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The Interpreted Type-Free Modal Calculus MC^{∞} II.

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Part 2
Foundations of MC^{∞}

Chapter 3
AXIOMS OF MC^{∞} AND BASIC CONSEQUENCES

15. Introduction to Part 2.

We construct the modal analogue MC^{∞} of the calculus EC^{∞} [Chap. 1] which is the extension to the case where individuals exist, of the extensional calculus on which [IST]—i.e. [3]—is based. In Chapter 3 we lay down the axioms for MC^{∞} and state their main consequences. We write explicitly the proofs of those among these consequences that are essentially modal. In chapter 4 we study relations and functions syntactically; of course we take care especially of their modal properties; in particular these properties are important in connection with the various modal kinds of functions introduced in Part 1, n. 13. This

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has counterparts on the notions of equipotence and class exponentiation [nn. 27, 29].

Now let us describe the content of Part 2, divided into chapters 3 and 4, in more detail. As far as Chapter 2 is concerned, in nn. 16, 17 we state a set of logical axioms valid in ML^{∞}. For the calculus MC^{∞} that thus arises, the well known metatheorems for first order calculi hold [n. 16].

The axioms of MC^{∞}, especially those for individuals, sets, and classes, are as close as possible to those of EC^{∞} [nn. 2, 4], hence to the axioms of the extensional calculus of classes, without individuals, considered in [IST], and in part to Suppes' extensional type-free logic with individuals [7]. Nearly all axioms in [IST] for classes and sets want some more or less relevant modal changes (indipendent of the fact that MC^{∞} unlike [IST] deals with individuals). For instance our class building axiom A17.3 is practically the one in [IST], while our regularity axiom A17.7 has some modal features that make it stronger than the direct analogue for ML^{∞} of the regularity axiom in [IST]. This difference has counterparts on some consequences of A17.7 such as theorems (19.1). Axiom A17.11 on classes and sets has no extensional analogue; it is the analogue for ML^{∞} of AS12.19 in [GIMC], i.e. [1].

In n. 18 we consider some basic theorems on Λ, In, El, and class existence, a part of which has no extensional analogue. For instance we prove that if $\mathscr{A}$ can be an element, then $\mathscr{A}$ must be an element. Lambda expressions for properties [relations and functions] are dealt with in connection with the usual substitution properties in n. 18 [in n. 23]. In n. 19 we show how the elementary algebra of classes for EC^{∞}, hence the one in [IST], can be carried over to ML^{∞} by performing very slight and standard changes on [IST, Sect. 2]. Only a few essentially modal thorems must be proved anew.

Among the main consequences of the essentially modal axiom A17.11 and either A17.1 on the non-existing object a^*, or the version A17.5$'$ of the pairing axiom, there is the possibility of introducing an analogue $\mathscr{E}$l, of the notion of elementary possible cases (Γ) within MC^{∞} itself. We can deal with $\mathscr{E}$l [n. 20] with substantially the same words as in [GIMC, nn. 47-49], so that hints suffice. One of the main consequences of our theorems on $\mathscr{E}$l is the assertion that for any set X, its modal sum $X^{\cup}$ also is a set [Theor. 21.1].

In n. 22 we briefly show that the intensional description operators ι_u and $\imath_u$, and the real case ϱ (and ι_ϱ and $\imath_u$) can be dealt with in MC^{∞} with practically the same words as in [GIMC].

In Chap. 4 relations and functions are studied. We have two main kinds of functions; *intensional functions*, briefly *functions*, and *extensionally univalent* or *extensionally invariant functions*. The former are useful to carry over to MC^{∞} the set theory in [IST], with very few and rather standard changes. The latter constitute a special case of the former, and are more similar to ordinary functions (such as the i-th co-ordinate $x_i(\tau)$ of a moving particle in a given inertial reference frame at the instant τ). In case the domain and range of an intensional function, f, are absolute [D12.6] which certainly occurs in pure number theory), then f also is extensionally invariant. The aforementioned pluralism of function kinds has counterparts of a modal nature in connections with mappings [n. 25], class exponentiation [n. 27], and equipotence [nn. 27, 29], equivalence relations [n. 30], and ordering [n. 31]. This pluralism also gives rise to two versions of the replacement axioms: a weaker one, A17.9, and a stronger version, theorem $(24.6)_2$.

In order to relate the two kinds of entities being considered with one another the (n-ary) intensionalization $A^{(nI)}$ [D14.4, D24.1] and the intrinsic extension class $A^{(E)}$ [D29.3] of any class A are studied [nn. 27, 29] as well as the notion of weakly separated properties ($WSep$) [D29.2], the strict subset class $S^{\cap}A$ [D20.5], and the modally constant subset class $S^{mc}A$ [D29.1] of any class A—cf. n. 29.

Another source of essentially modal theorems is connected with the fact that e.g. $\{a\}$ is not an element in ML^{∞}. Hence it is natural to search for suitable explicata of the intrinsic (n-ary) extensionalization of any attribute [n. 26]. Rank-preserving extensionalization will be considered in Part 3 after ranks are defined within MC^{∞}.

In connection with (intensional) functions, in Chap. 4 we carry over to MC^{∞} sections 3-8 in [IST] very quickly, as in part has already been said. We are referring to algebra of relations [n. 23] and functions [n. 25], infinite Boolean operations and direct products [n. 28], power classes [n. 29], equivalence relations [n. 30], and ordering [n. 31].

16. Asions for the lower predicate calculus, identity, and descriptions in MC^{∞}.

Modus ponens is the only inference rule in MC^{∞}. We take the analogues for ML^{∞} of AS12.1-3 in [GIMC]—i.e. AS1-3 in [6, p. 212]—as axioms of the extensional propositional calculus. As axioms for the lower predicate calculus $LPC(ML^\infty)$ for ML^∞ we take these analogues and axioms A16.1-4 below (1).

A16.1 $$(N)(\forall V_i)\Phi(V_i) \supset \Phi(\Delta)$$

—cf. Conv. 3.5.

A16.2 $$(N)(\forall V_i)(p \supset q) \supset [(\forall V_i)p \supset (\forall V_i)q]\,.$$

A16.3 $$(N)(\forall V_i)(p \supset q) \supset [p \supset (\forall V_i)q]$$

where V_i does occur free in p.

A16.4 $$(N)Np \supset p\,.$$

A16.5 $$(N)N(p \supset q) \supset (Np \supset Nq)\,,$$

A16.6 $$(N)N(p \supset q) \supset (p \supset Nq)$$

where p is modally closed,—cf. [GIMC, Def. 4.3]—i.e. *is constructed starting out of matrices such as $Np_1, \ldots, Np_n$ by means of $\wedge, \sim, N$, and $(\forall V_i)$.*

It is evident that AA16.1-6 are (logically) valid and that modus ponens preserves validity.

The obvious analogues for ML^∞ of the theorems and metatheorems considered in [GIMC, nn. 31-33] for the lower predicate calculus LPC(MC^ν) of the modal ν-sorted calculus MC^ν hold and can be stated by substantially the same procedures. Among others let us mention the deduction, duality, equivalence, and replacement theorems; let us add the generalization theorem and the theorem legitimating the formal analogue of an act of choice (rule C)—cf. [GIMC, Theor. 33.1]—i.e. the theorems on ordinary or modal rules G and C:

THEOR. 16.1. If $p_1, \ldots, p_n \vdash_{\overline{C}} q$, i.e. *q can be deduced from $p_1, \ldots, p_n$ (possibly) using rules G and C according to Defs.* 33.1,2 *in* [GIMC], *and no variable introduced by any application of rule C occurs free in q, then* $p_1, \ldots, p_n \vdash q$.

(1) A16.1-6 are axiom schemes correponding to AS12.4-9 in [GIMC]. As well as other axiom schemes, they will be called here axioms for the sake of simplicity.

Axioms A16.7-10 below on identity and A16.11 on descriptions substantially are As12. 10-13, 18 in [GIMC].

A16.7-9 $(N)x =^{\cap} x\,, \quad (N)x = y \supset y = x\,, \quad (N)x = y = z \supset x = z\,.$

A16.10 $(N)x =^{\cap} y \supset [\Phi(x) \equiv \Phi(y)]$

—cf. Conv. 3.5.

A16.11 $(a)\, p(\exists_1 x)p \supset (\imath x)p = x\,, \quad (b) \sim (\exists, x)p \supset (\imath x)p = a^*\,.$

Let us mention the following essentially modal axiom—cf. [GIMC, AS12.23]:

A16.12 $\exists_{\mathscr{A},\mathscr{B}}(\mathscr{A} =^{\cup} \mathscr{B} \wedge \diamond \mathscr{A} \neq \mathscr{B})\,.$

The validity of A16.7-11 in ML^∞ can be checked on the basis of rules (δ_1), (δ_2), and (δ_4) to (δ_7) [n. 5], substantially as the validity in ML^ν of the corresponding axioms for MC^ν in [GIMC, n. 12] (2). Furthermore the obvious analogues for ML^∞ of the theorems on $=$ and $\imath$ that are stated in [GIMC nn. 34-39] and do not specifically concern attributes, also hold in MC^∞, up to the obvious very slight changes due to the difference between ML^ν and ML^∞ concerning types, and can be proved in substantially the same way.

17. Axioms for individuals, sets, and classes.

We list a main set of axioms, AA17.1-11, for individuals, sets, and classes and we also consider some alternative sets of axioms. The validity in ML^∞ of all these axioms in the semantical system $\Sigma_\vartheta(ML^\infty)$—cf. (8.9)—is long but easy to prove on the basis of rules (δ_1) to (δ_7) [n. 9]—cf. the proof of (10.5).

We do not use all aforementioned axioms immediately. E.g. in n. 18 we derive some properties for Λ, In, and El, and some replacement properties for lambda expressions on the basis of only AA17.1-6 and a part of A17.8 below. After this in n. 19 we use the other axioms to extend elementary algebra of classes to MC^∞. Incidentally we in-

(2) Use Theor. 9.4 and rule (δ_7) in N9 in connection with A16.11—cf. Theors. 11.1-3 in [GIMC].

clude lambda expressions only in the parts of AA17.1-11 that are not used in n. 18.

Let us note that A17.1 is the modal analogue for MC^∞ of A2.1 for EC^∞, that AA17.2-11 have the same connection with axioms A4.1-7, 8′, 9, 10 for EC^∞ and that axioms AA17.2,3 [AA17.10,11] are the type-free analogues of axioms AS12.14-17 [AS12.20,19] in [GIMC] for the calculus ML^ν.

Let us remember that Λ is a primitive constant and the definitions DD2.6-8 of class, individual, and set are understood.

A17.1 (*non-existing object*) $a^* \in In$.

A17.2 (*Intensionality*) $(N)\forall_{\mathscr{A}}(\mathscr{A} \in F \equiv \mathscr{A} \in G) \equiv F = G$.

A17.3 (*Class-building*) $(N)\exists_B \forall_{\mathscr{A}}(\mathscr{A} \in B \equiv \mathscr{A} \in El \wedge p]$ *where the class variable B has no free occurrences in the matrix p.*

A17.4 (*Power set*) $(N)\exists_v \forall_{\mathscr{A}}(\mathscr{A} \subseteq^{\cap} u \supset \mathscr{A} \in v)$.

A17.5 (*Pairing*) $(N)\exists_u(a \in u \wedge b \in u)$.

A17.6 (*Union set*) $(N)\exists_v \forall_X(X \in u \supset X \subseteq v)$.

A17.7 (*Regularity*)

$$(N)X \neq \Lambda \supset \exists_a \sim \exists_b(a \in X \wedge b \in^{\cup} a \wedge b \in X) .$$

A17.8 (*Infinity*)

$$\exists_u[\forall_{\mathscr{B}}(N\forall_{\mathscr{A}} \mathscr{A} \notin \mathscr{B} \supset \mathscr{B} \in u) \wedge \forall_x(x \in u \supset x \cup \{x\}^{(i)} \in u)] .$$

A17.9 (*Replacement*)

$$(N)F \in Fnc \wedge \forall_b \exists_a(b \in \mathscr{V} \supset^{\cap} a \in u \wedge (a, b) \in F) \supset \mathscr{V} \in El .$$

A17.10 (*Intensional relational axiom of choice*)

$$(N)\ \exists_{\mathscr{F}}(\mathscr{F} \in Fn \wedge \mathscr{F} \subseteq \mathscr{R} \wedge \mathrm{Dmn}\,\mathscr{F} = \mathrm{Dmn}\,\mathscr{R}] \text{ [DD11.11,12, D13,3].}$$

A17.11 (*Modally constant class existence*)—cf. AS12.19 in [GIMC]

(I) $(N)\exists_G(G^{\cap} = G^{\cup} = F)$,

(II) $(N)\exists_y(y^{\cap} = y^{\cup} = x)$;

or

(I′) $(N)\exists_F[F^{\cap} = F^{\cup} \wedge \forall_{\mathscr{A}}(\mathscr{A} \in F \equiv \mathscr{A} \in El \wedge p)]$,

(II′) $(N)x = Y^{\cap} = Y^{\cup} \supset Y \in El$.

Someone might prefer to replace A17.1 with

A17.1′ $a^* = {}^{\frown}\Lambda$ or A17.1″ $a^* \in El$.

The first helps considering Λ as a primitive term—cf. D2.5. However it is confusing in my opinion—cf. fn. 4 in Part 1. Either of the latter axioms is compatible, unlike A17.1, with $In = \Lambda$. Therefore in order to be able to prove some basic theorems such as (21.8) we propose, in case A17.1 is replaced in the aforementioned way to turn A17.5 into

A17.5′ (*strong pairing*) $(N)\exists_u \forall_{\mathscr{C}}[N(\mathscr{C} = a \vee \mathscr{C} = b) \supset \mathscr{C} \in u]$.

Obviously A17.5′ yields the weaker axiom A17.5. In n. 20 —cf. $(20.3)_1$— we show that both follow from AA17.1,7-9; and neither seems to follow from AA17.8,9 and either of AA17.1′, 1″.

Now let us remark that $\mathscr{A} \subseteq {}^{\frown}u$ cannot be replaced in A17.4 with $\mathscr{A} \subseteq u$. Furthermore the infinity axiom A17.8 is a direct modal analogue of A4.8′, which is equivalent to the usual infinity axiom in case no individual exists.

Let us add that we cannot substitute *Fnc* for *Fn* [DD13.3,4] in A17.10. Indeed the resulting assertion would be false e.g. at the value-assignments for which $\mathscr{R} = {}^{\frown}\{(0,1), (a,2)\}^{(i)} \wedge a = {}^{\smile}0 \wedge \Diamond a \neq 0$ holds.

