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OPTIMIZATION BY A METHOD OF MAXIMUM SLOPE  
IN THE COMPLEX PLANE  
AND ITS APPLICATION TO THE TRANSPORTATION PROBLEM

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For the sake of completeness we include some results from a previous paper on regular graphs, which has been submitted for publication elsewhere, on the basic ideas of the maximization of a special simple linear functional by the method of maximum slope for vectors in the complex plane.

The new part consists of considerations of multiplicities of points in the complex plane which, when translated to applications to graphical problems, introduces multiple arcs. In the following particular application, the multiple arcs are those of a bipartite graph.

Maximization by the Method of Steepest Slope.

Consider the following problem: Given  $m$  distinct points  $z_1, z_2, z_3, \dots, z_m$  in the open sector  $D$  of the complex plane consisting of the first quadrant, excluding the real and imaginary axes, to find  $h \leq m$  of the points whose vector sum will have a maximum slope (or argument).

If  $z = x + iy$ , define  $\|z\| = \frac{y}{x}$  so that for  $z \in D$ ,  $\|z\| \neq 0$ , and  $\|\lambda z\| = \|z\|$  for  $\lambda > 0$ .

**THEOREM 1.** (CAUCHY)  $\|z_1\| \geq \|z_1 + z_2\| \geq \|z_2\|$  if and only if  $\|z_1\| \geq \|z_2\|$ .

**PROOF.** If  $\|z_1\| \geq \|z_2\|$  then  $\frac{y_1}{x_1} \geq \frac{y_2}{x_2}$ . Since all the elements are

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positive, it is easy to verify that

$$\frac{y_1}{x_1} \geq \frac{y_1 + y_2}{x_1 + x_2} \geq \frac{y_2}{x_2}.$$

COROLLARY.  $\max \| z_i \| \geq \| \sum_i z_i \| \geq \min \| z_i \|.$

PROOF. Order the  $z_i$ 's with respect to their slopes so that after relabelling the points

$$\begin{aligned} \| u_1 \| &= \max \| z_i \| = \| z_i \|, \\ \| u_2 \| &= \max_{i, i \neq i_1} \| z_i \| = \| z_i \|, \text{ etc.} \end{aligned}$$

so that

$$\| u_1 \| \geq \| u_2 \| \geq \| u_3 \| \geq \dots \geq \| u_m \|.$$

By Theorem 1,

$$\| u_1 \| \geq \| u_1 + u_2 \| \geq \| u_2 \|.$$

Considering  $u_1 + u_2$  as a single term, again by Theorem 1,

$$\| u_1 \| \geq \| u_1 + u_2 + u_3 \| \geq \| u_3 \|, \text{ etc.}$$

Thus  $\| u_1 \| \geq \| u_1 + u_2 + \dots + u_m \| \geq \| u_m \|.$

COROLLARY.  $\| z_1 + z_2 \| \leq \| z_1 \| + \| z_2 \|.$

The proof is immediate from Theorem 1. Thus the triangular law holds so that we shall call  $\| z \|$  a quasi-norm of  $z$ , and hereafter we shall refer to it as simply the norm of  $z$ .

LEMMA. A binomial can be of maximum norm if and only if it contains as one term the element of maximum norm.

PROOF. As before, we assume that

$$\| u_1 \| \geq \| u_2 \| \geq \| u_3 \| \geq \dots \geq \| u_m \|$$

and let

$$\max \| u_i + u_j \| = \| u_{i_1} + u_{j_1} \|;$$

then, by Theorem 1,

$$\begin{aligned} \|u_1\| &\geq \max(\|u_{i_1}\|, \|u_{j_1}\|) \geq \|u_{i_1} + u_{j_1}\| \geq \\ &\|u_1 + u_2\| \geq \|u_2\| \geq \min(\|u_{i_1}\|, \|u_{j_1}\|) \end{aligned}$$

or

$$\max(\|u_{i_1}\|, \|u_{j_1}\|) \geq \|u_1 + u_2\| \geq \|u_2\|.$$

Thus

$$\max(\|u_{i_1}\|, \|u_{j_1}\|) = \|u_2\| \text{ if } \|u_1\| = \|u_2\|$$

so that, in general, we must have

$$\max(\|u_{i_1}\|, \|u_{j_1}\|) = \|u_1\|.$$

Conversely,

$$\|u_1\| \geq \max\|u_1 + u_i\| = \|u_1 + u_{i_2}\| \geq \|u_1 + u_2\| \geq \|u_2\| \geq \|u_{i_2}\|,$$

so that  $\|u_1 + u_{i_2}\|$  is equal to the maximal-norm binomial above.

