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Projectivities in finite p-groups

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PROJECTIVITIES IN FINITE p-GROUPS

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In [2] the projectivity groups of quotients in a finite abelian p-group were determined ([2], theorem 5.13). In the present paper, we study projectivity groups and certain important subgroups of projectivity groups in finite p-groups. In particular theorem 5.13 [2] is generalized to include the case of abelian quotients in a non-abelian p-group. The notational conventions of [2] are used throughout this paper. The commutator subgroup of a group H will be denoted by H'.

1. Analogues of Position.

The notion of position of a quotient was defined in [2], definition 4.2, for quotients in an abelian p-group. In this section we give two characterizations of this notion which have applications in more general situations.

DEFINITION 1.1. Suppose A/B is an abelian quotient in the p-group G. Define

$$t_G(A/B) = \max\{\exp F/F_{(1)} \mid F_{(1)} = \\ = \langle\langle F, C\rangle', D\rangle,\ F \in M_G(C/D),\ C/D \in [A/B]\}.$$

LEMMA 1.2. Suppose A/B is a quotient in the abelian p-group G and that G has type $(\lambda_1, \ldots, \lambda_k)$. Then $P_G(A/B) = t$ if and only if $t_G(A/B) = p^{\lambda_t}$ and $\lambda_t < \lambda_{t-1}$.

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PROOF. Suppose first that $P_G(A/B=t$ and let A/B have type $(\mu_1, \ldots, \mu_l)$. $\lambda_{t-1} \geqslant \mu_{t-1} > \lambda_t$, since $P_G(A/B)=t$, and hence it suffices to show that $t_G(A/B)=p^{\lambda_t}$. By definition 1.1 there exist elements $C/D \in [A/B]$ and $F \in M_G(A/B)$ with the property that $\exp F/F_{(1)} = t_G(A/B)$, where $F_{(1)}=\langle E', D\rangle = D$ and $E=\langle F, C\rangle$. By theorem 4.5 (i), [2],

$$\mho_{\lambda_t}(F) \leqslant \mho_{\lambda_t}(E) = \mho_{\lambda_t}(C).$$

Hence $\mho_{\lambda_t}(F) \leqslant F \cap C = D$ and we have

$$(1) \ \ldots \qquad t_G(A/B) \leqslant p^{\lambda_t}.$$

On the other hand by corollary 5.6, [2] A/B is projective to a quotient C/D, where $C \cap \langle z\rangle = 1$ for some element $z \in G$ of order p^{λ_t} Hence $t_G(A/B) \geqslant p^{\lambda_t}$ and by (1), $t_G(A/B)=p^{\lambda_t}$ Conversely, let $t_G(A/B) = p^{\lambda_t}$, where $\lambda_t < \lambda_{t-1}$. By the preceding argument, if $P_G(A/B)=s$ then $t_G(A/B)=p^{\lambda_s}$. Therefore $\lambda_s=\lambda_t$ and $P_G(A/B)=t$ since $\lambda_t<\lambda_{t-1}$.

EXAMPLE 1.3. As an illustration, suppose G is the group generated by elements x and y subject to the relations

$$x^{p^3}=y^{p^2}=1 \text{ and } x^y=x^{1+p}:$$

If A/B is any cyclic quotient of G of order p^2 then $t_G(A/B) \geqslant p$ since the center of G is elementary abelian of order p^2. Hence $t_G(A/B)=p$ as otherwise G would possess an abelian quotient of order p^4.

The following two results provide necessary conditions for abelian quotients in a p-group to be projective.

LEMMA 1.4. Suppose G is a p-group and A/B is an abelian quotient in G with $t_G(A/B)=p^\alpha$. If C/D is a quotient in G and $(A/B \rightarrow C/D)$ is a projection up then

$$\langle \mho_\alpha(C), C'\rangle = \langle \mho_\alpha(A), C'\rangle.$$

PROOF. Write $A_1=\langle A, C'\rangle$ and $B_1=\langle B, C'\rangle$ and denote by $\overline{X}$ the quotient X/C' for all subgroups X of C which contain C'. By definition 1.1 there exists a quotient $\overline{E}/\overline{F}$ in $\overline{C}$ projective to $\overline{A}_1/\overline{B}_1$ and $\overline{H} \in M_{\overline{C}}(\overline{E}/\overline{F})$ such that $\exp \overline{H}/\overline{F} = t_{\overline{C}}(\overline{A}_1/\overline{B}_1)$. By lemma 5.10, [2] $E/F \in A_1/B_1$.

