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ON EXTENSIONS OF AN ELEMENTARY ABELIAN GROUP
OF ORDER 2^5 BY $GL(5, 2)$

ULRICH DEMPWOLFF *)

1. Introduction.

In [2] the author considered groups G with the following properties

- i) $V \triangleleft G$ and $|V| = 2^n$, V is elementary abelian
- ii) $G/V \cong GL(n, 2)$
- iii) $C_G(V) \subseteq V$

and combining this result with known ones we know the possibilities for G in all cases with the sole exception $n=5$. If we consider V as the standard \mathbb{F}_2 -module belonging to $GL(n, 2)$ we may state the result for $n \neq 5$ in terms of the second cohomology group

$$\dim_{\mathbb{F}_2} H^2(GL(n, 2), V) = \begin{cases} 0 & \text{if } n \neq 3, 4 \\ 1 & \text{if } n = 3 \text{ or } 4 \end{cases}$$

In this note we now inspect the remaining case $n=5$ and state

THEOREM A. *Let G be a finite group which satisfies the properties i) to iii) and where $n=5$. Then if G does not split over V the multiplication table of G is uniquely determined.*

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In terms of the degree 2 cohomology group we state

THEOREM B. $\dim_{\mathbb{F}_2} H^2(GL(5, 2), V) \leq 1.$

2. Proof of the Theorem.

Assume G is a group satisfying the conditions of Theorem A. For $\nu \in V^\#$ we set $H = \mathbb{C}_G(\nu)$. As the assertions of [2; (1) and (2)] are true in the case $n=5$ we may assume that H is a faithful extension of an extraspecial 2-group A of width 4 and type (+) by A_8 . A result in [1; (2.2)] tells us:

Let V be a regular orthogonal \mathbb{F}_2 -space of type (+) and dimension 8. Assume that $X \cong GL(4, 2)$ is a subgroup of $O(V)$ such that X normalizes an isotropic subspace U of V with $\dim U=4$.

Then one of the following possibilities occurs.

a) *There is a X -admissible, isotropic subspace W of V such that $U \oplus W = V$.*

b) *X acts on V reducibly but not completely reducibly. The action of X on V is uniquely determined.*

Therefore we may first consider.

CASE 1. H/A acts completely reducibly on $A/\langle \nu \rangle$. Using the same argument as in [2; (3)] we can find a subgroup H_0 of H such that $H_0A = H$ and $H_0 \cap A = \langle \nu \rangle$. We know that A contains an elementary abelian H_0 -admissible group V_1 such that $VV_1 = A$ and $V \cap V_1 = \langle \nu \rangle$. As $H^1(GL(4, 2), V(4, 2)) = 0$ we can find H_0 -admissible subspaces W of V and W_1 of V_1 such that $V = \langle \nu \rangle \oplus W$ and $V_1 = \langle \nu \rangle \oplus W_1$.

If H_0 would split over $\langle \nu \rangle$ say $H_0 = H_1 \langle \nu \rangle$, $H_1 \cap \langle \nu \rangle = 1$ then $W_1 H_1 \cap A = 1$ and G would split by a result of Gaschütz. So we may assume $H_0 \cong \widehat{A}_8$, the covering group of A_8 . Take $x_0 \in H_0 - \langle \nu \rangle$ such that $x_0^2 = \nu$. Then $\mathbb{C}_V(x_0)$ has order 2^3 . Further $[H_0, V] \subseteq W$. So the coset x_0V can not contain an involution. Now take an involution x_1 in $H_0 - \langle \nu \rangle$. As H_0 acts on W_1 as the full automorphism group, there is a $x_2 \in V_1$ such that $\mathbb{C}_V(x_1 x_2)$ has order 2^3 and $x_1 x_2 = x_2 x_1$. But this is a contradiction as $x_1 x_2 V \underset{G}{\sim} x_0 V$. So we now can restrict our attention to

CASE 2. H/A acts indecomposable on $A/\langle \nu \rangle$. Let F/V be a subgroup of H/V such that $F/V \cap A/V = 1$ and $F/V \cong A_8$ then there is always a hyperplane V_0 of V such that $V_0 \oplus \langle \nu \rangle = V$ and V_0 is F/A -admissible.