By repeated applications of the lemma, we have

**THEOREM 2.** A maximal-normed  $h$ -termed,  $h \leq m$ , sum can be found by sequentially maximizing the norms of the partial sums of points in  $D$ , starting with the element of maximal norm.

Thus Theorem 2 describes the solution to the problem of how to find a subset of  $h \leq m$  elements from the given collection  $z_1, z_2, \dots, z_m$  whose sum will have maximal slope.

Suppose as before that the points  $z_1, z_2, \dots, z_m$  are distinct but that we now allow integral multiplicities; i.e.,  $z_1$  can be counted up to  $a_1 > 0$  times,  $z_2$  up to  $a_2$  times, etc. We now ask how one obtains a linear combination of some subset of  $h \leq m$  points which will have a maximum slope when the sum of all the coefficients actually used is a given constant.

Here we must now determine two different sets of quantities; viz., the subset of  $h \leq m$  points in the complex plane, and the proper set of corresponding integral coefficients satisfying the constraints that go along with them.

Note that when the points in  $D$  are weighted, the variation of the weights distributes the norm of the linear combination from one extre-

me to the other: e.g., if

$$\|u_1\| > \|u_2\|, \text{ then } \|u_1\| > \|c_1u_1 + c_2u_2\| > \|u_2\|$$

and the relative weights  $c_1 > 0$  and  $c_2 > 0$ ; when varied, vary the mean value between the indicated bounds. Thus, when one has a descending chain of norms, in order to maintain a maximum norm of partial sum of linear combination of vectors, it is necessary that the dominant terms are weighted maximally and yet maintain a lower bound which is as great as possible. If there are several points of maximal norm, then the one with the greatest magnitude is the one with the maximal inherent weight.

LEMMA 1. If, in a binomial of maximum norm in the Lemma of Theorem 2, the element of maximal norm is repeated  $e_1$  times, and the second term  $u_{i_2} = v_2$  in which we assume  $v_2 \neq u_2$ , can be repeated several times, then the maximum number of repetitions of the second term is  $e_2$  if the binomial is to maintain the same greatest lower bound  $\|u_2\|$ , as in the Lemma of Theorem 2, where  $e_2 = [\lambda_2]$ ; i.e., the greatest integer contained in  $\lambda_2$  and where

$$\lambda_2 = \frac{e_1[I(u_2)R(u_1) - I(u_1)R(u_2)]}{I(v_2)R(u_2) - I(u_2)R(v_2)}$$

The  $I$ 's and  $R$ 's are the imaginary and real parts of the arguments.

PROOF. Let

$$\|u_1\| \geq \|u_2\| \geq \dots \geq \|u_m\|.$$

Then  $\max_{i \in I'} \|e_1u_1 + u_i\| = \|e_1u_1 + u_{i_2}\|$ , where  $I'$  denotes a deleted subset of the index set  $I$ .

Let  $u_1 = v_1$  and  $u_{i_2} = v_2$  so that

$$\|e_1v_1 + v_2\| \geq \|u_2\| \geq \|u_3\| \geq \dots \geq \|u_{i_2-1}\| \geq \|u_{i_2+1}\| \geq \dots \geq \|u_m\|.$$

The greatest lower bound is attained if

$$\|e_1v_1 + \lambda_2v_2\| = \|u_2\|$$

or

$$\frac{e_1 I(u_1) + \lambda_2 I(v_2)}{e_1 R(u_1) + \lambda_2 R(v_2)} = \frac{I(u_2)}{R(u_2)}$$

whence

$$\lambda_2 = \frac{e_1 [I(u_2)R(u_1) - I(u_1)R(u_2)]}{I(v_2)R(u_2) - I(u_2)R(v_2)}.$$