Clearly $\overline{H}/\overline{F}$ is isomorphic to H/F and $H \in M_G(E/F)$. Hence

(1) ... $$t_G(A/B) \geqslant t_{\overline{C}}(\overline{A}_1/\overline{B}_1).$$

Suppose $t_{\overline{C}}(\overline{A}_1/\overline{B}_1) = p^\beta$ and that $\overline{C}$ has type $(\lambda_1, \ldots, \lambda_k)$. If $P_{\overline{C}}(\overline{A}_1/\overline{B}_1) = s$ then by lemma 1.2 $p^{\lambda_s} = p^\beta$. Hence by theorem 4.5 (i), [2] and (1) we have $\mho_\alpha(\overline{A}_1) = \mho_\alpha(\overline{C})$ from which the lemma follows.

THEOREM 1.5. Suppose G denotes a p-group and A/B is an abelian quotient in G with $t_G(A/B) = p^\alpha$. Let $H = \langle E \mid E/F \in [A/B] \rangle$. If C/D is a quotient in G projective to A/B then

(i) $\langle \mho_\alpha(A), H' \rangle = \langle \mho_\alpha(C), H' \rangle$

and

(ii) $\langle \mho_\alpha(A) \cap B, H' \rangle = \langle \mho_\alpha(C) \cap D, H' \rangle$.

PROOF. It suffices to treat the case when $(A/B \to C/D)$ is a projection up. Assertion (i) is an immediate consequence of lemma 1.4. Also by lemma 1.4

$$\mho_\alpha(C) \cap D \leqslant \langle \mho_\alpha(A), C' \rangle \cap D = \langle \mho_\alpha(A) \cap D, C' \rangle = \langle \mho_\alpha(A) \cap B, C' \rangle.$$

Hence

$$\langle \mho_\alpha(C) \cap D, H' \rangle \leqslant \langle \mho_\alpha(A) \cap B, H' \rangle$$

and (ii) holds.

Unlike the case for abelian p-groups, the converse of theorem 1.5 is not true. As an example, let G be the direct product of groups Q and H where $Q = \langle x, y \mid x^4 = 1, x^2 = y^2, x^y = x^{-1} \rangle$ and $H = \langle z \mid z^2 = 1 \rangle$ and take $A/B = \langle x \rangle$, $C/D = \langle y \rangle$. Then conditions (i) and (ii) of theorem 1.5 hold but $\langle x \rangle$ and $\langle y \rangle$ are not projective.

$t_G(A/B)$ is not defined for non-abelian quotients. However the following related quantity allows us to obtain several more geenral results.

DEFINITION 1.6. Let F and D be subgroups of a p-group G. Write $\exp F \bmod D = \min\{p^\alpha \mid x^{p^\alpha} \in D, \text{ for all } x \in F\}$. If A/B and C/D are quotients in G write

$$s_G(C/D) = \max\{\exp F \bmod D \mid F \in M_G(C_1/D_1),\ C_1/D_1 \in [C/D]\}$$

$$s_G[A/B]=\max\{s_G(C/D) \mid C/D\in[A/B]\}.$$

If A/B is an abelian quotient then $s_G[A/B]\geqslant t_G(A/B)$. $s_G[A/B]$ may be strictly greater than $t_G(A/B)$. For example, choose $A/B=\langle y\rangle$ in example 1.3.

LEMMA 1.7. Suppose that G is an abelian p-group and A/B is a quotient in G. Then

$$s_G[A/B]=t_G(A/B).$$

PROOF. Let $t_G(A/B)=p^\alpha$ and suppose C/D is any quotient in G projective to A/B. If $C_1/D_1\in[C/D]$ and $F\in M_G(C_1/D_1)$ then $\mho_\alpha(F)\leqslant \leqslant\mho_\alpha(E)\cap F$, where $E=\langle F, C_1\rangle$. Also $\mho_\alpha(E)\cap F=\mho_\alpha(C)\cap D$ by lemma 1.4 and theorem 4.5 (ii), [2]. Therefore, $\exp F \bmod D\leqslant p^\alpha$ and hence $s_G(C/D)\leqslant t_G(A/B)$ for all C/D of $[A/B]$.

THEOREM 1.8. Suppose A/B is a quotient in the p-group G and that $s_G[A/B]=p^\alpha$. If C/D and E/F are elements of $[A/B]$ then $\langle\mho_\alpha(A), F, H'\rangle=\langle\mho_\alpha(C), F, H'\rangle$ where $H=\langle C_1 \mid C_1/D_1\in[A/B]\rangle$.

PROOF. It suffices to assume that $(A/B\rightarrow C/D)$ is a projection up. Trivially, $\langle\mho_\alpha(A), F, H'\rangle\leqslant\langle\mho_\alpha(C), F, H'\rangle$. On the other hand, $\mho_\alpha(C)\leqslant\langle\mho_\alpha(A), \mho_\alpha(D), H'\rangle$. $\mho_\alpha(D)\leqslant F$ since $s_G[A/B]=p^\alpha$ and so the theorem holds.

