiv \sim \mathfrak{m} \approx \mathfrak{n}\,, \quad \vdash (\exists_1 \mathfrak{m})\, \mathfrak{m} \approx x\,,$$

$$(\vdash (\exists_1 \mathfrak{m}) p \equiv (\exists_1^{\cap} \mathfrak{m}) p)\,,$$

$$(41.2)\qquad \vdash x \approx y \equiv |x| = |y|\,, \quad \vdash x \approx z \approx y \supset |x| = |y|\,, \quad \vdash x \approx |x|\,.$$

Remark that $|x| \in \mathrm{Card}$ or $|x|^E \in \mathrm{Card}$ is true only for special choices of x. More precisely

$$(41.3)\qquad \vdash |x| \in \mathrm{Card}^{(e)}\,, \quad \vdash |x|^E \in \mathrm{Card}^{(e)}, \quad \vdash x \in \mathrm{MConst} \supset |x| \in \mathrm{Card},$$

and $|x|$ cannot be replaced by $|x|^E$ in $(41.3)_3$.

By $(33.2)_{3,4}$ DD41.1,2 yield easily the theorems

$$(41.4)\qquad \begin{cases} \vdash \mathrm{Card} \in Abs\,, \quad \vdash \mathrm{Card} \subseteq Abs\,, \quad \vdash |\alpha| = |\alpha|^E\,, \\ \vdash |\alpha| = (\imath \mathfrak{m})\, \mathfrak{m} \approx \alpha\,, \quad \vdash |\alpha| \in \mathrm{Card} \end{cases}$$

which say that the two notions of power coincide in pure number theory and that this theory can be developed in MC^{∞} practically the way it is in extensional logic. Remembering (38.3) and $(36.1)_2$ we have—cf. [IST, (18.6)]:

$$(41.5)\qquad \vdash |\alpha| \leqslant \alpha\,, \quad \vdash \alpha \approx A \supset |A| \leqslant \alpha\,, \quad \vdash \alpha \in \mathrm{Card} \equiv |\alpha| = \alpha\,,$$

$$\vdash \omega \leqslant \alpha \supset |\alpha + 1| = |\alpha|\,.$$

Theor. 18.7 in [IST] has, so to say, the intensional and extensional (or total) versions below.

$$(41.6)\qquad \vdash x \subseteq y \supset |x| \leqslant |y|\,, \quad \vdash x \subseteq y \supset |x|^E \leqslant |y|^E \qquad (18.7).$$

The analogous versions for [IST, Theor, 18.8] can be written—cf. D27.1 and D13.3—in the forms

$$(41.7)\qquad \vdash |x| \leqslant |y| \equiv \exists_F (F \in {}^x y \wedge \breve{F} \in Fn) \equiv 0 = x \vee \exists_F (F \in {}^y x \wedge \mathrm{Rng}\, F = x)$$

and—cf. D39.6 and DD27.1,2

$$(41.8)\qquad \vdash |x|^E \leqslant |y|^E \equiv \exists_F(F \in {}^{v^{(\mathscr{E})}} x^{(\mathscr{E})} \wedge \breve{F} \in Fn) \equiv$$
$$\equiv 0 = x \vee \exists_F[F \in x^v \wedge (\mathrm{Rng}\, x)^{(e)} = x^{(e)}]\,,$$

where $x^{(\mathscr{E})}$ and $y^{(\mathscr{E})}$ can be replaced by $x^{(E)}$ and $y^{(E)}$ respectively [D29.3].

The usefulness of the notion $|x|^E$ appears from $(41.9)_1$ below.

$$(41.9)\qquad \vdash |x|^E = |y|^E \equiv x \approx^{(e)} y\,,$$
$$\vdash |x^{(E)}|^E = |x|^E = |x^{(\mathscr{E})}|^E \text{ [D14.3, D29.3, D39.6].}$$

This theorem is easy to prove—cf. D41.3 and D14.3—and yields $(41.9)_2$ by $(29.8)_1$ and $(39.9)_1$. By D39.6, D29,2, and D14.1 the first of the theorems

$$(41.10)\qquad \vdash x \in \mathrm{WSep} \supset x \approx x^{(\mathscr{E})}\,,\ \vdash x \in \mathrm{WSep} \supset |x| = |x|^E\,,\ \vdash |x|^E \leqslant |x|$$

holds. It yields the second by $(41.2)_1$ and D41.3.

To prove the third we deduce from (29.5), using rule C with y, (a) $y \in \mathrm{WSep}$, (b) $y \subseteq x$, and $y^{(e)} = x^{(e)}$; hence $y \approx^{(e)} x$, which by $(41.9)_1$ yields (c) $|x|^E = |y|^E$. From (a) and $(41.10)_1$ we have $y \approx y^{(\mathscr{E})}$, which by $(41.2)_1$ and D41.3 yields (d) $|y|^E = |y|$.