Thus,

$$\| e_1 v_1 + e_2 v_2 \| \geq \| u_2 \|.$$

If the denominator for  $\lambda_2$  vanishes, then  $\| u_2 \| = \| v_2 \|$ , and  $\lambda_2$  is unbounded, so we take  $e_2 = a_{i_2}$ ; i.e.,

$$e_2 = \min(a_{i_2}, [\lambda_2]).$$

LEMMA 2. In order that the vector  $u_{i_1}$  determined by

$$\max_i \| u + u_i \| = \| u + u_{i_1} \|,$$

where  $u$  is any vector, also be the one determined by

$$\max_i \| u + \lambda u_i \|; \text{ i.e., } \max_i \| u + \lambda u_i \| = \| u + \lambda u_{i_1} \|,$$

it is sufficient to take  $1 \leq \lambda < \lambda_g$  where

$$\lambda_g = \frac{I(u_g)R(u) - I(u)R(u_g)}{I(u_g)R(u_{i_1}) - I(u_{i_1})R(u_g)},$$

and where  $\| u_g \|$  is the greatest lower bound of  $\| u + u_i \|$ .

PROOF. Since  $\| u + u_{i_1} \| \geq \| u_g \|$ , the extreme  $\lambda$  is determined when

$$\| u + \lambda u_{i_1} \| = \| u_g \|;$$

i.e., when  $\lambda$  satisfies

$$\frac{I(u) + \lambda I(u_{i_1})}{R(u) + \lambda R(u_{i_1})} = \frac{I(u_g)}{R(u_g)}$$

Thus

$$\lambda_g = \frac{I(u_g)R(u) - I(u)R(u_g)}{I(u_g)R(u_{i_1}) - I(u_{i_1})R(u_g)}.$$

If  $\|u_{i_1}\| = \|u_g\|$ , then any finite positive  $\lambda$  will do.

**THEOREM 3.** If  $\|u_1\| \geq \|u_2\| \geq \dots \geq \|u_m\|$  and  $u_k$  can be counted up to  $a_{i_k}$  times, then there exist  $h \leq m$  positive integers  $e_k \leq a_{i_k}$ ,  $k=1, 2, \dots, h$  with  $\sum_{k=1}^h e_k = K$  a constant, such that  $\|\sum_{k=1}^h e_k v_k\|$  is maximal over all possible subsets of  $h$  vectors  $v_i$  of the  $mu_i$ 's.

**PROOF.** By Lemma 1, we have that the binomial has dominant norm; viz.,

$$\begin{aligned} \|e_1 v_1 + e_2 v_2\| &\geq \|u_2\| \geq \|u_3\| \geq \dots \\ &\geq \|u_{i_2-1}\| \geq \|u_{i_2+1}\| \geq \dots \geq \|u_m\|. \end{aligned}$$

If  $v_2 = u_2$ , then  $e_2 = a_{i_2}$ , and the sequence would be

$$\|e_1 v_1 + e_2 v_2\| \geq \|u_3\| \geq \|u_4\| \geq \dots \geq \|u_m\|.$$

(Here, obviously,  $\|e_1 v_1 + e_2 v_2\| \geq \|u_2\|$ , but let us not display in the descending sequence of norms that element which is already contained in the linear combination partial sum expression on the left).

By the Lemma of Theorem 2, in the last two descending sequences of norms, the binomial of maximal norm must necessarily contain the expression of maximal norm considered as a single vector. Thus

$$\begin{aligned} \max_{i \in I'} \|(e_1 v_1 + e_2 v_2) + u_i\| &= \|e_1 v_1 + e_2 v_2 + u_{i_3}\| = \\ &= \|e_1 v_1 + e_2 v_2 + v_3\| \end{aligned}$$

with  $u_{i_3} = v_3$ . Again  $e_3$  is chosen so that Lemma 2 is applicable; viz.,

$$e_3 = \min(a_{i_3}, [\lambda_3]),$$

where  $\lambda_3$  is determined analogously, as in the Lemma. Thus we obtain the modified descending sequence of norms with the maximal normed vector being  $e_1 v_1 + e_2 v_2 + e_3 v_3$ . In general, proceeding in this manner, we determine the maximal normed  $r$ -termed linear combination by taking the  $r-1$ -termed maximal-normed expression and finding