2. The Projectivity Groups of Abelian Quotients.

LEMMA 2.1. Suppose A/B is an abelian quotient in the p-group G and that $G=\langle A, F\rangle$, where F is some element of $M_G(A/B)$. Let $A_1=\langle A, G'\rangle$, $B_1=\langle B, G'\rangle$ and denote by $\overline{X}$ the quotient X/G' for all subgroups X of G containing G'. Then

(i) B_1 is a normal subgroup of F.

(ii) $t_{\overline{G}}(\overline{A}_1/\overline{B}_1)=t_G(A/B)$ if $t_G(A/B)=\exp F/B_1$.

PROOF:

(i) holds since G/F is abelian. F is an element of $M_{\overline{G}}(\overline{A}_1/\overline{B}_1)$ and $\overline{F}/\overline{B}_1$ is isomorphic to F/B_1, hence

(1) ... $$t_{\overline{G}}(\overline{A}_1/\overline{B}_1)\geqslant t_G(A/B).$$

On the other hand, by definition 1.1, there exists a quotient $\bar{C}/\bar{D} \in [\bar{A}/\bar{B}]$ and $\bar{H} \in M_{\bar{G}}(\bar{C}/\bar{D})$ such that $\exp \bar{H}/(\bar{H})_{(1)} = t_{\bar{G}}(\bar{A}_1/\bar{B}_1)$, where $(\bar{H})_{(1)} = \langle\langle \bar{H}, \bar{C}\rangle', \bar{D}\rangle = \bar{D}$. Thus $D \lhd H$, $H \in M_G(C/D)$ and $\bar{H}/(\bar{H})_{(1)}$ is isomorphic to H/D. Hence $t_{\bar{G}}(\bar{A}_1/\bar{B}_1 \leqslant t_G(A/B)$ and (ii) follows now from (1).

LEMMA 2.2. Let G denote a p-group. Suppose A/B is a quotient in G and that C/D is projective to A/B. Then for any integer α

$\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B)) \leqslant \pi_G(A/B)$ if and only if

$\mathrm{Aut}\,(C/D;\ \Omega_\alpha(C/D)) \leqslant \pi_G(C/D)$.

THEOREM 2.3. Suppose G is a p-group and A/B is an abelian quotient in G with $t_G(A/B) = p^\alpha$. Then

$$\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B)) \leqslant \pi_G(A/B).$$

PROOF. We use induction on the order of G. By definition 1.1 there exists a quotient $C/D \in [A/B]$ and an element $F \in M_G(C/D)$, with $t_G(A/B) = \exp F/F_{(1)}$, where $F_{(1)} = \langle E', D\rangle$ and $E = \langle F, C\rangle$. By lemma 2.2 and the induction hypothesis we may assume that $A/B = C/D$ and that $G = E$.

Suppose $G' \neq 1$. Denote by $\bar{X}$ the quotient X/G', for all subgroups X of G containing G', and write $A_1 = \langle A, G'\rangle$ and $B_1 = \langle B, G'\rangle$. By lemma 2.2 we may assume that $A/B = A_1/B_1$. By lemma 2.1 (ii) $t_{\bar{G}}(\bar{A}/\bar{B}) = t_G(A/B)$. Hence by induction we have

$$\mathrm{Aut}\,(\bar{A}/\bar{B};\ \Omega_\alpha(\bar{A}/\bar{B})) \leqslant \pi_{\bar{G}}(\bar{A}/\bar{B})$$

and the theorem follows from lemma 5.10, [2]. Finally, if $G' = 1$, then G is an abelian p-group, say of type $(\lambda_1, \ldots, \lambda_k)$. If $P_G(A/B) = t$ then by lemma 1.2, $t_G(A/B) = p^\alpha = p^{\lambda^t}$ and the result follows from theorem 5.13, [2].

COROLLARY 2.4. Suppose G is a p-group and A/B is an abelian quotient with $t_G(A/B) = \exp A/B$. Then

$$\pi_G(A/B) = \mathrm{Aut}\,(A/B).$$

DEFINITION 2.5. Suppose that H is a p-group and that K is a subgroup of H. Define

$$\mathrm{Aut}(H;\ \alpha;\ K)=\{\theta\in \mathrm{Aut}(H)\mid [x,\ \theta]^{p^\alpha}\in K, \text{ for all } x\in H\}.$$

We list without proof some of the properties of $\mathrm{Aut}(H;\ \alpha;\ K)$.

LEMMA 2.6. Suppose H is a p-group and K is a normal subgroup of H. If H/K is a regular p-group then

(i) $\mathrm{Aut}(H;\ \alpha;\ K)$ is a subgroup of $\mathrm{Aut}(H)$.

(ii) $\mathrm{Aut}(H;\ \alpha;\ K)$ is a normal subgroup of $\mathrm{Aut}(H)$ if K is a characteristic subgroup of H.