Axiom A17.11 (I′) is equivalent to AA17.3,11 (I). Moreover by A17.11 (I) the alternatives (II) and (II′) in A17.11 are equivalent; but if (I) is disregarded, then (II) yields (II′) by A16.10 (3), while the converse is false.

The admissible axioms A17.11 (I′), (II′) can be regarded as theorems in MC^{∞} to be proved after having stated the usual substitution properties for lambda expressions [n. 18].

A17.11 (I′) is a direct analogue for MC^{∞} of AS12.19 in [GIMC] in that it is substantially turned into AS12.19 by replacing the type-free variable F with a variable having an arbitrary attribute type.

Now remark that in ML^{∞} classes are the only primitive attributes

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(3) The substitution properties of λ in n. 18 easily yield (*a*) $\vdash (\exists^{(1)\frown} Y)(x = Y^{\frown} = Y^{\smile})$ [D10.8]. Now assume A17.11(II) and start with (*b*) $x = Y^{\frown} = Y^{\smile}$. By A17.11(II) and rule C with y we get (*c*) $x = y^{\smile} = y^{\smile}$. By (*a*) this and (*b*) yield $y = {}^{\frown}Y$. This, the theorem $\vdash y \in El$, and A16.10 yield $Y \in El$. We conclude that A17.11(II′) holds..

and that there are substantially two (cumulative) types of classes: classes and sets. Furthermore we prefer A17.11 (I) to A17,11 (I′). Then it is quite natural to use, as an analogue of AS12.19, the parts (I) and (II) of A17.11, which substantially are the versions of a same assertion for classes and sets respectively.

18. Consequences of AA17.1-8 concerning classes existence, lambda expressions for properties, Λ, In and El.

The analogue for MC^∞ of [GIMC, Th. 40.1] holds and can be proved in the same way, up to the usual very slight changes. We write explicitly the main parts of this analogue: (18.1,2) below; furthermore for the ease of the reader we explicitly prove the first of these parts as an example of the aforementioned changes between corresponding theorems in MC^ν and MC^∞.

By AA17.2,3 and D10.9, *in case the variable B does not occur free in p*

$$(18.1) \qquad \vdash (\exists_1 B)\Phi(B) \qquad \textit{where } \Phi(B) \equiv_D \forall_{\mathscr{A}}(\mathscr{A} \in B \equiv \mathscr{A} \in El \wedge p)\,.$$

Then by A16.11 (a) we get (a) $\vdash \forall_B[\Phi(B) \supset B = (\imath B)\Phi(B)]$ (4). Furthermore by A17.3 and rule C we deduce $\Phi(B)$. Then $B = (\imath B)\Phi(B)$ holds, which yields (b) $\forall_{\mathscr{A}}[\mathscr{A} \in El \wedge p \equiv \mathscr{A} \in (\imath B)\Phi(B)]$, where B does not occur free. Then, first, we can use the modal rule G—cf. [GIMC, Def. 33.1]—by which we obtain (c) $\vdash_{\overline{C}} N(b)$ and, second, by Theor, 16.1 we can turn (c) into $\vdash \forall_{\mathscr{A}}[\mathscr{A} \in El \wedge p \equiv^{\cap} \mathscr{A} \in (\imath B)\Phi(B)]$. Now it is easy to deduce the following strengthened versions of A17.3, where X *does not occur free in* p:

$$(18.2) \qquad \vdash \exists_X \forall_{\mathscr{A}} N[\mathscr{A} \in X \equiv \mathscr{A} \in El \wedge p]\,, \qquad \vdash \exists_X \forall_a (a \in X \equiv^{\cap} p)\,.$$

The basic substitution properties for lambda expressions

$$(18.3) \qquad \vdash \mathscr{A} \in (\lambda \mathscr{A})p \equiv \mathscr{A} \in El \wedge p \qquad \text{or} \qquad \vdash \forall_a[a \in (\lambda_a)p \equiv p]$$

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(4) If in a proof we write « (a) p » and then « (a) $\vdash q$ », we mean that $p \vdash q$. In case (a) had not been used to denote any matric, by « (a) $\vdash q$ » we mean that $\vdash q$ and that we denote this fact by « (a) ».

are substantially proved as (45) in [GIMC, n. 40]. Incidentally

(18.4) $\vdash \forall_{\mathscr{A}} \sim p \supset (\lambda \mathscr{A})p = \Lambda$[D3,1, D2.4], $\vdash (\lambda \mathscr{A})\Phi(\mathscr{A}) = (\lambda a)\Phi(a)$,
where $\Phi(\mathscr{A})$ is $\binom{a}{\mathscr{A}}\Phi(a)$ [Convs. 3.4,5].

Of course, if p has the form $\mathscr{A} \in \mathscr{B} \wedge q$ we can drop $\mathscr{A} \in El$ from $(18.3)_1$. E.g. in connection with DD3.2,3,5 and DD10.11,13,14 we have

$$(18.5)\quad \begin{aligned} &\vdash \mathscr{A} \in (\bar{\Delta})^{\cup} \cap \Delta_1^{(e)} \equiv \Diamond \mathscr{A} \notin \Delta \exists_{\mathscr{B}}(\mathscr{A} = \mathscr{B} \in \Delta_1)\,, \\ &\vdash \mathscr{A} \in \Delta \cup \Delta_1^{\cap} \equiv \mathscr{A} \in \Delta \vee \mathscr{A} \in^{\cap} \Delta_1\,. \end{aligned}$$

Theorem (3.2) can be proved in MC^{∞} in substantially the same way as in EC^{∞}. From AA17.4,8 we deduce, as analogues of A4.1 and $(4.5)_2$—cf. DD2.1,7,8 and Convs. 3.2 and 10.2

$$(18.6)\qquad \vdash In^{\cap} \subseteq El\,, \qquad \vdash In^{\cap} \subseteq St\,.$$

Incidentally $In \subseteq El$ is false. By AA17.5,11 (II) and Convention 10.1, and by A17.11 (I) respectively we obtain

$$(18.7)\qquad \vdash (\exists u) a \in^{\cap} u\,, \qquad \vdash \mathscr{A} \in F \supset \exists_G \mathscr{A} \in^{\cap} G\,.$$

The latter result yields the important theorem—cf. Convs. 10.2 and 12.1

$$(18.8)\qquad \vdash El \in MConst, \text{ i.e. } \quad \vdash \Diamond \mathscr{A} \in El \supset N \mathscr{A} \in El\,.$$

On its basis we can simplify formulas (10.2) on restricted variables. Formulas (10.2,3) on class and set variables cannot be simplified for the notion of class Cl is extensional.

The theorems

$$(18.9)\qquad \vdash (N)\mathscr{V}_1 \notin \Lambda\,, \qquad \vdash (N)\forall_{\mathscr{A}} \mathscr{A} \notin X \equiv X = \Lambda$$

can be proved (in MC^{∞}) in a way that is similar to and easier than the corresponding proofs in EC^{∞} [n. 2] and is more similar with Suppes' [7, Sect. 2.2] (because, like [7] and unlike EC^{∞}, MC^{∞} contains Λ as a primitive notion). Indeed by (10.1), A17.3, and rule C with X we get

$(a) X \in^{\cap} Cl$ and $(b) \forall_{\mathscr{A}}(\mathscr{A} \in X \equiv \mathscr{A} \in El \wedge N\mathscr{A} \neq \mathscr{A})$, hence $N\forall_{\mathscr{A}} \mathscr{A} \notin X$. This, (a), and D2.6 yield $X =^{\cap} \Lambda$. Then by A16.10 we have $\forall_{\mathscr{A}} \mathscr{A} \notin \Lambda$. So $(18.9)_1$ holds. Thence we easily deduce $(18.9)_2$ by D2.6 and A17.2.

By DD11.1,2 and (18.3)

$$(18.10) \quad \vdash b \in \{a_1, \ldots, a_n\} \supset \bigvee_{i=1}^{n} b = a_i , \qquad \vdash b \in \{a_1, \ldots, a_n\}^{(i)} \supset \bigvee_{i=1}^{n} b =^{\cap} a_i .$$

Lastly remark that by DD2.7,8 and Convs. 3.1,2

$$(18.11) \qquad \vdash \mathrm{El} = El , \qquad \vdash \mathrm{St} = St .$$

19. Extension to MC^{∞} of elementary algebra of classes and relations.

We briefly show that most of the elementary set theory presented in [IST, Chap. 1] can be carried over to MC^{∞} and that generally this can be done by slight and, so to say, standard changes. The most important of these changes concern restricted variables and the use of $\{\ldots\}^{(i)}$ and $=^{\cap}$ instead of $\{\ldots\}$ and $=$ respectively [DD11.1,2] in several theorems. Some of the analogues for MC^{∞} of the theorems proved in [IST, Sect. 1] are explicitly written below. After each of them the number of the corresponding item or items in [IST] is placed ([5]). Very few of these analogues are explicitly proved as examples.

Among them are $(19.1)_{1,2}$ below in that e.g. $(19.1)_1$ is similar to the consequence (1.19) in [IST] of the regularity axiom: however the versions of this axiom in [IST] and MC^{∞} [A17.7] are not very similar and the same holds for the aforementioned theorems and for their proofs. Theorem $(19.1)_6$ is essentially modal.

$$(19.1) \begin{cases} \vdash \mathscr{A}_1 \in^{\cup} \mathscr{A}_2 \in^{\cup} \ldots \in^{\cup} \mathscr{A}_n \supset \sim \mathscr{A}_n \in^{\cup} \mathscr{A}_1 \qquad (\text{for } n \leqslant 4 \text{ cf. } (1.19),) \\ \vdash \mathscr{A}_1 \in^{\cup} \ldots \in^{\cup} \mathscr{A}_{n-1} \in \mathscr{A}_n \supset \mathscr{A}_n \neq \mathscr{A}_1 , \qquad \vdash \sim \mathscr{A} \in \mathscr{A} , \\ \vdash a \in \mathscr{A} \supset a \neq \mathscr{A} , \qquad \vdash 0 \neq 1 , \qquad \vdash \exists_{aA}(a \in A \wedge a =^{\cup} A) . \end{cases}$$

([5]) In [IST] there is a single numbering for axioms, definitions, theorems, and corollaries. When we write the analogue for MC^{∞} of one among the items above, we place its number in [IST] at the end of it. E.g. by « (1.20,22) » at the end of (19.2) we mean that the theorems (19.2) are the analogues for MC^{∞} of theorems (1.20) and (1.22) in [IST].

The matrix $\mathscr{A}_{n-1} \in \mathscr{A}_n$ cannot be replaced by $\mathscr{A}_{n-1} \in^{\cup} \mathscr{A}_n$ in $(19.1)_2$.

To prove $(19.1)_1$ we set $(a)\ X =_D \{\mathscr{A}_1, \ldots, \mathscr{A}_n\}^{(i)}$ and assume $(b)\ \mathscr{A}_n =^{\cap} \mathscr{A}_0 \in^{\cup} \mathscr{A}_1 \in^{\cup} \ldots \in^{\cup} \mathscr{A}_n$ as an hypothesis for reductio ad absurdum. Then by A17.7 and rule C with a we have $(c)\ a \in X$ and $(d)\ a^{\cup} \cap X = \Lambda$. By (a) and (c) we have $a =^{\cap} \mathscr{A}_i$ for some $i \in \{1, \ldots, n\}$. Furthermore (a), (b), and (18.8) yield $(e)\ \mathscr{A}_{i-1} \in X$ and $\mathscr{A}_{i-1} \in \mathscr{A}_i^{\cup}$ $(i = 1, \ldots, n)$. Then $\mathscr{A}_{i-1} \in a^{\cup} \cap X$ for some $i \in \{1, \ldots, n\}$, which contradicts (d). Thus $(19.1)_1$ has been proved.

To prove $(19.1)_2$ assume $\mathscr{A}_m = \mathscr{A}_1 \in^{\cup} \ldots \in^{\cup} \mathscr{A}_{m-1} \in \mathscr{A}_m$ instead of (b). Then $\mathscr{A}_{m-1} \in \mathscr{A}_1$, hence $\mathscr{A}_{m-1} \in^{\cup} \mathscr{A}_1 \in^{\cup} \ldots \in^{\cup} \mathscr{A}_{m-1}$ in contrast to $(19.1)_1$ for $n = m - 1$). We conclude that $(19.1)_2$ holds. For $n = 1$ it becomes $(19.1)_4$, which in turn yields $(19.1)_5$ as a particular case by $(13.6)_{1,2}$.

$\mathscr{A} \in \mathscr{A}$ yields $\mathscr{A} \in^{\cup} \mathscr{A} \wedge \mathscr{A} = \mathscr{A}$ in contrast to $(19.1)_2$ for $n = 1$. Hence $(19.1)_3$ holds.

We have $\vdash p \wedge \Diamond \sim p$ for some p by A16.12. Furthermore by A16.7 and rule C with A we can write $A =^{\cap} (\imath b)(b = 1 \wedge p \vee b = = 0 \wedge \sim p)$. Hence $A = 1 \wedge A =^{\cup} 0$, so that $0 \in A \wedge 0 =^{\cup} A$ by $(13.6)_{1,2}$: We conclude that $(19.1)_6$ also holds.