$$\max_{i \in I'} \left\| \sum_{k=1}^{r-1} e_k v_k + u_i \right\| = \left\| \sum_{k=1}^{r-1} e_k v_k + u_{i_r} \right\| = \left\| \sum_{k=1}^{r-1} e_k v_k + v_r \right\|$$

with  $u_{i_r} = v_r$ . Now take

$$e_r = \min(a_{i_r}, [\lambda_r])$$

where

$$\lambda_r = \frac{I(u_g) \sum_{k=1}^{r-1} e_k R(v_k) - R(u_g) \sum_{k=1}^{r-1} e_k I(v_k)}{I(v_r)R(u_g) - R(v_r)I(u_g)}$$

where  $u_g$  is a vector not contained in the linear combination  $\sum_{k=1}^{r-1} e_k v_k$  and  $\|u_g\|$  is its greatest lower bound; i.e.,

$$\left\| \sum_{k=1}^{r-1} e_k v_k \right\| \geq \|u_g\|.$$

Finally, the  $h$ -termed linear combination of maximum norm with  $\sum_{k=1}^h e_k = K$  is  $\sum_{k=1}^h e_k v_k$  where  $e_h = K - \sum_{k=1}^{h-1} e_k$ , and  $v_h$  is determined by

$$\begin{aligned} \max_{i \in I} \left\| \sum_{k=1}^{h-1} e_k v_k + e_h u_i \right\| &= \left\| \sum_{k=1}^{h-1} e_k v_k + e_h u_{i_h} \right\| = \\ &= \left\| \sum_{k=1}^{h-1} e_k v_k + e_h v_h \right\| = \left\| \sum_{k=1}^h e_k v_k \right\|, \end{aligned}$$

where  $u_{i_h} = v_h$ . To show that this last expression is the linear combination with maximal norm and is unique up to equivalent norms, assume that another form exists such that

$$\left\| \sum_{i=1}^h f_i y_i \right\| > \left\| \sum_{i=1}^h e_i v_i \right\|$$

for another choice of  $h$  vectors  $y_i$  of the original  $m$   $z_i$ 's with different positive integral coefficients  $f_i$  with  $\sum_{i=1}^h f_i = K$ .

Let  $\max_i \|y_i\| = \|Y_1\|$ . By the Lemma of Theorem 2, we find



that we must have  $Y_1 = v_1$ ; moreover, for maximality of the norm of any linear combination containing  $Y_1$ , we must weight it maximally:  $f_1 = e_1$ . Also, they both must contain the element  $v_2$ , since  $\|e_1 v_1 + v_2\| > \|f_1 Y_1 + y_i\|$ , unless  $y_i = v_2$ . Thus,

$$\|e_1 v_1 + v_2\| = \max_{i \in I'} \|f_1 Y_1 + y_i\| = \|f_1 Y_1 + Y_2\| = \|f_1 v_1 + v_2\|.$$

The only way in which  $\|f_1 Y_1 + f_2 Y_2\| > \|e_1 v_1 + e_2 v_2\|$  is if  $f_2 < e_2$ .

Applying Lemma 2 in determining the maximal trinomial and using the binomials as  $u$  (in Lemma 2), we determine  $v_3$ , which is  $= Y_3$  again. The norm of the linear form will exceed the norm of the original; viz.,

$$\|f_1 v_1 + f_2 v_2 + f_3 v_3\| > \|e_1 v_1 + e_2 v_2 + e_3 v_3\|$$

if and only if  $f_3 < e_3$ , if  $f_2 = e_2$ . We see that the repeated application of Lemma 2 determines the same vectors; i.e.,  $v_i$ 's; and that for  $\|\sum f_i y_i\| > \|\sum e_i v_i\|$  we must have  $f_i \leq e_i$  with at least one of them  $f_{i_0} < e_{i_0}$ . Thus we see that  $\sum f_i < \sum e_i = K$ , which contradicts our assumption.

We now apply the foregoing theory to the transportation problem.