By $(41.6)_1$ (b) yields $|y| \leqslant |x|$. Thence, by (c) and (d), we have $|x|^E \leqslant |x|$ where the variable y does not occur. Now we easily conclude that $(41.10)_3$ holds.

42. Further hints at the theory of cardinals in MC^∞; universes in MC^∞.

Now it is a straightforward matter of routine to enunciate the analogues for MC^∞ of the theorems in [IST] concerning the pure number theory, and to prove them. This holds in particular in connection with cardinal addition, cardinal multiplication, cardinal exponentiation, and regular and singular cardinals—cf. [IST, Secs. 20-24].

As far as universes are concerned, the extension to MC^∞ of the treatment in [IST] is less straighforward by two reasons. Since in MC^∞ individuals are taken into account, this extension has to be made via n. 6 on EC^∞. Furthermore non-absolute classes also have to be taken into account, which requires less straightforward changes of a modal nature.

We now define *universe* in MC^∞ using the ordinary language as well as in the analogous definition in EC^∞ [n. 6].

DEF. 42.1. *We say that* $\mathscr{A}$ *is a universe* ($\mathscr{A} \in \text{Univ}$) *in case* (42.1) *below holds*—cf. fn. 5 in Part 2, n. 19, and Convs. 3.1 and 10.1:

$$(42.1)\quad \left|\begin{array}{l} \mathscr{A} \in \text{MConst}\,, \qquad \omega \in \mathscr{A}\,, \qquad \text{In} \in \mathscr{A}\,, \\ (N)\, x \in \mathscr{A} \supset \boldsymbol{S}^{\frown} x \in \mathscr{A}\,, \qquad (N)\, x \in \mathscr{A} \supset \boldsymbol{S}^{mc} x \in \mathscr{A}\,, \\ (N)\, x \in \mathscr{A} \supset x \subseteq \mathscr{A}\,, \qquad (N)\, x \subseteq^{\frown} \mathscr{A} \wedge \sim |x| =^{\smile} |\mathscr{A}| \supset x \in \mathscr{A}\,. \end{array}\right.$$

Remark that $(42.1)_{4,5}$ are two analogues of $(6.1)_5$.
From Def. 42.1 and D12.5 we deduce $(42.2)_1$ below.

$$(42.2)\quad \left\{\begin{array}{ll} \vdash \text{Univ} \in MConst\,, & \vdash \text{Univ} \subseteq \text{MConst}\,, \\ \vdash \text{Univ} \in Abs\,, & \vdash \text{Univ} \in MSep\,. \end{array}\right.$$

From (42.1) we easily see that $(42.2)_{1,2}$ hold, which by the syntactical analogue of $(12.2)_2$ yields $(42.2)_3$. By $(42.2)_3$ and D12.6 we have $(42.2)_4$.

THEOR. 42.1. *Sentences* (42.3-8) *below are syntactical consequences in* MC^∞ *of the assumption that* $\mathscr{A}$ *is a universe*:

$$(42.3)\quad (N)\, \mathscr{A} \in St\,, \qquad (N)\, x \in^{\smile} \mathscr{A} \supset x \subseteq^{\frown} \mathscr{A}\,, \qquad |\mathscr{A}| \in \text{Card},$$

$$(42.4)\quad (N)\, x \in \mathscr{A} \supset |x| < |\mathscr{A}|, \quad \omega < |\mathscr{A}|, \quad (N)\, x \subseteq^{\frown} y \wedge y \in \mathscr{A} \supset y \in^{\frown} \mathscr{A}\,,$$

$$(42.5)\quad (N)\, a, b \in^{\smile} \mathscr{A} \supset \{a, b\}^{(i)}, (a, b) \in^{\frown} \mathscr{A}, \quad (N)\, x, y \in^{\smile} \mathscr{A} \supset x \times y \in^{\frown} \mathscr{A}$$

$$(42.6)\quad (N)\, x \in^{\smile} \mathscr{A} \wedge f \in^{\frown} x \to \mathscr{A} \supset (f, \text{Rng}\, f \in \mathscr{A})\,, \qquad |\mathscr{A}|^E = |\mathscr{A}|\,,$$

$$(42.7)\quad (N)\ |\mathscr{A}| \text{ is strongh inaccessible—cf. [IST, p. 159]}\,, \qquad (N)\, x \in \mathscr{A} \supset \bigcup x \in \mathscr{A}\,,$$

$$(42.8)\quad (N)\, I \in \mathscr{A} \wedge x \in Fn \wedge \forall_i (i \in I \supset x_i \in \mathscr{A}) \supset \bigcup_{i \in I} x_i \in \mathscr{A} \wedge P_{i \in I}\, x_i \in \mathscr{A}.$$

where $P_{i \in I}$ *denotes direct product*—cf. [IST, p. 55].