The following theorems are more similar with their correspondents in [IST] and the same holds for their proofs.

$$(19.2)\quad \vdash x \in \mathfrak{S}u \equiv x = u \vee x \in^{\cap} u\,, \qquad \vdash A \notin \overline{El} \supset \mathfrak{S}A = A \qquad (1, 20, 22)\,,$$

where the metalinguistic definition D4.1 is understood. From D2.1,7 and D4.1 we easily deduce

$$(19.3)\quad \vdash a \in In \cup \{0\} \supset \mathfrak{S}a = \{a\}^{(i)}\,, \qquad \vdash \mathscr{A} \in In \cap El \supset \mathfrak{S}\mathscr{A} = 0\,.$$

Since $\Vdash In \notin El$, « a » cannot be replaced by « $\mathscr{A}$ » in $(19.3)_1$. Obviously

$$(19.4)\quad \vdash \{a, b\}^{(i)} \in St \quad (1.25)\,, \qquad \vdash \{a_1, \ldots, a_n\}^{(i)} \in St \qquad (2.21)\,.$$

The formula $(\ )\{a, b\} \in St$ is not valid. To prove $(19.4)_1$ we first deduce $N(a, b \in u)$ from AA17.5,11 (II), by rule C with u. Then by $(18.10)_2$ we get $\{a, b\}^{(i)} \subseteq^{\cap} u$, which by A17.4 and rule C with v yields $\{a, b\}^{(i)} \in v$. Then by DD2.1,8 and Theor. 16.1 we easily conclude that $(19.4)_1$ holds. From $(18.10)_2$ we deduce

$$(19.5)\quad \vdash \{a, b\}^{(i)} = \{c, d\}^{(i)} \equiv a =^{\cap} c \wedge b =^{\cap} d \vee a =^{\cap} d \wedge b =^{\cap} c \qquad (1.30)$$

and the analogue for the (proper) classes $\{a, b\}$ and $\{c, d\}$ also holds and can proved in substantially the same way. Let us now note the following analogues for MC^∞ of some theorems in [IST] to be proved in substantially the same way as the latter:

$$(19.6)\quad \vdash (a, b) \in St \;\; (1.32)\,, \qquad \vdash (a_1, \ldots, a_n) \in St\,,$$

$$(19.7)\quad \vdash (a_1, \ldots, a_n) = (b_1, \ldots, b_n) \supset \bigwedge_{i=1}^{n} a_i =^{\cap} b_i \;\; (1.33)\,,$$

$$(19.8)\quad \vdash A \in St \;\; (2.1)\,, \qquad \vdash A \subseteq^{\cap} u \supset A \in St \;\; (2.2)\,, \qquad \vdash \bigcup a \in St \;\; (5.3)\,,$$

$$(19.9)\quad \vdash u \cap A \in St \;\; (2.7)\,, \qquad \text{hence } \vdash u^{\cap} \in St\,,$$

Let us only add that the proof of $(19.8)_2$ is independent of theorem $(19.8)_1$; this theorem follows from $(19.8)_2$ and A17.8 and is a mate of $(18.6)_2$ in the present theory where individuals are dealt with. Theorems $(19.8)_{2,3}$ are basic as well as

$$(19.10)\quad \vdash A \subseteq u \supset \exists_v (v = A \wedge v \in MConst)\,, \qquad \text{i.e. } \vdash Su \subseteq \mathrm{MConst}^{(e)}$$

$$(19.11)\quad \vdash A \subseteq^{\cup} u \wedge A \in MConst \supset A \in^{\cap} St, \qquad \text{i.e. } \vdash (Su)^{\cap} \cap MConst \subseteq St^{\cap}\,;$$
$$\vdash El^{(e)} \cap MConst \subseteq St^{(e)}$$

that are essentially modal. To prove $(19.10)_1$ assume (a) $A \subseteq u$, whence (b) $A = A \cap u$. By $(19.9)_1$, A17.11 (II), and rule C with v we obtain (c) $v = A \cap u \wedge v \in MConst$. Hence (b) yields (d) $\exists_v (A = v \in MConst)$. Thus $(a) \vdash (d)$. Hence $(19.10)_1$ holds.

To prove $(19.11)_1$ start with (e) $A \in MConst$ and $\diamond\,(a)$, whence $\diamond\,(c)$. By D12.5 this and (e) yield $A =^{\cap} v$, which by A16.10 and the analogue for ML^∞ of Convention 3.3—cf. $(10.1)_1$—yields $A \in^{\cap} St$. So $(19.11)_1$ holds.

Now, to prove $(19.11)_3$, we assume (f) $A \in El^{(e)}$ and (g) $A \in MConst$. Then (h) $A = u$ for some u, so that, by $(19.10)_1$, $A = v \in MConst$ for some v. Then, by (g) and (h), $v =^{\cap} A$. Hence $A \in^{\cap} St$. We conclude that $(19.11)_3$ holds.

Theorems (19.12-18) below concern class operations. Our aim is to show that all such theorems in [IST] hold for MC^∞ provided we use class variables, but that some of them, such as $(19.12)_2$, $(19.16)_2$, or $(19.17)_3$ also hold for general variables $\mathscr{A}$, $\mathscr{B}$, ... Let us add that theorem $(19.17)_1$ below will be strengthened into $(24.7)_{1,2}$.

(19.12) $\vdash A \subseteq B \wedge B \subseteq C \supset A \subseteq C$, $\vdash \mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C} \supset \mathscr{A} \subseteq \mathscr{C}$ (2.3,v),

(19.13) $\vdash 0 \subset A \equiv A \neq 0$ ($0 \subset \mathscr{A} \equiv \mathscr{A} \neq 0$ is false) (2.5,i),

(19.14) $\vdash (N) \exists_u x \notin u$, $\vdash (N) \exists_u x \subset u$ (2.6)

(19.15) $\vdash (N) u \neq v \supset [u \cap v = A \equiv \forall_{\mathscr{A}\mathscr{B}} (\mathscr{A}, \mathscr{B} \in \left\{ \begin{array}{l} \{u, v\} \\ \{u\ v,\}^{(i)} \end{array} \right. \wedge \mathscr{A} \neq$ $\neq \mathscr{B} \supset \mathscr{A} \cap \mathscr{B} = A)$ (2.10,iv),

(19.16) $\vdash u \cap v \in St$, $\vdash a \cup b \in St$ (2.12), $\vdash \bigcup v \in St$,

(19.17) $\vdash V \notin El$ (2.16), $\vdash u - A \in St$ (2.18) $(\vdash (N)\ \bar{u} \in St)$

(19.18) $\vdash \bar{V} = 0$, $\bar{0} = V$, $\bar{\bar{A}} = A$ ($\bar{\bar{\mathscr{A}}} = \mathscr{A}$ is not valid),

(19.19) $\vdash Su \subseteq St^{(e)}$.

20. On $\{a, b\}^{\cap}$. The analogue $\mathscr{E}$l of the class Γ of elementary possible cases, defined within MC^{∞} itself. Consequences on N and $\diamond$.

After stating some preliminaries on $\{a, b\}^{\cap}$, of interest in themselves, we show that the treatment of the analogue $\mathscr{E}$l of Γ and the one of the real case ϱ presented in [GIMC, nn. 47-49] can be carried over to MC^{∞} in a straightforward way. From the results on $\mathscr{E}$l in this section some basic properties of $\mathscr{E}$l and El (and $F^{\cup}$) will be derived in n. 21.

By A17.1 and DD2.4,7 we obtain the first of the theorems

(20.1) $\vdash N0 \neq a^*$, $\vdash \{0, a^*\}^{\cap} \in St$.

Now let us derive $\exists_u \{0, a^*\}^{\cap} \in St$ from AA17.1,8, $(19.8)_2$, and D2,7. Furthermore $\vdash \{0, a^*\}^{\cap} \in MConst$. Hence $(19.11)_1$ yields $(20.1)_2$. By DD13.1-4

(20.2) $\vdash F \in Fnc$ where $F =_D (\lambda x, y) N(x = 0 \wedge y = a \vee x = a^* \wedge y = b)$.

Furthermore by $(20.2)_2$ and DD11.11,12 $\vdash \text{Dmn}\, F = \{0, a^*\}^{\cap}$ and $\vdash \text{Rng}\, F = \{a, b\}$. Hence by A17.9 *we have* A17.5′:

(20.3) $\vdash \{a, b\}^{\cap} \in St$, in particular $\vdash \{0, 1\}^{\cap} \in St$.

Let us observe that our deduction of (20.3) is based on A17.1 via (20.1) and is independent of A17.5 which obviously follows from A17.5′.

Now we consider the notions of *proper range* (PR), *subrange* (Sub$R$), *elementary range* ($\mathscr{E}lR$), and (*absolute*) *elementary range* ($\mathscr{E}l$)—cf. [GIMC, Defs. 47.1.2,3 and Def. 48.1]:

DD20.1,2 $PR =_D \{0, 1\}^\cap \cap \{0\}^\cup \cap \{1\}^\cup$, $\quad \text{Sub}R =_D \{(a, b) | a, b \in$
$\in PR \wedge \alpha \subseteq^\cap b\}$,

D20.3 $\mathscr{E}lR =_D (\lambda a)\{\alpha \in PR \forall_b [(b, a) \in \text{Sub}R \supset a =^\cap b]\}$,

D20.4 $\mathscr{E}l =_D \mathscr{E}lR^{(I)}$ [D14.4].

Let us only remark that by D20.1 and $(13.6)_{1,2}$, « $x \subseteq^\cap y$ » in D20.2 means $Na \leqslant b$. Now let us define $S^\cap$, i.e. *strict power set*, and $S^\cup$:

DD20.5,6 $S^\cap \mathscr{A} =_D (\lambda a) a \subseteq^\cap \mathscr{A}$, $\quad S^\cup \mathscr{A} =_D (\lambda a) a \subseteq^\cup \mathscr{A}$.

From A17.4 we deduce the first of the theorems

(20.4) $\vdash S^\cap u \in St$ (6.7), $\quad S^\cap\{0, 1\}^\cap \in St$

which by $(20.3)_2$ yields $(20.4)_2$ By DD20.1,3-5

(20.5) $\vdash PR \cup \mathscr{E}lR \subseteq^\cap \{0, 1\}^\cap$, $\quad \vdash \mathscr{E}l \subset^\cap S^\cap\{0, 1\}^\cap$.

By $(19.8)_2$ from $(20.5)_2$ and $(20.4)_2$ [$(20.5)_1$ and $(20.3)_2$] we have

(20.6) $\vdash \mathscr{E}l \in El \quad [\vdash PR, \mathscr{E}lR \in El]$.

Thus we have substantially proved in MC^∞ that the *elementary possible cases* (Γ-cases, cf. [GIMC, nn. 48, 49]) form a set. Though not hard to prove, this result is important. Some basic consequences of its will be shown in n. 21.

21. Validity in MC^ν of some theorems on $\mathscr{E}l$, N, and $\diamond$ already stated in MC^ν. Basic modal consequences on El and $u^\cup$.

First we introduce the matrix $|_u$ that means: *the case* $u(\in \mathscr{E}l)$ *is taking place*—cf. [GIMC Def. 48.2]:

D21.1 $|_u \equiv_D u \in \mathscr{E}l \wedge 1 = (\imath a) a \in u$.

Furthermore in (21.1-6) below we list the respective analogues for MC^∞ of the following theorems on $\mathscr{E}1$, $|_u$, N, and $\Diamond$ proved for MC^ν in [GIMC, nn. 48, 49]: $(82)_{2,3}$, $(83)_{1,3}$, $(83)_{4,6}$, (89), $(\overline{89})$, (90), (91), and $(\overline{92})$. They can be proved for MC^∞ in substantially the same way as for MC^ν.

$$(21.1) \quad \vdash \mathscr{E}1 \in MSep\,, \qquad \vdash \mathscr{E}1 \in Abs\,, \qquad \vdash \exists_u |_u\,,$$

$$(21.2) \quad \vdash (\exists_1 u)\, |_u\,, \qquad \vdash u \in \mathscr{E}1 \supset \Diamond |_u\,, \qquad \vdash (\exists_1^\cap u)\, |_u\,,$$

$$(21.3) \quad \vdash u \in \mathscr{E}1 \supset [\Diamond(|_u p) \equiv (|_u \supset^\cap p)]\,, \qquad \vdash \Diamond(|_u p) \equiv u \in \mathscr{E}1\, (|_u \supset^\cap p)\,,$$

$$(21.4) \quad \vdash \forall_u[u \in \mathscr{E}1 \supset \Diamond(|_u p)] \equiv Np \equiv \forall_u(|_u \supset^\cap p)$$

if u does not occur free in p; under the same assumption

$$(21.5) \qquad \vdash (\exists_u)[u \in \mathscr{E}1(|_u \supset^\cap p)] \equiv \Diamond p \equiv (\exists_u)\Diamond(|_u p)\,.$$

We also have

$$(21.6) \qquad \vdash \Diamond[|_u \varphi(p_1, \dots, p_n)] \equiv u \in \mathscr{E}1 \wedge \varphi[\Diamond(|_u p_1), \dots, \Diamond(|_u p_n)]$$

where $\varphi(p_1, \dots, p_n)$ is any matrix constructed out of p_1 to p_n by means of connectives $(\sim, \wedge)$ and quantifiers $((\forall V_1), (\forall V_2), \dots)$.

Now we can prove, on the basis of (20.6), some basic essentially modal properties of El and $u^\cup$. Since D21.1 yields $|_u \supset u \in \mathscr{E}1$, by $(21.3)_2$, we easily get

$$(21.7) \qquad \vdash |_u \supset [\Diamond(|_u p) \equiv p]\,.$$

TEOR. 10.1. *If an object, $\mathscr{A}$, necessarily equals an element, then $\mathscr{A}$ is an element and (in case $\mathscr{A}$ is a set) its modal sum $\mathscr{A}^\cup$* [D.10.14] *is a set. More precisely (in MC^∞) we have*

$$(21.8) \quad \vdash \mathscr{A} \in^\cap El^{(e)} \supset \mathscr{A}^\cup \in \mathrm{El}\,, \qquad \vdash u^\cup \in St\,, \qquad \vdash El^{(e)\cap} \subseteq El\,.$$

PROOF. To prove $(21.8)_1$ we start with (a) $\mathscr{A} \in^\cap El^{(e)}$ and $|_u$. By rule C with $\mathscr{B}$, we deduce (b) $\mathscr{B} =^\cap (\lambda a) \Diamond (|_u a \in \mathscr{A})$ from the well known theorem $\vdash \forall_u \exists_{\mathscr{B}} \mathscr{B} =^{\cap} \mathscr{U}$. Furthermore by (21.7), $|_u$ yields $\forall_a[a \in \mathscr{A} \equiv \diamond(|_u \wedge a \in \mathscr{A})]$ so that by (b) and $(18.3)_1$ $\mathscr{A} = \mathscr{B}$; thence by ($a$) and D10.11 we easily get $\exists_v \mathscr{B} \subseteq v$. In addition ($b$) yields $\mathscr{B} \in MConst$. Then by $(19.11)_1$ we easily deduce $\mathscr{B} \in \mathrm{St}$, which by ($b$) yields ($c$) $(\lambda a) \diamond (|_u a \in \mathscr{A}) \in El$. We conclude that ($a$), $|_u \vdash$ (c). Since (a) is modally closed, by the generalization theorem—cf. [GIMC, Theor. 32.2]—whose validity for MC^{∞} was asserted in n. 16 and by Theor. 16.1 we have $(a) \vdash \forall_u [|_u \supset^{\cap} (c)]$. By $(21.4)_2$ this yields $(a) \vdash N(c)$. Hence by the deduction theorem

$$(21.9) \qquad \mathscr{A} \in^{\cap} \mathrm{El}^{(e)} \vdash (\lambda a) \diamond (|_u a \in \mathscr{A}) \in^{\cap} El\,.$$