Given two finite sets of points  $A = \{A_1, A_2, \dots, A_m\}$  and  $B = \{B_1, B_2, \dots, B_n\}$  called the source or origin points, and the destination or terminal points, respectively, such that at each source point  $A_i$ , there are exactly  $\alpha_i$  items or commodities to be transported to some, one, or all the destination points; and at each destination point  $B_j$ , exactly  $\beta_j$  items or commodities must be received from one, some or all the origin points. The cost of shipping one item from  $A_i$  to  $B_j$  is  $c_{ij}$ . What should be the allocation of goods from all the source points to the destination points such that the total cost of transportation of all items  $A$  to  $B$  is a minimum? Thus the classical formulation for a linear-programming solution is to maximize the linear functional

$$\sum c_{ij} x_{ij},$$

where the  $x_{ij}$ 's are the integral number of items transported from  $A_i$  to  $B_j$ , over all possible assignments under the linear constraints of the problem.

As is well-known, even for a moderate-size matrix, the linear programming solution is very lengthy and time-consuming. We now reformulate the problem in such a way that a direct solution can be found with relative ease by use of the above theory.

Since  $A_i$  is stocked with  $\alpha_i$  items and  $B_j$  can receive  $\beta_j$  items, let

$$\xi_i = \sum_j c_{ij} \beta_j$$

and

$$\eta_j = \sum_i c_{ij} \alpha_i .$$

Define the weight of a single connection or arc between  $A_i$  and  $B_j$  by

$$w_{ij} = \xi_i + \eta_j - c_{ij} .$$

Summing the weights of the individual arcs for any permissible assignment of transporting all of the items, we have

$$W = \sum \xi_i + \sum \eta_j - C$$

where  $C$  is the total cost of the transportation and  $W$  is the sum of the weights of the individual connections or arcs of the bipartite graph (counting the multiplicities of arcs). We note that the conservation principle holds in that

$$\sum \xi_i = \sum \eta_j = K, \text{ a constant}$$

so that we have

$$W = 2K - C.$$

Thus if we can maximize  $W$  over all possible assignments, we can minimize  $C$ , or equivalently, since

$$\frac{W}{C} = \frac{2K}{C} - 1,$$

if we can maximize the ratio  $\frac{W}{C}$  over all possible assignments, we can minimize  $C$ , since  $K$  is a constant.

Now consider each number pair  $(c_{ij}, w_{ij}) = z_{ij}$  as a point in sector

$D$  of the complex plane. Let

$$\begin{aligned} \max_{i, j} \| z_{ij} \| &= \| z_{i_1 j_1} \| = \| u_1 \| \\ \max_{i, j \in I'} \| z_{ij} \| &= \| z_{i_2 j_2} \| = \| u_2 \| \\ &\text{etc.} \end{aligned}$$

so that

$$\| u_1 \| \geq \| u_2 \| \geq \dots \geq \| u_{mn} \| .$$

Associated with every segment corresponding to  $z_{ij}$  we have a number pair  $(\alpha_i, \beta_j)$  indicating the original maximum possible degree of the end points  $A_i$  and  $B_j$ ; (the degree of a point is the number of arcs emanating from it) viz.,  $\alpha_i$  and  $\beta_j$  respectively. With  $z_{i_1 j_1} = u_1$  is associated the number pair  $(\alpha_{i_1}, \beta_{j_1})$ . To simplify the notation, let  $\alpha_{i_1} = a_1$ ,  $\beta_{j_1} = b_1$ ,  $u_1 = v_1$  and  $e_1 = \min(a_1, b_1)$ . Then  $e_1 v_1$  is the maximum-normed term with the heaviest possible weight  $e_1$ .