Now consider elements t_y, t_w, t_x, \dots of G such that $t_y^2 = t_w^2 = t_x^2 = \dots = 1$ and t_y, t_w, t_x, \dots centralize a fixed hyperplane U of V and $[t_y, V] = y, [t_w, V] = w, [t_x, V] = x, \dots$ where x, y, w, \dots are in $V^\#$

Assume we have for three different elements x, y, w in $V^\#$ the relations $[t_x, t_w] = 1$ and $[t_y, t_w] \neq 1$. As $t_y t_w$ acts as a transvection on V it follows that for $s \in V - U$ the element $st_y t_w$ has order 2 and so $[t_y, t_w] = yw$. So either

$$[t_x, t_y] = \begin{cases} yw \\ \text{or} \\ xw \end{cases} \text{ and further } [t_y, t_x t_w] = [t_y, t_w][t_x, t_y] = \begin{cases} 1 \\ \text{or} \\ xy \end{cases} .$$

But then there is certainly no involution in the coset $t_y t_x t_w V$. So we may conclude if there is at least one pair of elements y, w such that $[t_y, t_w] = yw$ then this equation must be true for arbitrary pairs. But [1; (2.2)] shows us that there is indeed a pair t_y, t_w such that $[t_y, t_w] \neq 1$.

Now we introduce a basis ν_1, \dots, ν_5 of V . Define transvections $\tau_{ik} \in G - V$ such that $\nu_i^{\tau_{ik}} = \nu_i \nu_k^2$ where $i \neq k, 1 \leq i, k, l \leq 5$. Then $G = \langle \tau_{ik} \mid 1 \leq i, j \leq 5; i \neq j \rangle V$ and it is well known that

$$B = \langle \tau_{ik} \mid i < j \rangle V/V \text{ and } N = \langle \tau_{i, i+1}^{\tau_{i+1, i}} \mid 1 \leq i \leq 4 \rangle V/V$$

is a (B, N) -pair for $GL(5, 2) \cong G/V$. We now choose involutions $t_{ij} \in \tau_{ij} V$ in such a way that

$$[t_{ij}, t_{ik}] = \nu_j \nu_k$$

and

$$[t_{ji}, t_{ki}] = 1$$

for $1 \leq i, j, k \leq 5$ and $k \neq i, j \neq i$.

As the group $\langle t_{ij} \mid i \neq j \neq l \neq i \rangle V/V$ acts indecomposable on $\langle t_{il} \mid 1 \leq i \leq 5; i \neq l \rangle V / \langle \nu_l \rangle$ it follows by [1; (2.2)] that there exist numbers $\alpha(ij, kl), \beta(ij, kl), \gamma(ijl)$ (here different letters mean different

numbers) such that

$$[t_{ij}, t_{kl}] = v_s v_j^{\alpha(ij, kl)} v_l^{\beta(ij, kl)}$$

where $\{s\} = \{1, \dots, 5\} - \{i, j, k, l\}$ and

$$[t_{ij}, t_{jl}] = v_i v_j v_r^{\alpha(ij, rl)} v_s^{\alpha(ij, sl)} v_l^{\gamma(iil)} t_{ie}$$

where $\{r, s\} = \{1, \dots, 5\} - \{i, j, l\}$.

This follows from the fact that $V\langle t_{il} \mid 1 \leq i \leq 5; l \neq i \rangle$ is the central product of the 4 dihedral groups

$$\langle v_1, t_{1l} \rangle, \dots, \langle v_5, t_{5l} \rangle$$

and

$$[t_{ij}, t_{kl}] \in t_{ii}^{\epsilon} j^k V \text{ for } i \neq j, k \neq l \text{ and } 1 \leq i, j, k, l \leq 5.$$

We note that $\alpha(ij, kl) = \beta(kl, ij)$. From the table of the indecomposable action in [1; (2.2)] we read of

$$\alpha(ij, st) = 1 + \alpha(sj, it)$$

$$\alpha(ij, st) = \alpha(sk, it) \text{ where } \{k\} = \{1, \dots, 5\} - \{i, j, s, t\}$$

$$\alpha(ij, st) = 1 + \alpha(ik, st) \text{ where } \{k\} = \{1, \dots, 5\} - \{j, i, s, t\}.$$