Now let us remark that from the strengthened version (18.2) of A17.3 and rule C with $\mathscr{F}$ we get

$$(21.10) \quad (N)\, \mathscr{F} \in Cl \wedge \{\mathscr{U} \in \mathscr{F} \equiv \exists_{u\mathscr{B}}[(u, \mathscr{B}) \in \mathscr{U} \wedge u \in \mathscr{E}1 \wedge \mathscr{B} =^{\cap} =^{\cap} (\lambda a) \diamond (|_u a \in \mathscr{A})]\}\,,$$

which by D13.3 and the consequence $\vdash \mathscr{E}1 \in \mathrm{MConst}$ of $(21.1)_2$ easily yields (d) $\mathscr{F} \in Fn$ and by $(21.1)_1$, D12.4 (for $n = 1$), and D13.2 yields $\mathscr{F} \in EUn$. By (d) and D13.4 this yields $\mathscr{F} \in Fnc$. In addition from (21.10) and D21.1 we easily deduce $\mathrm{Dmn}\, \mathscr{F} = \mathscr{E}1$. Hence by $(20.6)_1$ and A17.9 we have (e) $\mathrm{Rng}\, \mathscr{F} \in El$. By D11.12 and (21.10) we easily deduce $\bigcup \mathrm{Rng}\, \mathscr{F} =^{\cap} (\lambda a) \exists_u [u \in \mathscr{E}1 \diamond (|_u a \in \mathscr{A})]$. Furthermore (21.6) for $n = 1$ and for $\varphi(p_1) = p_1$ yields $\vdash u \in \mathscr{E}1 \diamond (|_u a \in \mathscr{A}) \equiv \diamond (|_u a \in \mathscr{A})$, and $(21.5)_2$ yields $\diamond(|_u a \in \mathscr{A}) \equiv \diamond\, a \in \mathscr{A}$. Then—cf. D10.14

$$(21.11) \quad \bigcup \mathrm{Rng}\, \mathscr{F} =^{\cap} (\lambda a) \exists_u \diamond (|_u a \in \mathscr{A}) =^{\cap} (\lambda a) \diamond a \in \mathscr{A} =^{\cap} \mathscr{A}^{\cup}\,.$$

Hence by (e) and $(19.8)_3$ we get (f) $\mathscr{A}^{\cup} \in \mathrm{El}$. We conclude that $(a) \vdash (f)$. Now we easily see that thesis $(21.8)_1$ holds.

By D2.8 and Conv. 3.3 modified according to (10.3), $(21.8)_1$ yields $(21.8)_2$:

Lastly by D10.14 $\vdash A \subseteq^{\cap} A^{\cup}$ which by $(19.8)_2$ yields $\vdash A^{\cup} \in El \supset \supset A \in El$. This and $(21.8)_1$ yields (g) $\vdash A \in^{\cap} El^{(e)} \supset A \in El$. Hence $\mathscr{A} \in^{\cap} \mathrm{Cl} \vdash \mathscr{A} \in El^{(e)\cap} \supset \mathscr{A} \in El$.

Assume (h) $\mathscr{A} \in^{\cap} El^{(e)}$ and set $A =_D (\imath\mathscr{B})[\mathscr{B} = \mathscr{A} \in \mathrm{Cl} \vee \mathscr{B} = \Lambda \wedge \mathscr{A} \notin \mathrm{Cl}]$ hence $A \in^{\cap} El^{(e)}$ which by (g) yields $A \in El$.

By AA17.1,8 $\mathscr{C} \in El$ where $\mathscr{C} =_D (\imath a)[a = \mathscr{A} \notin Cl \vee a = a^* \wedge \mathscr{A} \in Cl]$. Then $\{A, \mathscr{C}\}^{\cap} \in El$ by $(20.3)_1$. Furthermore $\mathscr{A} \in \{A, \mathscr{C}\}^{\cap}$ obviously. Hence (i) $\mathscr{A} \in El$. We conclude that $(h) \vdash$ (i). Hence $(21.8)_3$ holds. q.e.d.

22. Hints at the intensional description operators ι_u and $\imath_u$; and the calculus MC_{ϱ}^{∞} capable to deal with the real case ϱ.

The operators ι_u and $\imath_u$ depending on the parameter u $(\in \mathscr{E}1)$ can be introduced in MC^{∞} in the same way as in MC^{ν}—cf. [GIMC, D50.1, D51.1], D10.10:

D22.1 $$(\iota_u \mathscr{V}^{\cdot})p =_D (\imath \mathscr{V}^{\cdot}) \diamond [|_u (p \exists^{(1)\cap} \mathscr{V}^{\cdot})p],$$

D22.2 $$(\imath_u \mathscr{V}^{\cdot})p =_D (\imath \mathscr{V}^{\cdot})\{\diamond [|_u(\exists_1^{\cap} \mathscr{V}^{\cdot})p] \wedge \mathscr{V}^{\cdot} = = (\iota_u \mathscr{V}^{\cdot})p \vee \sim \diamond [|_u(\exists_1^{\cap} \mathscr{V}^{\cdot})p] \wedge \mathscr{V}^{\cdot} = (\imath \mathscr{V}^{\cdot})p\}.$$

All properties of ι_u and $\imath_u$ considered in [GIMC] and in particular Theors. 50.1 and 51.1 hold without any change in connection with MC^{∞}, for unrestricted variables.

Now let ϱ be a constant of MC^{∞} not yet used. Let MC_{ϱ}^{∞} be the calculus obtained from MC^{∞} by adding the axiom

A21.1 $$\varrho \in \mathscr{E}1 \quad \text{—cf. [GIMC, A52.1]},$$

which allows us to interpret ϱ as representing the *real* (*elementary*) *case*.

Now we can turn ι_u and $\imath_u$ into the parameterless description operators ι_{ϱ} and $\imath_{\varrho}$. Furthermore we can define p *occurs in the real case* ($\mathfrak{R}p$):

D22.3 $$\mathfrak{R}p \equiv_D \diamond(|_{\varrho} p). \quad \text{— cf. [GIMC, Def. 52.1]}.$$

All considerations on ϱ and $\mathfrak{R}$ made in [GIMC, nn. 52-54] hold for MC^{∞} in practically the same form. Furthermore one can use MC^{∞} [MC_{ϱ}^{∞}] instead of MC^{ν} [MC_{ϱ}^{ν}] in the applications of the latter calculus to the philosophical puzzles presented in [GIMC nn. 54,55].

CHAPTER 4

ON RELATIONS AND FUNCTIONS

23. Relations and functions, corresponding lambda expressions.

The whole treatment of relations presented in [IST, Sec. 3] holds for MC^{∞} up to some simple and standard changes such as the use of $\{\}^{(i)}$ instead of $\{\}$ in accordance with definition D11.3 of ordered couple, and the use of $=^{\cap}$ instead of $=$ —cf. ft. 5 in n. 19 as far as e.g. item (3.1) in [IST] corresponding to (23.1) is concerned.

Remembering D11.9 we have

(23.1) $\vdash 0 \in V^n$ (3.1) , $\vdash V^{n+1} \subset V^n$,

(23.2) $R, S \subseteq V^2 \vdash R \subseteq S \equiv \forall_{a,b}[aRb \supset aSb]$ (3.2,i)

where as well in the sequel we use, after [IST], the following

CONVENTION 23.1. *«* aRb *» stands for «* $(a, b) \in R$ *» in case* $R \subseteq V^2$, *and «* $R(a_1, \ldots, a_n)$ *» stands* for *«* $(a_1, \ldots, a_n) \in \mathscr{R}''$ $(n = 2, 3, \ldots,)$.

The *intensionalityprinciples for relations and functions* read

(23.3) $R, S \subseteq V^n \vdash R = S \equiv \forall_{a_1,\ldots,a_n}[(a_1, \ldots, a_n) \in R \equiv (a_1, \ldots, a_n) \in S]$,

(23.4) $F, G \in Fn_n \vdash F = G \equiv \text{Dmn}\, F = \text{Dmn}\, G \wedge \forall_{a_1,\ldots,a_n} F'(a_1, \ldots, a_n) =^{\cap}$
$=^{\cap} G'(a_1, \ldots, a_n)$ [D4.2]

respectively—$(23.3)_1$ for $n = 2$ is (3.2,ii) in [IST]. By DD11.8,9

$$(23.5)\quad \begin{cases} u \times v \in St \ (3.12)\,, \quad \text{hence } \vdash u^n \in St\,, \\ R \subseteq V^2 \vdash R \subseteq \text{Dmn}\, R \times \text{Rng}\, R \ \ (3.13, xi)\,. \end{cases}$$

We now introduce the *relative product* $R|S$ of (the relations) R and S:

D23.1 $\mathscr{R}|\mathscr{S} =_D \{(a, b) | \exists_c (a\mathscr{R}c \wedge c\mathscr{S}b)\}$.

Hence

(23.6) $$\vdash \mathscr{R} \in In \cup \{0\} \vee \mathscr{S} \in In \cup \{0\} \supset \mathscr{R}|\mathscr{S} = 0 .$$

All algebraic properties of the relative product and cartesian product proved in [IST, Sec. 3] hold for classes in MC^{∞}, and the proofs are substantially the same. We only mention some of them as examples.

(23.7) $$\begin{cases} \vdash R|(S|T) = (R|S)\,T , \quad \vdash (R \cup S)^{-1} = R^{-1} \cup S^{-1} , \\ \vdash R \subseteq V^2 \supset (R^{-1})^{-1} = R , \quad \vdash (R|S)^{-1} = S^{-1}|R^{-1} \text{ (3.5, i, vi-viii)} , \end{cases}$$

(23.8) $$\begin{cases} \vdash \text{Dmn}\,(R \cup S) = \text{Dmn}\,R \cup \text{Dmn}\,S , \\ \qquad \vdash \text{Rng}\,(R \cup S) = \text{Rng}\,R \cup \text{Rng}\,S , \\ \vdash \text{Dmn}\,R|S \subseteq \text{Dmn}\,R , \\ \qquad \vdash \text{Rng}\,(R|S) \subseteq \text{Rng}\,S \quad \text{(3.8, i, viii)} , \end{cases}$$

(23.9) $$\vdash r, s \subseteq V^2 \supset (r|s,\ r^{-1},\ \text{Dmn}\,r,\ \text{Rng}\,r,\ \text{Fld}\,r \in \text{St}) \text{ (3.14)} ,$$

(23.10) $$\vdash R''(A \cap B) \subseteq R''A \cap R''B \text{ (3.16, iv) } (^6) \qquad \vdash r''A \in St \text{ (3.17)} .$$

By D4.2, D11.6.7, and the replacement theorems (18.3,4) of lambda expressions for properties, it is now easy to deduce their following analogues for n-ary relations and functions, where p and $\Phi(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ are matrices and Δ is a term:

(23.11) $$\begin{cases} \vdash (\mathscr{A}_1, \ldots, \mathscr{A}_n) \in (\lambda \mathscr{A}_1, \ldots, \mathscr{A}_n) p \equiv p \bigwedge_{i=1}^{n} \mathscr{A}_i \in El , \\ \qquad \vdash [(\lambda a_1, \ldots, a_n) p](a_1, \ldots, a_n) \equiv p \text{ —cf. Conv 23.1} , \end{cases}$$

(23.12) $$\vdash (\lambda \mathscr{A}_1, \ldots, \mathscr{A}_n)\ \Phi(\mathscr{A}_1, \ldots, \mathscr{A}_n) = (\lambda a_1, \ldots, a_n)\ \Phi(a_1, \ldots, a_n),$$

(23.13) $$\bigwedge_{i=1}^{n} \mathscr{A}_i \in El \vdash [(\lambda \mathscr{A}_1, \ldots, \mathscr{A}_n) \Delta]'(\mathscr{A}_1, \ldots, \mathscr{A}_n) = \Delta ,$$
$$\vdash [(\lambda a_1, \ldots, a_n) \Delta]'(a_1, \ldots, a_n) = \Delta .$$

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(6) « R^*A » is used in [IST] for the R-transform of A instead of our $R''A$ [D3.6].

24. n-ary intensionalization. Strengthening of the replacement axiom and a theorem on V.

We define the *n-ary intensionalization* $F^{(nI)}$ of F by

D24.1 $\mathscr{R}^{(nI)} =_D \{(\{a_1\}^{(i)}, \ldots, \{a_n\}^{(i)}) | (a_1, \ldots, a_n) \in \mathscr{R}\}$—cf. D.14.4

Hence

(24.1). $$\vdash F^{(nI)} = (F \cap V^n)^{(nI)}, \quad \vdash F^{(1I)} = F^{(I)}.$$

Furthermore by D12.4 and DD11.11,12

(24.2) $$\begin{cases} \vdash \mathscr{R}^{(nI)} \in MSep_n, \ \vdash \mathrm{Dmn}\, \mathscr{R}^{(n+1I)} \in MSep_n, \ \vdash \mathrm{Rng}\, \mathscr{R}^{(nI)} \in Msep, \\ \vdash \mathrm{Dmn}\, \mathscr{R}^{(n+1I)} = (\mathrm{Dmn}\, \mathscr{R})^{(nI)}, \ \vdash \mathrm{Rng}\, \mathscr{R}^{(n+1I)} = (\mathrm{Rng}\, \mathscr{R})^{(I)} \\ \vdash \mathscr{R} \in MConst \supset \mathscr{R}^{(nI)} \in Abs, \ \vdash \mathscr{R} = S \supset \mathscr{R}^{(nI)} = S^{(nI)} \\ \qquad (n = 1, 2, \ldots), \end{cases}$$

By D24.1, D3,10, and D20.5, D11.2 yields

(24.3) $$\vdash F \in MConst \supset F^{(I)} \subseteq S^\cap F, \quad \vdash F \subseteq \bigcup F^{(I)}$$

and D11.3 yields—cf. the proof of (3.14) in [IST]

(24.3′) $$\vdash F \in MConst \supset F^{(2I)} \subseteq S^\cap S^\cap S^\cap \bigcup \bigcup F,$$
$$\vdash F \in MConst \wedge F \subseteq V^2 \supset F \subseteq S^\cap S^\cap \bigcup \bigcup \bigcup F^{(2I)}$$

To prove the first of the theorems

(24.4) $$\vdash G \in El \supset G^{(nI)} \in El, \quad \vdash G \subseteq^\cap V^n \wedge G^{(nI)} \in El \supset G \in El$$

for $n = 1, 2$ we assume (a) $G \in El$. By A.17.11(I) and rule C with F we obtain (b) $F = G$ and (c) $F \in MConst$: By (b), $G \subseteq^\cap F$ which by (a) and $(19.11)_1$ yields $F \in^\cap \mathrm{St}$, hence (d) $F \in El$. By (c) and $(24.3)_1$ $[(24.3')_1]$ $F^{(nI)} \subseteq S^\cap F$ for $n = 1$ $[F^{(nI)} \subseteq S^\cap S^\cap S^\cap \bigcup \bigcup F$ for $n = 2]$. By $(19.16)_3$ and $(20.4)_1$ this and (d) yield (e) $F^{(nI)} \in El$. From (b) and

$(24.2)_7$ we have (f) $F^{(nI)} = G^{(nI)}$ which by (e) yields (g) $G^{(nI)} \in El^{(e)}$. We conclude that $N(a) \supset N(g)$. Furthermore by (18.8), $\vdash (a) \supset N(a)$ and by $(21.8)_3$ $\vdash N(g) \supset G^{(nI)} \in El$. Hence $\vdash (a) \supset G^{(nI)} \in El$, i.e. $(24.4)_1$ holds.