Applying Theorem 3 for the case of the binomial, we have

$$\| e_1 v_1 + e_2 v_2 \| \geq \| u_2 \| \geq \dots \geq \| u_{i_2-1} \| \geq \| u_{i_2+1} \| \geq \dots \geq \| u_{mn} \| .$$

If  $A(v)$  and  $B(v)$  denote the end point of the segment corresponding to  $v$  in the set  $A$  and  $B$  respectively, then

$$e_2 = \min(a_2^*, b_2^*, [\lambda_2])$$

where  $a_2^*$  and  $b_2^*$  are the residual degrees of the ends of the segment corresponding to  $v_2$ ; i.e.,

$$\begin{aligned} a_2^* &= \begin{cases} a_2 & \text{if } A(v_2) \neq A(v_1) \\ a_2 - e_1 & \text{if } A(v_2) = A(v_1) \end{cases} \\ b_2^* &= \begin{cases} b_2 & \text{if } B(v_2) \neq B(v_1) \\ b_2 - e_1 & \text{if } B(v_2) = B(v_1) \end{cases} \end{aligned}$$

and  $\lambda_2$  as in Theorem 3.

To see with clarity how the residual degrees of the ends of segments behave, we write out the trinomial case.

The maximal-normed trinomial will then be

$$e_1v_1 + e_2v_2 + e_3v_3$$

where

$$e_3 = \min (a_3^*, b_3^*, [\lambda_3])$$

where

$$a_3^* = \begin{cases} a_3 & \text{if } A(v_3), A(v_2), A(v_1) \text{ are all distinct} \\ a_3 - e_1 = a_1 - e_1 & \text{if } A(v_3) = A(v_1) \text{ but } \neq A(v_2) \\ a_3 - e_2 = a_2 - e_2 & \text{if } A(v_3) = A(v_2) \text{ but } \neq A(v_1) \\ a_3 - (e_1 + e_2) = a_2 - (e_1 + e_2) = a_1 - (e_1 + e_2) & \text{if } A(v_1) = A(v_2) = A(v_3) \end{cases}$$

and

$$b_3^* = \begin{cases} b_3 & \text{if } B(v_1), B(v_2), B(v_3) \text{ are all distinct} \\ b_3 - e_1 = b_1 - e_1 & \text{if } B(v_3) = B(v_1) \text{ but } \neq B(v_2) \\ b_3 - e_2 = b_2 - e_2 & \text{if } B(v_3) = B(v_2) \text{ but } \neq B(v_1) \\ b_3 - (e_1 + e_2) = b_2 - (e_1 + e_2) = b_1 - (e_1 + e_2) & \text{if } B(v_1) = B(v_2) = B(v_3). \end{cases}$$

Thus by Theorem 3 we ultimately, after  $N$  iterations, obtain the maximal-normed linear combination,

$$\left\| \sum_{i=1}^N e_i v_i \right\| = \max \frac{W}{C},$$

where  $N < K$  is determined by the iterative process. We have, in general, that

$$e_k = \min (a_k^*, b_k^*, [\lambda_k])$$

where

$$a_k^* = \begin{cases} a_k & \text{if } A(v_i), i = 1, 2, \dots, k \text{ are distinct points} \\ a_k - \sum_{i=1}^{k-1} e_i & \text{if } A(v_i), \text{ for two or more } i\text{'s are not} \\ & \text{distinct points} \end{cases}$$

and

$$b^*_k = \begin{cases} b_k & \text{if } B(v_i), i=1, 2, \dots, k \text{ are all distinct points} \\ b_k - \sum_{i=1}^{k-1} e_i & \text{if } B(v_i) \text{ for some two or more } i\text{'s are not} \\ & \text{distinct points} \end{cases}$$

and where  $\Sigma'$  indicates summation omitting certain  $e$ 's, depending upon how many and which end points coincide.

Thus  $e_i$  is the number of items transported from the source point corresponding to the segment represented by  $v_i$  (which is a well-determined  $z$ ) to the terminal point of the segment corresponding to  $v_i$  for  $i=1, 2, \dots, N$ , which minimizes the total cost of transportation, and the  $v_i$ 's correspond to the segments which define the optimum scheduling. The actual minimal cost is given by

$$R\left(\sum_{i=1}^N e_i v_i\right)$$

where  $R$  is the real part of the complex argument.

Note that when all of the segments' end points are of degree 1; i.e.,  $\alpha_i = \beta_j = a_i = b_j = 1$ , then all the  $e_i$ 's = 1; we then have  $K=N$ , and the problem degenerates to the simple assignment problem of assigning  $N$  people to  $N$  jobs optimally.

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