Remark that $(42.6)_2$ has no extensional analogue, whereas the others among the consequences (42.3-8), except $(42.3)_3$, are the analogues for MC^∞ of the assertions $(6.1)_{1,3}$ and (6.2-6) in EC^∞—cf. [IST, Def. 23.12(i) and Theor. 23.13].

PROOF of Theor. 42.1. In connection with several theses we give only directions for [IST, p. 160] can be followed.

By Convention 3.1 and D12.5, $(42.1)_1$ yields $(42.3)_1$. We deduce $(42.3)_2$ from $(42.1)_{1,6}$, and $(42.3)_3$ from $(42.2)_1$ and $(41.3)_3$.

We can deduce (42.4) to $(42.6)_1$ following the proof of Theor. 23.13 in [IST], with the usual changes. E.g. to prove $(42.4)_1$ remark that, if $x \in \mathscr{A}$, then $S^\frown x \in \mathscr{A}$ by $(42.1)_4$. Hence $S^\frown x \subseteq^\frown \mathscr{A}$ by $(42.3)_2$. Then $|x| < |S^\frown x| \leqslant |\mathscr{A}|$ by $(41.6)_1$, by the theorem $\vdash \sim (u \approx S^\frown u)$—whose proof is quite similar with the one of the stronger assertion $(29.9)_1$—, by $\vdash u \approx u^{(I)}$—cf. $(27.5)_1$—, and by $\vdash u^{(I)} \subseteq S^\frown u$ [D14.4].

To show another example, let us prove $(42.5)_1$—cf. (iv) in [IST, Theor. 23.13). To this end we remark that $x, y \in^\smile \mathscr{A}$ yields $x, y \in^\frown \mathscr{A}$ by $(42.1)_1$; hence $\{x, y\}^{(i)} \subseteq^\frown \mathscr{A}$ and $N|\{x, y\}^{(i)}| \neq |\mathscr{A}|$, since $\omega < |\mathscr{A}|$ by $(42.4)_2$. Likewise $N|(x, y)| \neq |\mathscr{A}|$ and $(x, y) \subseteq^\frown \mathscr{A}$. Then $(42.5)_1$ holds by $(42.1)_7$.

To deduce $(42.6)_1$, first obtain $x = y$ and $y \in \mathrm{MConst}$ from A17.11(II), using rule C with y. Then practically repeat the deduction of (vii) in [IST, Theor. 23.13, p. 161].

Similar preliminary steps are useful to state (42.8). The deduction of $(42.7)_1$ is substantially the one of (viii) in [IST, Theor. 23.13].

To deduce (from $\mathscr{A} \in \mathrm{Univ}$) the essentially modal assertion $(42.6)_2$ remark that by (27.5) and $(29.6)_3$ we have $\mathscr{A} \approx \mathscr{A}^{(I)} \in \mathrm{WSep}$, which by $(41.2)_1$ and $(41.10)_2$ yields $|\mathscr{A}| = |\mathscr{A}^{(I)}| = |\mathscr{A}^{(I)}|^E$. Furthermore, by $(42.5)_1$ (for $a =^\frown b$) and $(42.1)_4$ we have $\mathscr{A}^{(I)} \subseteq^\frown \mathscr{A}$ which by $(41.6)_2$ yields $|\mathscr{A}^{(I)}|^E \leqslant |\mathscr{A}|^E$. Hence $|\mathscr{A}| \leqslant |\mathscr{A}|^E$, which by $(41.10)_3$ yields $(42.6)_2$.

Assertions (42.7,8) can be deduced practically as (viii)-(x) in [IST, Theor. 23.13] with the changes hinted at above, or the introduction of a suitable modally constant entity on the basis of A17.11. q.e.d.

[IST, Theor, 23.14] can be extended to MC^∞ via Theor. 6.1 into

THEOR. 42.2-$\mathscr{A}$ *is a universe iff* $\mathscr{A} =^\frown \boldsymbol{M}_\vartheta$ [D39.2] *for some strongly inacessible cardinal* ϑ *larger than* $|In|$.

The proof of this theorem is an easy matter of routine on the basis of the proof of [IST, Theor. 23.14] and the hints at the proof of Theor. 6.1. From this proof we see—cf. the italicized remark below Theor. 6.1—that

$$(42.9) \qquad \vdash \mathscr{A} \in \mathrm{Univ} \supset \mathscr{A} =^\frown \boldsymbol{M}_\vartheta \quad \textit{where} \quad \vartheta =_D |\mathscr{A}| .$$

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