To prove $(24.4)_2$ for $n = 2$, assume (h) $G \subseteq^\cap V^n$ and (i) $G^{(nI)} \in El$. Then we get (b), (c), and (f) again. From (c) we deduce (l) $F^{(2I)} \in MConst$. Furthermore (f) yields $F^{(nI)} \subseteq^\cup G^{(nI)}$, so that by $(19.11)_1$ we obtain $F^{(nI)} \in^\cap St$, hence (m) $F^{(nI)} \in El$. Furthermore (b) and (i) yield $F \subseteq V^2$. Hence by $(24.3')_2$ we have $F \subseteq S^\cap S^\cap \bigcup\bigcup\bigcup F^{(2I)}$ which by (m), $(19.16)_3$, and $(20.4)_1$ yields $F \in El$. This and (b) yield $G \in El^{(e)}$. Furthermore $\vdash (a) \wedge (c) \equiv N(a) \wedge N(c)$. Hence $G \in^\cap El^{(e)}$ by the modal rule G; thence we deduce $G \in El$ by $(21.8)_1$. We conclude that $(24.4)_2$ holds for $n = 2$. Its validity for $n = 1$ can be proved in a similar and easier way.

The proofs of $(24.4)_{1,2}$ for $n = 3, 4, \ldots$ are similar with those for $n = 2$. They are based on some suitable generalizations of $(24.3')$.

By D13.3 $\mathscr{F} \in Fn_n$ yields $\mathscr{F} \in MConst$, hence $\mathscr{F}^{(n+1I)} \in MConst$. Then by $(24.2)_1$ and D13.5 we deduce $\mathscr{F}^{(n+1I)} \in AFn_n$. Thus the first of the theorems

$$(24.5) \qquad \vdash \mathscr{F} \in Fn_n \supset \mathscr{F}^{(n+1I)} \in AFn_n\,,$$
$$\vdash Abs_{n+1} \cap Un \subset EUn^\cap_{n+1}\,, \qquad \vdash AFn_n \subset Fnc^\cap_n$$

has been proved. The second is the syntactical analogue $SA(13.2)_2$ of $(13.2)_2$ (7). It follows easily from DD13.1,2 and DD12.4-6. At this point $SA(13.2)_1$, i.e. $\vdash AFn_n \subset Abs_{n+1}$ is easy to prove. This theorem and DD13.1-5 easily yield $(24.5)_3$.

By $(19.8)_2$ $\vdash \mathscr{V} \subseteq^\cap \text{Rng}\, F \wedge \text{Rng}\, F \in El \supset \mathscr{V} \in El$. Hence *the replacement axiom* A17.9 *is equivalent to the first of the theorems*

$$(24.6) \qquad \vdash F \in \left\{ \begin{array}{l} Fnc^\cap \\ Fn \end{array} \right. \wedge \text{Dmn}\, F \in \text{St} \supset \text{Rng}\, F \in \text{St}\,.$$

Now let us prove that *the replacement axiom can be strengthned into* $(24.6)_2$. To this end we start with (a) $F \in Fn$ and (b) $\text{Dmn}\, F \in St$. This (b) and $(24.4)_1$ for $n = 1$ yield $(\text{Dmn}\, F)^{(I)} \in El$. Thence by $(24.2)_4$

(7) If e.g. (r.s.) labels a semantical theorem, $\Vdash p$ or $p \Vdash q$, then by « $SA(r, s)$ » we denote its syntactical correspondent $\vdash p$ or $p \vdash q$ respectively.

we get (*c*) $\mathrm{Dmn}\, F^{(2I)} \in St$. From (*a*) and $(24.5)_1$ we deduce $F^{(2I)} \in AFn$. Thence by $(24.5)_3$ for $n = 1$ we get $F^{(2I)} \in Fnc^{\frown}$ which by $(24.6)_1$ and (*c*) yields $\mathrm{Rng}\, F^{(2I)} \in St$; thence by $(24.2)_5$ we have $(\mathrm{Rng}\, F)^{(I)} =^{\frown}$ $=^{\frown} \mathrm{Rng}\, F^{(2I} \in El$. Then by $(24.4)_2$ for $n = 1$, $\mathrm{Rng}\, F \in El$; hence (*d*) $\mathrm{Rng}\, F \in St$. We conclude that (*a*), (*b*) $\vdash$ (*d*). So $(24.6)_2$ holds by the deduction theorem.

TEOR. 24.1. *Assertion* $(19.17)_1$ *can be strengthened into the modal theorems*

$$(24.7) \quad \vdash V \notin El^{(e)}, \qquad \vdash A^{(e)} = V \supset A \notin El^{(e)},$$

$$(24.8) \quad \vdash F \neq \Lambda \supset F^{(e)} \notin El^{(e)}, \qquad \vdash \{a_1, \ldots, a_n\} \notin El^{(e)} \qquad (n = 1, 2, \ldots)\,.$$

PROOF. Since $\vdash V^{(e)} = V$ [D10.11], $(24.7)_1$ follows from $(24.7)_2$ [8]. Therefore we now prove the latter theorem. To this end we assume (*a*) $A^{(e)} = V$ and (*b*) $A = \mathscr{B} \wedge \mathscr{B} \in El$ as an hypothesis for reduction ad absurdum.

Thence we get (*c*) $\mathscr{B}^{(e)} = V$. Now remark that $\vdash \{a\}^{(i)} = \{a, a\}^{(i)}$, so that $a \in V$ yields $\{a\}^{(i)} \in St$ by $(19.4)_1$; hence (*c*) yields (*d*) $\exists_y (y \in \mathscr{B} \wedge y = \{a\}^{(i)})$. Let us set

$$(24.9) \qquad \mathscr{C} =_D \{y \,|\, y \in \mathscr{B} \wedge \exists_a (y =^{\frown} \{a\}^{(i)})\}, \qquad \text{hence } \vdash \mathscr{C} \subseteq^{\frown} \mathscr{B}\,.$$

By A17.11 (I) and rule *C* with *F* we have

$$(24.10) \qquad F = \{(y, a) \,|\, y \in \mathscr{C} \wedge y =^{\frown} \{a\}^{(i)}\}\,, \qquad F \in MConst\,.$$

Then $F \in Fn$ [D13.3], $\mathrm{Dmn}\, F = \mathscr{C}$ and—cf. (*d*)—$\mathrm{Rng}\, F = V$ hold. Hence, by the strengthened version $(24.6)_2$ of the replacement axiom A17.9, $\mathscr{C} \in El \supset V \in El$. Then $\mathscr{C} \notin El$ by $(19.17)_1$, and $\mathscr{B} \notin El$ by $(19.8)_2$ and $(24{,}9)_2$, which contrasts to (*b*). Hence $\vdash \sim [(a) \wedge (b)]$, which is $(24.7)_2$.

To prove $(24.8)_1$ we assume (*e*) $F \neq \Lambda$, and (*f*) $F^{(e)} \in El^{(e)}$ as an hypothesis for reduction ad absurdum. Then $G = F^{(e)} \wedge G \in El$ for some *G*. Hence A17.11 (II) yields (g_1) $H = G = F^{(e)}$, (g_2) $H \in El$, and (g_3) $H \in MConst$ for some *H*.

---

[8] To prove $(24.7)_1$ directly, assume (*a*) $V = \mathscr{A}$ and (*b*) $\mathscr{A} \in \mathscr{B}$ as hypotheses for reductio ad absurdum. Since $\vdash \mathscr{B} \subseteq V$, by (*a*) we have $\mathscr{B} \subseteq \mathscr{A}$. Hence (*b*) yields $\mathscr{A} \in \mathscr{A}$ which contrasts with $(19.1)_3$. Hence $\vdash \sim [(a) \wedge (b)]$ which by D2.1, D10.11, and Convention 10.2 yields $(24.7)_1$.

From A16.12 we deduce $(h)\ \sim p_1 \Diamond p_1$ for some p_1. By (e), $a \in F$. Hence $\forall_b \xi_b \in F^{(e)}$ where $\xi_b =_D (\imath c)(p_1 c = b \wedge \sim p_1 c = a)$. Then $p_1 \supset^\cap \forall_b \xi_b = b$ and, by (g_1) and (g_3), $\forall_b \xi_b \in^\cap H$. So $p_1 \supset^\cap H^{(e)} = V$. Hence by $(24.7)_2$ $p_1 \supset^\cap H \notin El^{(e)}$. Then (h) yields (i) $\Diamond H \notin El^{(e)}$, hence $\Diamond H \notin El$. By (f), (g_1), and (g_3), $H \in El^{(e)} \cap MConst$ which by $(19.11)_3$ yields $H \in St^\cap$. This contrasts to (i). We conclude that $(e) \vdash \sim (f)$. Hence $(24.8)_1$ holds. Thence $(24.8)_2$ follows. q.e.d.

25. On elementary algebraic properties of functions.

Some theorems in MC^∞ asserting algebraic properties of functions (Fn or Fnc) have a direct analogue in extensional logic. Others are essentially modal. Many among the latter are stated via some semantical theorems in n. 13. At this point *we can easily prove the syntactical analogues in MC^∞ of all semantical theorems asserted in* n. 13. Here are some examples of the aforementioned algebraic properties —cf. D4.3, D11.10

$$(25.1) \qquad F^{-1} \in Fn \vdash \begin{cases} F''(A \cap B) = F''(A) \cap F''(B), \\ F''(A - B) = F''(A) - F''(B). \end{cases}$$

By $(23.7)_3$, (25.1) is equivalent to theorem (4.1) in [IST]. By DD13.1,3 $(a, c), (b, c) \in F \wedge F^{-1} \in Fn$ yields $a =^\cap b$. With this in mind the proof of (25.1) appears substantially the same as the one of (4.1) in [IST]. Similar remarks can be made on theorems $(25.2)_{1,2}$ below, which correspond to (4.3, i, ii, iii) in [IST]. Some differences among the former and the latter—cf. $(25.2)_2$ below—are due to different conventions about the functional notations, by which in [IST] e.g. V has the role of our a^*—compare Def. 4.2 in [IST] where individuals are not taken into account, with D4.2. These differences cause only trivial changes in the proofs.

$$(25.2) \quad F \in Fn \vdash \begin{cases} a \in \mathrm{Dmn}\, F \supset \forall_b[(a, b) \in F \equiv b =^\cap F'a] F'a \in El\,, \\ a \notin \mathrm{Dmn}\, F \supset F'a =^\cap a^* \wedge F'a \in El\,. \end{cases}$$

The following form of the intensionality principle for functions is similar to (23.4) and constitutes a direct analogue of (4.4,i) in [IST]:

(25.3) $\vdash F, G \in Fn \supset [F = H \equiv \operatorname{Dmn} F =$

$$= \operatorname{Dmn} G \wedge \forall_a (a \in \operatorname{Dmn} F \supset^{\cap} F'a = G'a)] .$$

Theorem $(25.4)_1$ below is $SA(13.5)_2$—cf. *ft.* (6).

(25.4) $\vdash I^{\cap} \in Fn$ [D5.1], $\vdash (I^{\cap})'a = a = I'a$ [D4.2] (4.5).

(25.5) $\begin{cases} \vdash F, G \in Fn \wedge \operatorname{Dmn} F \cap \operatorname{Dmn} G = \Lambda \supset F \cup G \in Fn , \\ \vdash F, G \in Fnc \wedge (\operatorname{Dmn} F)^{(e)} \cap (\operatorname{Dmn} G)^{(e)} = \Lambda \supset F \cup G \in Fnc . \end{cases}$

Thus [IST, Th. 4.6] has two analogues for MC^{∞}. Remark that Fn cannot be replaced by Fnc in $(25.5)_1$. The analogue of [IST, Theor. 4.8] for MC^{∞} is

(25.6) $F, G \in Fn \vdash$ (i) to (v) where —cf. D4.3

(i) $\operatorname{Dmn}(G|F) = \breve{G}''(\operatorname{Dmn} F) \subseteq \operatorname{Dmn} G$,

(ii) $\operatorname{Rng}(G|F) = F''(\operatorname{Rng} G) \subseteq \operatorname{Dmn} F$,

(iii) $a \in \operatorname{Dmn}(G|F) \supset (G|F)'a =^{\cap} F'(G'a)$,

(iv) $\breve{F} \in Fn \equiv \forall_{a,b}(a, b \in \operatorname{Dmn} F \wedge F'a =^{\cap} F'b \supset a =^{\cap} b)$ (4.5) ,

(v) $\breve{F} \in Fnc \equiv \forall_{a,b}(a, b \in \operatorname{Dmn} F \wedge F'a = F'b \supset a = b) \wedge \breve{F} \in Fn$,

Note that (iv) and (v) are two analogues of [IST, Th. 4.8, iv].

In parallel with DD14.1,2 it is natural to define « the n-ary function F (*intensionally*) *maps* [*totally maps*] A *into* B » ($A \to B$ [$A \to^{(t)} B$]) or « F is an *n-ary function* [*an extensionally univalent or extensionally invariant function*] *from* A *into* B » as follows:

DD25.1,2 $F \in \begin{cases} A \to_n B \\ A \to_n^{(t)} B \end{cases} \equiv_D \operatorname{Dmn} F = A \wedge \operatorname{Rng} F \subseteq B \wedge F \in$

$$\in \begin{cases} Fnc_n & (\to \;=_D \to_1) \\ Fn_n & (\to^{(t)} =_D \to_1^{(t)}) . \end{cases}$$

Then remembering DD11.11,12 and DD13,3,4 we deduce $(25.7)_{1-4}$ below

$$(25.7)\quad \begin{cases} \vdash A,\, B \in MConst \supset A \to B \in MConst\,, \\ \vdash A \to_{n+1} B \subseteq A \to_n B\,, \qquad \vdash A \to_{n+1}^{(t)} B \subseteq A \to_n^{(t)} B\,, \\ \vdash A \to_n^{(t)} B \subset A \to_n B\,, \qquad \vdash A, B \in MSep_n \supset A \to_n B = A \to_n^{(t)} B\,, \end{cases}$$

while $(25.7)_5$ holds by D12.4. By D12.6 $(25.7)_5$ shows that no duplication of the usual notion of mapping $A \to B$, very common in extensional set theory, arises in MC^∞ as far absolute concepts are concerned. Let us remark that this is the case with pure set theory in (MC^∞).