Now we study modifications of the elements t_{ik} by elements in V such that the relations $t_{ik}^2 = 1, [t_{ik}, t_{ij}] = v_k v_j$ and $[t_{ki}, t_{ji}] = 1$ are preserved. Fix first a number k . Let $\emptyset = (\varphi_{ij})$ be a symmetric 4×4 -matrix with entries in $\{0, 1\}$ and all diagonal elements are 0. Here we choose the indices i, j in the set $\{1, \dots, 5\} - \{k\}$ and order them in the natural way. Replace then t_{ik} by $v_1^{\varphi_{1i}} \dots v_5^{\varphi_{5i}} t_{ik}$. As $\varphi_{ii} = 0$ we have

$$(v_1^{\varphi_{1i}} \dots v_5^{\varphi_{5i}} t_{ik})^2 = 1$$

and

$$[v_1^{\varphi_{1i}} \dots v_5^{\varphi_{5i}} t_{ik}, t_{il}] = v_k v_l$$

and

$$[v_1^{\varphi_{1i}} \dots v_5^{\varphi_{5i}} t_{ik}, v_1^{\varphi_{1j}} \dots v_5^{\varphi_{5j}} t_{jk}] = v_k^{\varphi_{ij}} v_k^{\varphi_{ji}} = 1.$$

Changing in this way the t_{ij} for $1 \leq i, j \leq 5$ we determine an unique table for the $\alpha(ij, kl)$.

We have only to show that we can now determine the $\gamma(ijl)$ in a unique fashion and $(t_{ij}t_{ji})^3$ is uniquely determined. Clearly $(t_{ij}t_{ji})^3 \in V$. One checks easily that $t_{sr}(t_{ij}t_{ji})^3 = v_r^{\alpha(sr, ij) + \alpha(sr, ii)} t_{sr}$ where different letters mean different numbers. Note that always $\alpha(rs, ij) + \alpha(rs, ji) = \alpha(rk, ij) + \alpha(rk, ji)$. As $t_{ij}t_{ji}$ acts fixedpoint-free on $\langle v_i, v_j \rangle$ we may set

$$(t_{ij}t_{ji})^3 = v_s^{\alpha(sr, ij) + \alpha(sr, ji)} v_r^{\alpha(rs, ij) + \alpha(rs, ji)} v_k^{\alpha(ks, ij) + \alpha(ks, ji)}$$

where $\{1, \dots, 5\} - \{i, j\} = \{s, k, r\}$.

Now one can verify by a somewhat elaborate but trivial computation that

$$\begin{aligned} (t_{lr})^v t^{\gamma(ijl)} [t_{ij}, t_{jl}] &= \\ &= v_r^{\gamma(ijl) + \gamma(ijl) + \alpha(ir, jl) + \alpha(ij, lr) + \alpha(jl, ir)} \cdot \\ &\cdot v_s^{\alpha(is, rs) + 1} v_i v_j^{\alpha(jl, ir) + 1} v_l t_{lr} t_{ir}. \end{aligned}$$

But using

$$[t_{ij}, t_{jl}] = v_r^{\alpha(ij, rl)} v_i v_j v_s^{\alpha(ij, sl)} v_l^{\gamma(iil)} t_{il}$$

and

$$(t_{lr})^t u = v_i v_l v_r^{\gamma(ilr)} v_s^{\alpha(il, sr)} v_j^{\alpha(il, jr)} t_{lr} t_{ir}$$

it follows

$$(*) \quad \gamma(ijr) + \gamma(ijl) = \gamma(ilr) + \alpha(ir, jl) + \alpha(ij, lr) + \alpha(jl, ir).$$

Replacing if necessary t_{ik} by $v_k t_{ik}$ we may assume that

$$\begin{aligned} &\gamma(125), \gamma(135), \gamma(145), \gamma(154), \gamma(215), \gamma(235), \gamma(245), \\ &\gamma(254), \gamma(315), \gamma(325), \gamma(345), \gamma(354), \gamma(415), \gamma(425), \\ &\gamma(435), \gamma(453), \gamma(514), \gamma(524), \gamma(534), \gamma(543) \end{aligned}$$

are equal to 0.

We note that such a modification does not affect the relations in t_{ik} , $\alpha(ij, lk)$ which we have obtained so far. But then using (*) all $\gamma(ijl)$ are uniquely determined.

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