Let us note that the analogue of $(25.7)_1$ for $\to_n^{(t)}$ fails to hold, because, after setting

$$(25.8)\quad A =_D \{x, y\}^{(i)}, \qquad B =_D \{0, 1\}^{(i)}, \qquad F =_D \{(x, 0), (y, 1)\}^{(i)},$$

we easily seee that

$$(25.9)\quad x \neq y \wedge x =^{\cup} y \vdash F \in (Fnc - Fnc^{\cap}) \wedge F \in A \to^{(t)} B \wedge \Diamond\, F \notin A \to^{(t)} B\,.$$

By D11.8 and DD25.1,2 the first of the theorems

$$(25.10)\qquad \vdash (A \to_n B) \cup (A \to_n^{(t)} B) \subseteq S^{\cap}(A \times B)\,, \qquad \vdash u \to_n v,\, u \to_n^{(t)} v \in St$$

holds. By $(23.5)_1$ and $(24.1)_1$ it yields the second.

26. Basic theorems on $u^{(e)\cap}$, $u^{\cup(e)\cap}$, and intrinsic n-ary extensionalization.

By DD12.1,2 we respectively have

$$(26.1)\qquad \vdash A,\, B \in^{\cap} V^n \wedge NA =_n B \supset A =^{\cap} B\,, \qquad \vdash \mathscr{A}^{(ne)} \subseteq^{\cap} V^n\,.$$

Furthermore by $(25.10)_2$, $(20.6)_1$, and $(21.8)_2$

$$(26.2)\qquad \vdash F_{(v)} \in St\,, \qquad \text{where } F_{(v)} =_D \mathscr{E}1 \to v^{\cup}\,.$$

TEOR. 26.1. *By the definitions* $(26.2)_2$ *and*

$$(26.3)\qquad R_{(v)} =_D \{(f, a) | f \in F_{(v)} \wedge a \in v^{\cup(ne)\cap} \wedge \forall_u (|_u \supset^{\cap} f'u =_n a\}\}$$

—cf. DD12.1,2 *and* D4.2—*we have*

$$(26.4)\qquad \vdash R_{(v)} \in Fn\,, \qquad \vdash \operatorname{Dmn} R_{(v)} = F_{(v)}\,, \qquad \vdash \operatorname{Rng} R_{(v)} = v^{\cup(ne)\cap}\,, \qquad \vdash F_{(v)} \in El\,.$$

PROOF. Start with (f, a), $(f, a') \in R_{(v)}$. Thence by (26.3) we easily get (a) $\forall_u\big(|_u \supset^\cap a =_n a'\big)$ and $a, a' \in^\cap v^{\cup(ne)}$, which by $(26.1)_2$ yields (b) $a, a' \in^\cap V^n$. Since u does not occur (free) in $a =_n a'$, by $(21.4)_2$ (a) yields $Na =_n a'$, so that by (b) and $(26.1)_1$ we have $a =^\cap a'$. We conclude that $(26.4)_1$ holds.

We deduce $(26.4)_2$ from (26.3).

To prove $(26.4)_3$ we start with (c) $a \in v^{\cup(ne)\cap}$. Then we easily deduce

$$(26.5)\qquad \begin{cases} R \subseteq V^2\,, \qquad \operatorname{Dmn} R = \mathscr{E}1\,, \\ \text{where} \\ R =_D \big\{(u, b) | u \in \mathscr{E}1 \wedge b \in v^\cup \diamond \big(|_u b =_n a\big)\big\}\,. \end{cases}$$

By A17.10, $(26.5)_2$, and rule C with $\mathscr{F}$, we get

$$(26.6)\qquad \mathscr{F} \in Fn\,, \qquad \operatorname{Dmn} \mathscr{F} = \mathscr{E}1\,, \qquad \mathscr{F} \subseteq R\,.$$

Hence by (26.5) we get (d) $\forall_u[u \in \mathscr{E}1 \supset \mathscr{F}'u \in v^\cup \diamond \big(|_u \mathscr{F}'u =_n a\big)]$.

Thence we deduce $\operatorname{Rng} \mathscr{F} \subseteq v^\cup$, which by $(26.6)_2$ and $(26.2)_2$ yields (c) $\mathscr{F} \in F_{(v)}$. Furthermore by $(21.3)_1$ and D21.1, (d) yields (e) $\forall_u\big(|_u \supset^\cap \mathscr{F}'u =_n a\big)$. Then by (c) and (26.3) we have $(\mathscr{F}, a) \in R_{(v)}$, hence (f) $\exists_{\mathscr{F}}(\mathscr{F}, a) \in R_{(v)}$. We conclude that $(c) \vdash_{\overline{C}} (f)$. In addition $\mathscr{F}$ does not occur free in (f). Hence $(c) \vdash (f)$ by Theor. 16.1. This easily yields $\vdash v^{\cup(ne)\cap} \subseteq \operatorname{Rng} R_{(v)}$. In addition (26.3) yields $\vdash \operatorname{Rng} R_{(v)} \subseteq v^{\cup(ne)\cap}$. So $(26.4)_3$ holds. $(26.2)_2$, $(20.6)_1$, $(21.8)_2$, and $(25.10)_2$ yield $(26.4)_4$ q.e.d.

We now prove the following important corollaries of Theor. 26.1:

$$(26.7)\qquad \vdash u^{\cup(ne)\cap} \in St\,, \qquad \vdash u^{(ne)\cap} \in St\,.$$

From (26.4) and $(24.6)_2$ we deduce $(26.7)_1$. Since $u^{(ne)\cap} \subseteq^\cap u^{\cup(ne)\cap}$, $(26.7)_1$ and $(19.8)_2$ yield $(26.7)_2$:

Now we define the *n-ary intrinsic extensionalization* $F^{(nie)}$ of F —cf. D12.2:

$$\text{D.26.1}\qquad \mathscr{A}^{(nie)} =_D \mathscr{A}^{\cup(ne)\cap} \cap \mathscr{A}^{(ne)} \ (n = 1, 2, \ldots)\,, \qquad \mathscr{A}^{(ie)} = \mathscr{A}^{(1ie)}\,.$$

Thus by $(26.7)_1$ and $(19.8)_2$ we have the first of the theorems

$$(26.8)\qquad \begin{array}{ll} \vdash u^{(nie)} \in St\,, & \vdash A \subseteq V^n \supset A \subseteq A^{(nie)}\,, \\ \vdash [A^{(ne)}]^{(ne)} = A^{(ne)}\,, & \vdash A \subseteq^{\cap} B \supset A^{(nie)} \subseteq^{\cap} B^{(nie)}\,, \\ \vdash A^{(ne)} = (A \cap V^n)^{(ne)}\,, & \vdash A^{(nie)} = (A \cap V^n)^{(nie)}\,, \\ \vdash A^{(nie)} \subseteq [A^{(nie)}]^{(nie)}\,, & \vdash [A^{(nie)}]^{(ne)} = A^{(ne)}\,. \end{array}$$

Theorems $(26.8)_{2,\ldots,6}$ follow easily from D26.1 and DD12.1,2. Since $\vdash A^{(nie)} \subseteq^{\cap} V^n$, $(26.8)_2$ yields $(26.8)_6$.

By D26.1 $\vdash A^{(nie)} \subseteq^{\cap} A^{(ne)}$; hence by $(26.8)_3$ $\vdash [A^{(nie)}]^{(ne)} \subseteq^{\cap} [A^{(ne)}]^{(ne)} = A^{(ne)}$. Furthermore by $(26.8)_{2,6}$ $\vdash A \cap V^n \subseteq^{\cap} (A \cap V^n)^{(nie)} = A^{(nie)}$; hence by $(26.8)_{2,5}$ $\vdash A^{(ne)} = (A \cap V^n)^{(ne)} \subseteq [A^{(nie)}]^{(ne)}$. So $(26.8)_7$ hols.

Remark that $A = \Lambda \wedge \Diamond A \neq \Lambda \vdash A^{(ne)} = \Lambda \subset (A^{\cup(ne)\cap})^{(ne)}$. Furthermore $A \neq \Lambda \wedge A = {}^{\cup}\Lambda \vdash A^{(ne)\cap} = \Lambda \subset A$ —cf. $(14.4)_2$ (and the subsequent considerations). The comparison of these facts with $(26.8)_{8,2}$ affords a motive to prefer $A^{(nie)}$ to $A^{(ne)\cap}$ and $A^{\cup(ne)\cap}$ as an explicatum in Carnap's sense of the intuitive notion (explicandum) of n-ary intrinsic extensionalization.

27. Two modal exponentiation operations for classes. On equipotence relations.

We defined intensional and extensionally invariant n-ary functions [DD13.3,4] and the analogue was done with mappings [DD25.1,2]. In accordance with this we now introduce the symbol ${}^{A}B$ [B^{A}] of (intensional) *exponentiation* [*total exponentiation*] as follows [9]:

$$\text{DD17.1,2}\quad {}^{A}B =_{D} (\lambda f)\, f \in A \to B\,, \qquad B^{A} =_{D} (\lambda f)\, f \in A \to^{(t)} B\,.$$

Then by DD25.1,2

$$(27.1)\qquad \left\{ \begin{array}{ll} \vdash A, B \in MConst \supset {}^{A}B \in MConst\,, & \vdash B^{A} \subseteq {}^{A}B\,, \\ \vdash A, B \in MSep \supset B^{A} = {}^{A}B\,. & \end{array} \right.$$

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(9) The exponentiation operation for classe is denoted by « B^{A} » in most text books and by « ${}^{A}B$ » in [IST].

The analogue of $(27.1)_1$ for B^A fails to hold because under the definitions (25.8), (25.9) holds and $F \in St$ also does. By $(27.1)_3$ we have ${}^AB = B^A$ for A and B absolute, in accordance with the existence of only one exponential operation for classes in extensional logic. Hence ${}^AB = B^A$ in pure number theory (which deals with transfinite ordinals and cardinals).

The well known algebraic theorems on ${}^AB$ in extensional logic —cf. [IST, Theors. 4.10-16]—also hold for ${}^AB$ and B^A in MC^∞ (and can be proved in substantially the same way), up to the slight changes spoken of below. We now give some examples of these theorems.

$$(27.2)\quad \begin{cases} \vdash 0 \in 0 \to A\,, \quad \vdash F \in 0 \to A \supset F =^{\cap} 0\,, \\ \vdash F \in A \to 0 \supset A = 0 =^{\cap} F\,, \quad \vdash {}^0B = \{0\}^{(i)}, \\ \vdash A \neq 0 \supset {}^A0 = 0\,, \quad \vdash B \neq 0 \supset {}^aB \neq 0 \quad (4.10, \text{ii, iii, viii, ix})\,. \end{cases}$$

We can also assert the analogue of (27.2) (on $\to$ and left exponentiation) for $\to^{(t)}$ and right exponentiation (e.g. $\vdash 0 \in 0 \to^{(t)} A$, $\vdash B^0 = \{0\}^{(i)}$).

Here are some modal theorems concerning exponentiation outside pure number theory and the equivalence relations $\approx$, $\approx^{(t)}$, and $\approx^{(e}$ —cf. DD14.1-3.

By DD14.1-3 we have SA(14.1,2)—cf. ft. ([6]). D14.1-3 and $SA(13.1)_2$ easily yield

$$(27.3)\qquad \vdash A, B \in MConst \vee A \approx B \supset NA \approx B\,.$$

However $\approx$ cannot be replaced by $\approx^{(t)}$ or $\approx^{(e)}$ in (27.3) because of the satisfiability of the set

$$(27.4)\qquad \Diamond a \neq b\,, \quad a =^{\cup} b\,, \quad A =^{\cap} \{0, 1\}^{(i)}, \quad B =^{\cap} \{a, b\}^{(i)}\,.$$

By DD14.2-4 $X \approx^{(t)} X^{(I)}$ and $X \approx^{(e)} X^{(I)}$ are false except for very particular choices of X; however

$$(27.5)\qquad \vdash X \approx X^{(I)} \quad \left(\text{via } \left\{\left(a, \{a\}^{(i)}\right) | a \in X\right\}\right).$$

By $(24.6)_2$ and D14.1

$$(27.6)\qquad \vdash A \approx B \wedge A \in El^{(e)} \supset B \in El^{(e)},$$

but the analogue for $\approx^{(e)}$ does not hold. To prove the first of the theorems

$$(27.7) \qquad \vdash {}^A B \approx (B^{(I)})^{A^{(I)}} \approx^{(t)} {}^A B \approx^{(e)} (B^{(I)})^{A^{(I)}}$$

it suffices to remark that by D14.4 and DD27.1,2

$$(27.8) \qquad \begin{cases} \vdash F,\ \breve{F} \in Fn \wedge {}^A B = \text{Dmn}\, F \wedge (B^{(I)})^{A^{(I)}} = \text{Rng}\, F \\ \text{for} \\ F =_D \{(f, f^{(2I)}) \mid f \in {}^A B\}\,. \end{cases}$$

By DD13.3,4 $\vdash Fn,\ Fnc \subseteq Abs$. This, DD25.1,2, and DD27.1,2 yield $\vdash [f, \varphi \in {}^A B \vee f,\ \varphi \in (B^{(I)})^{A^{(I)}}] \wedge f = \varphi \supset f =^{\cap} \varphi$. Hence $\vdash h, \breve{h} \in Fn \wedge \text{Dmn}\, h = {}^A B \wedge \text{Rng}\, h \in (B^{(I)})^{A^{(I)}} \supset h,\ \breve{h} \in Fnc$. This, DD14.1,2, and $(27.7)_1$ yield $(27.7)_2$. Thence by $SA(14.1)_1$ we deduce $(27.7)_3$.

28. On infinite Boolean operations.

In correspondence with [IST, Theors. 5.2 (i-iii), 5.11 (i-iii)], by DD3.10,11 we have (28.1-3) below, where $(28.3)_1$ is $(19.8)_3$:

$$(28.1) \qquad \begin{cases} \vdash \bigcup 0 = 0\,, \quad \vdash \mathscr{A} \in In \supset \bigcup \mathscr{A} = 0\,, \\ \vdash \bigcup \{x\} = \bigcup \{x\}^{(i)} = x\,, \\ \vdash \bigcup \{a, b\} = \bigcup \{a, b\}^{\cap} = \bigcup \{a, b\}^{(i)} = a \cup b \qquad (5.2,\ \text{i-iii})\,, \end{cases}$$

$$(28.2) \qquad \begin{cases} \vdash \bigcap 0 = V\,, \quad \vdash \mathscr{A} \in In \supset \bigcap \mathscr{A} = V\,, \\ \vdash \bigcap \{x\} = \bigcap \{x\}^{(i)} = x\,, \\ \vdash \bigcap \{a, b\} = \bigcap \{a, b\}^{\cap} = \bigcap \{a, b\}^{(i)} = a \cap b \qquad (5.11,\ \text{i-iii})\,, \end{cases}$$

$$(28.3) \qquad \vdash \bigcup a \in St \ (5.3)\,, \quad \vdash \mathscr{A} \notin In \cup \{0\} \supset \bigcap \mathscr{A} \in St \ (5.15)\,,$$

$$(28.4) \qquad \vdash M \subseteq Fn \wedge M \in MConst \wedge \forall_{f,g} [f, g \in M \wedge (\text{Dmn}\, f \cap \text{Dmn}\, g) \upharpoonright f = (\text{Dmn}\, f \cap \text{Dmn}\, g) \upharpoonright g] \supset \bigcup M \in Fn \ [\text{D11.14}] \ (5.9)\,.$$

From D3.10,11 and D12.5 we easily deduce that

$$(28.5) \qquad \vdash A \in MConst \wedge A \subseteq MConst \supset \left(\bigcup A,\ \bigcap A \in MConst\right).$$

Our definitions of *first* and *second co-ordinate* of the ordered couple $\mathscr{A}$ (are meaningful, but have no interest, also in case $\mathscr{A}$ is not a couple; hey) read

DD28.1,2 $$1^{st}\mathscr{A} =_D (\imath a)\, \exists_b\, \mathscr{A} = (a, b)\,, \qquad 2^{nd}\mathscr{A} =_D (\imath b)\, \exists_a\, \mathscr{A} = (a, b) \qquad (5.13)\,,$$

and work also in EC^∞ unlike [IST (15.13)] for EC^∞ deals with individuals:

(28.6) $$\vdash 1^{st}(a, b) =^\cap a\,, \qquad \vdash 2^{nd}(a, b) =^\cap b \qquad (5.14, i, ii)\,.$$

We don't need write explicitly the well known theorems in [IST, Secs. 5, 6] on the *indexed product* $\bigcap_{i\in I} A_i$, *indexed sum* $\bigcup_{i\in I} A_i$, and *direct product* $\mathscr{P}A(=\mathscr{P}_{i\in I}A_i)$ of the family A, i.e. the class-valued function A whose domain contains I [I is not to be confused with $\boldsymbol{I}$]:

DD28.3,4 $$\bigcup_{i\in I}\mathscr{A}_i =_D \bigcup \mathrm{Rng}\,(I \upharpoonright \mathscr{A})\,, \qquad \bigcap_{i\in I}\mathscr{A}_i = \bigcap \mathrm{Rng}\,(I \upharpoonright \mathscr{A})\,,$$

DD28.5 $$\mathscr{P}\mathscr{A} =_D \{f|\mathscr{A}, f \in Fn \wedge \mathrm{Dmn}\,f = = \mathrm{Dmn}\,\mathscr{A} \wedge \forall_a(a \in \mathrm{Dmn}\,\mathscr{A} \supset f'a \in \mathscr{A}'a)\}\,.$$

Let us only mention that they hold not only for absolute entities. As examples of direct products we may write

(28.7) $$\begin{cases} \vdash \mathscr{P}0 = \{0\}^{(i)} & (6.2,i) \\ \vdash f \in Fn \wedge \forall_i(i \in \mathrm{Dmn}\,f \supset f_i \neq 0) \wedge f \in St \supset \mathscr{P}f \in St & (6.2, ii)\,. \end{cases}$$

29. On subset classes, modally constant subset classes, and equipotence.

The main analogue for ML^∞ of SA (power set, or subset class of A) is perhaps the strict power set $S^\cap A$ of A [D20.5]. Another acceptable analogue of SA—useful e.g. in the pure theory of ordinals and to construct analogues of QIs within MC^∞ itself—is the *modally constant subset class* $S^{mc}A$:

D.29.1 $$S^{mc}\mathscr{A} =_D (\lambda u)(u \subseteq \mathscr{A} \wedge u \in MConst)\,.$$

The modal analogue for $S^{\cap} A$ $[S^{mc} A]$ of a theorem on SA in [IST] whose proof is based on A4.4, holds in MC^{∞} and its proof is based on A17.4 [AA17.4,11]. Generally the direct modal analogue for S of a theorem in [IST] does not hold in MC^{∞}. As examples we may consider (29.1,7) below where $(29.1)_1$ seems to constrast with [IST (6.7)].

$$(29.1) \quad \vdash Sx \notin St^{(e)}; \quad \vdash S^{\cap} 0 = \{0\}^{(i)}, \quad \vdash S0 = \{0\} \quad (6.6, \text{ii}) .$$

To prove $(29.1)_1$ we remark that by D3.9 and D10.11

$$(29.2) \qquad \vdash A \in Sx , \quad \vdash Sx \neq A , \quad \vdash Sx = (Sx)^{(e)} .$$

From $(29.2)_{2,3}$ and $(24.8)_1$ we deduce $(29.1)_1$.

Direct proofs of $(29.1)_{2,3}$ are easy—cf. D3.9 and D20.5.

By A17.11 (II) and A17.4 we have the first among the theorems ([10])

$$(29.3) \begin{cases} \vdash S^{mc} u \in St , & \vdash S^{mc} A = (SA) \cap MConst , \\ \vdash A \in MConst \supset S^{mc} A \in \text{Abs} , & \vdash A \in MConst \supset S^{mc} A \subseteq S^{\cap} A , \\ \vdash (S^{\cap} A)^{(e)} = (S^{mc} A)^{(e)} = (SA)^{(e)}, & \vdash A = B \supset S^{mc} A = S^{mc} B . \end{cases}$$

D29.1 yields the second. To prove $(29.3)_3$ assume (a) $A \in MConst$ and (b) $x \in^{\cup} S^{mc} A$. Then (c) $x \subseteq^{\cup} A$ and $x \in$ MConst, so that $y \in x$ yields $y \in^{\cup} x$. This yields in turn $y \in^{\cup} A$ by (c), and $y \in A$ by (a). We conclude that (a), (b) $\vdash x \subseteq A$. Since (a) is equivalent to $N(a)$ —cf. SA$(12.1)_1$ and *ft.* ([6])— by the modal rule G, (a), (b) $\vdash x \subseteq^{\cap} A$. Thus by the deduction theorem (a) yields (d) $S^{mc} A \in MConst$. Furthermore by $(29.3)_2 \vdash S^{\ c} A \subseteq MConst$. Then by SA$(12.2)_2$ we get $S^{mc} A \in Abs$. Thus $(29.3)_3$ has been proved.

To prove $(29.3)_4$ assume (a) and $x \in S^{mc} A$. Hence (b). From (a) and (b) we have already deduced $x \subseteq\ A$, i.e. $x \in S^{\cap} A$. Hence (a) yields $S^{mc} A \subseteq S^{\cap} A$. Thus we have proved $(29.3)_4$. To prove $(29.3)_{5,6}$ is easy.

Remark that the antecedent in $(29.3)_4$ is essential because

$$p \diamond \sim p \vdash \{1\}^{\cap} \in (S^{mc} A - S^{\cap} A) \text{ for } A =_D (\imath u)(pu = \{0, 1\}^{\cap} \vee \sim pu = \{0\}^{\cap}),$$

hence $p \diamond \sim p \vdash S^{mc} A \not\subseteq S^{\cap} A$.

([10]) To prove $(29.3)_1$ we get $x = u$ and $x \in MConst$ by A17.11(II) and rule C with x. Then we have $\forall_{\mathcal{A}}(\mathcal{A} \subseteq^{\cap} x \supset \mathcal{A} \in v)$ by A17.4, and rule C with v. Then $S^{mc} u \subseteq v$ easily follows by D29.1. Hence $\vdash S^{mc} u \in El^{(e)}$. This and $(21.8)_3$ yields $(29.3)_1$.

The pure theory of ordinals and cardinals in MC^∞ [Part 3] will be based on Fn and $\approx$, while Fnc and $\approx^{(e)}$ are important to apply this theory to classes whatever [DD13.3,4 and 14.1,3]. In this situation it often happens that an assertion involving Fn and $\approx$ has an analogue for Fnc and $\approx^{(e)}$. To be able to deduce quickly the latter from the former in several cases, we first define n-ary *weakly separated attributes* ($WSep_n$) for $n = 1, 2, \ldots,$—cf. D12.1:

$$\text{D29.2} \quad \mathscr{A} \in WSep_n \equiv_D \forall_{a,b}(a, b \in \mathscr{A} \wedge a =_n b \supset a =^\cap b), \qquad WSep =_D WSep_1 .$$

By DD12.1,4, by D29.1, and by DD14.1-3 respectively we have

$$(29.4) \quad \begin{cases} \vdash MSep \subset WSep , \quad S^{mc}u \in WSep \\ \vdash A, B \in WSep \supset (A \approx B \equiv A \approx^{(t)} B \equiv A \approx^{(e)} B) . \end{cases}$$

We now prove that *every set u is equiextensionalizable with a weakly separated subset v of its*:

$$(29.5) \qquad \vdash \exists_v (v \in WSep \wedge v \subseteq u \wedge u^{(e)} = v^{(e)}) \qquad [\text{DD12.1,2}] .$$

By A17.11 (I) and rule C with R we have $R \in MConst \wedge R = \{(u \cap \{a\}, a) | a \in u\}$ so that $u = \operatorname{Rng} R$. Then by A17.10 and rule C with F and v we easily get $F \in Fn$, $F \subseteq R$, $\operatorname{Dmn} F = \operatorname{Dmn} R$, and $v = \operatorname{Rng} F$. Then $v \in WSep \wedge v \subseteq u \wedge u^{(e)} = v^{(e)}$. Hence (29.5) holds.

It must be remarked that *$WSep$ cannot be replaced by $MSep$ in* (29.5). Indeed the result of this replacement is incompatible with $p \Diamond \sim p$ for $u =_D (\imath x)(x = \{0, 1\}^\cap \vee \sim px = \{0\}^\cap)$.

By DD14.1-4 and D12.4 we have $(29.6)_{1\text{-}3}$ below:

$$(29.6) \quad \begin{cases} \vdash \approx^{(t)} \subset \approx , \quad \vdash \sim (\approx \subseteq \approx^{(e)}) , \quad \vdash A^{(I)} \in MSep \\ \vdash A \approx B \equiv A^{(I)} \approx B^{(I)} \equiv A^{(I)} \approx^{(e)} B^{(I)} , \quad \vdash A^{(e)} = B^{(e)} \supset A \approx^{(e)} B , \end{cases}$$

where $(29.6)_{1,2}$ are $\text{SA}(14.1)_{1,2}$. By $(29.6)_3$ and $(29.4)_{1,3}$ $\vdash A^{(I)} \approx B^{(I)} \equiv A^{(I)} \approx^{(e)} B^{(I)}$. To prove the remaining part of $(29.6)_4$ and $(29.6)_5$ is easy.

Remark that a suitable analogue of $\mathscr{A}^{(I)}$ for $\{\,\}$ is the notion of *intrinsic extension class* $\mathscr{A}^{(E)}$ of $\mathscr{A}$ (this notion is certainly a class for $\mathscr{A} \in St$):

D29.3 $\quad \mathscr{A}^{(E)} \equiv_D (\lambda x)\, \exists_a (a \in \mathscr{A} \wedge x = \{a\} \cap \mathscr{A} \wedge x \in MConst)$.

Then by DD12.4,6 and D29.1

$$(29.7) \quad \vdash u^{(E)} \in MSep\,, \qquad \vdash S^{mc} u^{(E)} \in MSep\,, \qquad u \in MConst \supset u^{(E)} \in Abs\,.$$

Furthermore by D14.3 we easily deduce the first of the theorems

$$(29.8) \quad \vdash u \approx^{(e)} u^{(E)}\,, \qquad \vdash u \approx^{(e)} v \equiv u^{(E)} \approx v^{(E)} \equiv u^{(E)} \approx^{(e)} v^{(E)}\,,$$
$$\vdash \exists_v (v \in Abs \wedge v \approx^{(e)} u)\,.$$

The second follows easily from $(29.8)_1$ and from $(29.7)_1$ and $(29.4)_{1,3}$. To prove $(29.8)_3$ remark that by A17.11 (I) and rule C with v we get (a) $v = u^{(E)}$ and (b) $v \in MConst$. We have (c) $v \approx^{(e)} u^{(E)}$ by (a), $(29.8)_1$, and the extensional character of $\approx^{(e)}$. Furthermore (a) and $(29.7)_1$ yield $v \in MSep$. Thence by (b) and D12.6 we get $v \in Abs$. By this and (c), $(29.8)_3$ holds.

Generally in pure mathematics only absolute classes and the modally constant subclasses of them are considered (so that these subclasses also are modally constant). Then the best modal version of Cantor's theorem [IST, (6.8)] is perhaps the first of the theorems

$$(29.9) \qquad \vdash \sim (u \approx S^{mc} u)\,, \qquad \vdash \sim [u^{(E)} \approx^{(e)} S^{mc} u^{(E)}]$$

in case u is regarded as absolute; for u arbitrary $(29.9)_2$ seems the most satisfactory among the aforementioned versions.

The following proof of $(29.9)_1$ for any u is similar with the one of [IST (6.8)]. To prove precisely that

$$(29.10) \qquad \vdash \sim \exists_F (\breve{F} \in Fn \wedge F \in S^{mc} v \to v)$$

we start with (a) $\breve{F} \in Fn \wedge F \in S^{mc} v \to v$. Then by A17.11 and rule C with y we deduce $y \in MConst$ and $y = \{F'x | F'x \notin x \wedge x \in S^{mc} v\}$. Hence $y \in S^{mc} v$. Then $F'y \notin y$ yields $F'y \in y$ and conversely. So assumption (a) is absurd, which allows us to assert (29.10), hence $(29.9)_1$.

From $(29.9)_1$ we deduce $\sim [u^{(E)} \approx S^{mc} u^{(E)}]$ which by $(29.7)_{1,2}$ and $(29.4)_{1,3}$ yields $(29.9)_2$.

30. On equivalence relations.

In nn. 30, 31 we consider the analogues for MC^{∞} of Sects. 7 and 8 in [IST]. This is trivial as far as absolute concepts are referred to, and is also straightforward in the general case, provided our usual changes be performed in connection with restricted variables, collections ($\{\ \} \to \{\ \}^{(i)}$), and identity ($= \to =^{\cap}$), and provided functions be turned into *Fn s* and one to one functions be turned into $\mathscr{F}$'s for which $\mathscr{F}, \breve{\mathscr{F}} \in Fn$ holds. These changes imply e.g. the use of (*intensionally*) *reflexive relations* (*Rfl*), (*intensional*) *partial order* (*POrd*) [n. 31], and so on—see D30.1 and D31.1 below. However outside pure number theory the corresponding total notions (based on *Fnc* and not on *Fn*) may be of interest. Among them is the notion of *totally or extensionally reflexive relation* ($Rfl^{(t)}$). In connection with the aforementioned total notions some essentially modal considerations are required to extend Sects. 7 and 8 in [IST] to MC^{∞}. I think that nn. 30, 31 and in particular the proofs of (30.2) and (31.5) below will make it quite clear how to handle in MC^{∞} with similarity of order structures.

In accordance with tradition we should say (in ML^{∞}) that the relation R is *Rfl* [$Rfl^{(t)}$] in case for all $x, y \in \text{Fld}\, R$, $x =^{\cap} y$ [$x = y$] implies xRy [Conv. 23.1]. However in accordance with [IST, Th. 7.2, iv] it is equivalent (and shorter) to state the following definitions—cf. D.5.1 and D11.9:

DD30.1,2
$$R \in \left\{ \begin{array}{l} \text{Rfl} \\ \text{Rfl}^{(t)} \end{array} \right. =_D R \subseteq V^2 \wedge \left\{ \begin{array}{l} (\text{Fld}\, R) \upharpoonright \boldsymbol{I}^{\cap} \\ (\text{Fld}\, R)^2 \cap \boldsymbol{I} \end{array} \right. \subseteq R\,.$$

Transitive relations (*Trns*), (*intensional*) *equivalences* (*Eqv*) or *extensional equivalences* (*EEqv*) *can be defined in* ML^{∞} *in the same* (customary) way as in EL^{∞}—cf. [IST, Theor. 7.2, i, iii]:

DD30.3-5
$$R \in Trns \equiv_D R \subseteq V^2 \wedge R|R \subseteq R\,, \qquad R \in Eqv \equiv_D R|\breve{R} = R\,,$$
$$EEqv =_D Eqv \cap Rfl^{(t)}.$$

For the notions *Rfl*, *Trns*, and *Eqv* we can prove in MC^{∞} the theorems that are well known in extensional logic, in particular the following:

(30.1) $$\vdash R \in Eqv \equiv R \in Trns \wedge \breve{R} = R .$$

By DD30.1,2 the first of the essentially modal theorems

(30.2) $\vdash Rfl^{(t)} \subseteq Rfl$, $\vdash Rfl^{(t)} \cap Trns \subseteq Ext_2$,

$\vdash EEqv = Ext_2 \cap Eqv$ [D12.3]

holds. To prove $(30.2)_2$ we first assume (a) $R \in Rfl^{(t)}$, (b) $R \in Trns$, (c) $a, b, c, d \in \mathrm{Fld}\, R \wedge a = c \wedge b = d$, and aRb. From (a) and (c) we deduce cRa and bRd. Hence by (b) and aRb we have cRd. Now we easily conclude, by D12,3, that (a), (b) $\vdash R \in Ext_2$. Hence $(30.2)_2$ holds.

By (30.1), $(30.2)_2$, and D30.5, (d) $\vdash EEqv \subseteq Ext_2 \cap Eqv$. With a view to proving the converse of (d) we assume (e) $R \in Eqv$, (f) $R \in Ext_2$, and (g) $a, b \in \mathrm{Fld}\, R \wedge a = b$. Then, using rule C with c, we easily get aRc, so that by (g) and (f), bRc also holds. By (e) this yields aRb. Hence (e) and (f) yield $\forall_{a,b}[(g) \supset aRb]$. Now we easily conclude by D30.2 that (e), (f) $\vdash R \in Rfl^{(t)}$.

Hence by D30.5, $\vdash Eqv \cap Ext_2 \subseteq Rfl^{(t)} \cap Eqv = EEqv$. Now by ($d$) we have $(30.2)_3$. q.e.d.

Unlike the R-equivalence class represented by a, the *class* $a//R$ *of* R*-equivalence classes with representatives in* a must be redefined in ML^∞[DD5.2,3]. In correspondence with [IST, Def. 7.3, ii, iii] we state the definitions

DD30.6,7 $A//R =_D (\lambda x) \exists_a (a \in A \wedge x =^\cap a/R)$, $\pi_R =_D (\mathrm{Fld}\, R//R)$.

Now the well known theorems of Eqv, a/R. $a//R$, and π_R can be asserted in MC^∞. For instance [IST, Th. 7.4] holds in MC^∞ with no change.

31. Ordering.

Now we define *partial order* (*POrd*) *R-minimal element of* X (R-min El X), *well founded relation* (*WFnd*), *simple ordering* (*SOrd*), *well ordering* (*WOrd*), the corresponding *extensional* or *total* notions $POrd^{(t)}$ to $WOrd^{(t)}$, and A *is R-directed* ($A \in R$*-directed*) [n. 4]—cf.[IST, 8.1,6, 8, 10, 12] (11).

DD31.1,2 $$R \in \left\{ \begin{matrix} POrd \\ POrd^{(t)} \end{matrix} \right. \equiv_D R \in Trns \wedge R|\breve{R} \subseteq \left\{ \begin{matrix} I^{\cap} \\ I \end{matrix} \right. ,$$

DD31.3,4 $$b \in \left\{ \begin{matrix} R\text{-min El}\, X \\ R\text{-min El}^{(t)}\, X \end{matrix} \right. \equiv_D b \in X \forall_a [a \in X \wedge (a, b) \in R \supset \supset \left\{ \begin{matrix} a =^{\cap} b] \\ a = b] \end{matrix} \right. ,$$

DD31.5,6 $$R \in \left\{ \begin{matrix} WFnd \\ WFnd^{(t)} \end{matrix} \right. \equiv_D R \subseteq V^2 \forall_X [\Lambda \neq X \subseteq \text{Fld}\, R \supset \exists_b (b \in \in \left\{ \begin{matrix} R\text{-min El}\, X) \\ R\text{-min El}^{(t)}\, X) \end{matrix} \right. ,$$

DD31.7,8 $$R \in \left\{ \begin{matrix} SOrd \\ SOrd^{(t)} \end{matrix} \right. \equiv_D R \in \left\{ \begin{matrix} POrd \\ Pord^{(t)} \end{matrix} \right. \wedge R \cup \breve{R} = (\text{Fld}\, R)^2 \quad [\text{D11.9}],$$

DD31.9,10 $$\left\{ \begin{matrix} R \in WOrd \equiv_D R \in SOrd \wedge R \in WFnd \\ R \in WOrd^{(t)} \equiv_D R \in SOrd^{(t)} \wedge R \in WFnd^{(t)} \end{matrix} \right.$$

DD31.11 $A \in R\text{-}directed \equiv_D a, b \in A \supset \exists_c \big(c \in A \wedge (a, c), (b, c) \in R\big)$.

No change is required in the usual definitions of notions such as *R-lower bound* of X, *R-least element* of X, and *R-greatest lower bound* of X (R-g.l.b. X).

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(11) The direct analogues of the definitions [IST, (8.6,vii), (8.12,i)] of the R-minimal element of X and well founded relations are

D31.3′ $$b \in R\text{-min El}\, X \equiv_0 b \in X \wedge \sim \exists_a (a \in X \wedge aRb) ,$$

D31.5′, 6′ $$R \in \left\{ \begin{matrix} WFnd \\ WFnd^{(t)} \end{matrix} \right. \equiv_D R \subseteq V^2 \; \forall_X [\Lambda \neq X \subseteq \text{Fld}\, R \supset \exists_b b \in \in \left\{ \begin{matrix} (R - I^{\cap}) - \text{min El}\, X] \\ (R - I) \quad - \text{min El}\, X] \end{matrix} \right. .$$

We did otherwise in order to consider the two modal notions of R-minimal elements above.

Their well known properties, in particular the *fixed point theorem* [IST, Theor. 8.7] hold, and some useful characterizations of *WOrd* [IST, Theors, 8.12,13] can be proved in MC^{∞} as in EC^{∞}. The same holds for the following analogue of [IST, Th. 8.9]:

$$(31.1) \qquad \vdash A \subseteq \left| \begin{array}{l} Fn \\ Fnc \\ POrd \end{array} \right. \wedge A \in \subseteq\text{-}directed \supset \bigcup A \in \left| \begin{array}{l} Fn \\ Fnc \\ POrd \end{array} \right.$$

Incidentally

$$(31.2) \qquad \vdash \{(x, y) | x \subseteq y\} \in POrd^{(t)}, \qquad \vdash \{(x, y) | x \subseteq^{\cap} y\} \in \mathrm{POrd}\ (8.4,\ \mathrm{i})\ .$$

We now introduce the notions of an (*intensional*) *isomorphism* and a *total isomorphism from R onto S*:

D31.12 $\quad \mathrm{Ism}\,(F, R, S) \equiv_D R, S \in V^2 \wedge F, \breve{F} \in Fn \wedge \mathrm{Dmn}\, F =$
$= \mathrm{Fld}\, R \wedge \mathrm{Rng}\, F = \mathrm{Fld}\, S \wedge \forall_{a,b}\{a, b \in \mathrm{Fld}\, R \supset [aRb \equiv (F'a) R (F'b)]\} \quad (8.3).$

D31.13 $\quad \mathrm{Ism}^{(t)}\,(F, R, S) \equiv_D R, S \subseteq V^2 \wedge F, \breve{F} \in Fnc \wedge (\mathrm{Dmn}\, F)^{(e)} =$
$= (\mathrm{Fld}\, R)^{(e)} \wedge (\mathrm{Rng}\, F)^{(e)} = (\mathrm{Fld}\, S)^{(e)} \wedge$
$\wedge \forall_{a,b}\{a, b \in \mathrm{Fld}\, F \supset [(a, b) \in R^{(2e)} \equiv (F'a, F'b) \in S^{(2e)}\}\ .$

THEOR. 31.1. *It can proved in* MC^{∞} *that the notions POrd, WFnd, SOrd, and WOrd and the corresponding notions* $POrd^{(t)}$ *to* $\mathrm{WOrd}^{(t)}$ *are extensional, and that the notion* $A \in R$*-directed* [D31.11] *is extensional with respect to A and R while* Ism (F, R, S) *and* $\mathrm{Ism}^{(t)}$ (F, R, S) *are so with respect to R and S. For instance*

$$(31.3) \quad \vdash R = T \wedge T \in POrd \supset R \in POrd\ , \qquad T = R \wedge \mathrm{Ism}\,(F, T, S) \supset \supset \mathrm{Ism}\,(F, R, S)\ .$$

We proved in MC^{∞} that *any transitive and totally reflexive relation is innerly extensional*—cf. $(30.2)_2$. Hence by DD31.2, 8, 10

$$(31.4) \quad \vdash POrd^{(t)} \subseteq Ext_2\ , \qquad \vdash SOrd^{(t)} \subseteq Ext_2\ , \qquad \vdash WOrd^{(t)} \subseteq Ext_2\ .$$

On the basis of [IST] we can easily prove the following theorem, where « $x \subseteq y$ » can be replaced by « $x \subseteq^{\cap} y$ »—cf. D11.9:

(31.5) $\vdash R \in P\mathrm{Ord} \supset \exists_{u,F} \mathrm{Ism}(F, R, S)$ where $S =_D u^2 \cap \{(x, y) | x \subseteq y\}$.

PROOF. We assume (a) $R \in P\mathrm{Ord}$, i.e. $R \in POrd$ and $R \in El$ [Conv. 3.1]. By A17.11 and rule C with T we obtain (b) $T = R \wedge T \in MConst$, so that by (a), $(31.3)_1$, and $(19.11)_1$ we have (a') $T \in P\mathrm{Ord}$.

Now we can repeat on T the reasoning made in the proof of [IST, Th. 8.4, ii] on R. In particular we set

$$(31.6) \quad \begin{cases} & F =_D \{(a, \{b | (b, a) \in T\}) | a \in \mathrm{Fld}\, T\}, \\ \text{hence} & \\ & F \in Fn \wedge \mathrm{Dmn}\, F = \mathrm{Fld}\, T. \end{cases}$$

Thus for any $a \in \mathrm{Fld}\, R$ we have (c) $F'a = \{b | (b, a) \in T\}$. Set

$$(31.7) \qquad S =_D (\mathrm{Rng}\, F)^2 \cap \{(x, y) | x \subseteq y\}, \qquad \text{hence } \mathrm{Rng}\, F = \mathrm{Fld}\, S.$$

If $a, b \in \mathrm{Fld}\, T$ and $a \neq b$, then by antisymmetry $(a, b) \notin T$ or $(b, a) \notin T$, and hence $a \in (F'a - F'b)$ or $b \in (F'b - F'a)$. Then (d) $\breve{F} \in Fn$. We conclude that $(a) \vdash_C (d)$.

Now we also assume (e) $a, b \in \mathrm{Fld}\, T$ and (f) aTb. For any $c \in F'a$ we have cTa, which by (f) and (a') yields cTb, hence $c \in F'b$. We conclude that $F'a \subseteq F'b$ holds, hence (g) $(F'a) S (F'b)$. We easily obtain $\forall_{a,b}[(e) \wedge (f) \supset (g)]$.

Now we assume (e) and $\sim (f)$. Then $a \in (F'a - F'b)$ and so $F'a \not\subseteq F'b$, i.e. $(F'a, F'b) \notin S$, i.e. $\sim (g)$. We easily conclude that $\forall_{a,b}\{(e) \supset [(f) \equiv (g)]\}$, i.e. $\forall_{a,b}\{a, b \in \mathrm{Fld}\, T \supset [aTb \equiv (F'a) S (F'b)]\}$. Remembering (d), $(31.6)_2$, $(31.7)_2$, and D31.12, we have $\mathrm{Ism}(F, T, S)$ whence, by $(31.3)_2$, (b) yields (h) $\mathrm{Ism}(F, R, S)$, where T does not occur. We have $(a) \vdash_C (h)$ which by Th. 16.1 yields $(31.5)_1$. q.e.d.

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