Theorem 2.7. Suppose A/B is an abelian quotient in the p-group G. If $t_G(A/B)=p^\alpha$ then $\pi_G(A/B)\leqslant \mathrm{Aut}(A/B;\ \alpha;\ K/B)$, where

$$K=\langle \mho_\alpha(A)\cap H'\ B\rangle \text{ and } H=\langle E\mid E/F\in[A/\dot{B}]\rangle.$$

PROOF. It is sufficient to show, for any $\theta\in\pi_G(A/B)$ and $xB\in A/B$ that $(xB)\theta=(xB)(zB)$, where $zB\in A/B$ and $(zB)^{p^\alpha}\in K/B$. Suppose θ and xB are given. By lemma 2.4, [2], and theorem 2.3, [2], there exist quotients $A_1/B_1, \ldots, A_n/B_n$ of $[A/B]$, subgroups $F_1, \ldots, F_n$ of G with $F_i\in M_G(A_i/B_i)$ and elements $z_i\in F_i$, for $i=1, \ldots, n$, such that $(xB)\theta= =(xB)(zB)$, where $z=z_1 \ldots z_n$.

$z_i^{p^\alpha}\in \mho_\alpha(\langle A_i, F_i\rangle)\cap F_i$ and hence by theorem 1.5 (ii), $z_i^{p^\alpha}\in\langle \mho_\alpha(A)\cap \cap B,\ H'\rangle$ for all i. Also there exists an element ω of H' such that $z^{p^\alpha}=z_1^{p^\alpha}\ldots z_n^{p^\alpha}\omega$. Therefore,

$$z^{p^\alpha}\in\langle \mho_\alpha(A)\cap B,\ H'\rangle\cap\mho_\alpha(A)=\langle \mho_\alpha(A)\cap B,\ \mho_\alpha(A)\cap H'\rangle$$

and we conclude that $(zB)^{p^\alpha}\in K/B$.

DEFINITION 2.8. We say that an abelian quotient A/B in a p-group G has the trivial intersection property, or TI-property, if

$$\mho_\alpha(A)\cap H'\leqslant B,$$

where $t_G(A/B)=p^\alpha$ and $H=\langle E\mid E/F\in[A/B]\rangle$.

The next result shows that if A/B has the TI-property then any quotient projective to A/B also has this property.

LEMMA 2.9. Suppose A/B is an abelian quotient in the p-group G and that $t_G(A/B)=p^\alpha$. Let C/D be a quotient projective to A/B and suppose c is a projection chain between A/B and C/D. Write

$$H=\langle E \mid E/F\in[A/B]\rangle,\ K_1=\langle \mho_\alpha(A)\cap H',\ B\rangle$$

and

$$K_2=\langle \mho_\alpha(C)\cap H',\ D\rangle.$$

Then the image of K_1/B under $\alpha(c)$ is K_2/D. In particular, $\mho_\alpha(A)\cap H'\leqslant B$ if and only if $\mho_\alpha(C)\cap H'\leqslant D$.

PROOF. Without loss of generality we may assume that $c= =(A/B\to C/D)$ is a projection up. It suffices to show that $\langle K_1, D\rangle= =K_2$, for then $(K_1/B\to K_2/D)$ is a projection up. Clearly $\langle K_1, D\rangle\leqslant K_2$. On the other hand, by lemma 1.4, $\mho_\alpha(C)\leqslant\langle \mho_\alpha(A), C'\rangle$ and so

$$\mho_\alpha(C)\cap H'\leqslant\langle \mho_\alpha(A), C'\rangle\cap H'=\langle \mho_\alpha(A)\cap H', C'\rangle=\langle K_1, D\rangle.$$

As a corollary to theorems 2.3 and 2.7 we have:

THEOREM 2.10. Suppose A/B is an abelian quotient in the p-group G and that $t_G(A/B)=p^\alpha$. If A/B has the TI-property then

$$\pi_G(A/B)=\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B)).$$

We conclude this section with two examples of quotients which do not possess the TI-property. These show that the inequality in theorems 2.3 and 2.7 are best possible in the sense that both bounds may be attained.

I am indebted to Dr. D. W. Barnes for example 2.11.

EXAMPLE 2.11. Let G be the group of order p^4, where p is odd, generated by elements x, y and z and subject to the relations:

$$x^{p^2}=y^p=z^p=1,\ Z(G)=\langle x\rangle,\ [y, z]=x^p.$$

Choose A to be the subgroup generated by x. Clearly $t_G(A)=p$ and $G=\langle E \mid E/F\in[A]\rangle$. $G'=\langle x^p\rangle=\mho_1(A)$ and hence A does not possess the TI-property. It is difficult to verify that $\pi_G(A)=\mathrm{Aut}\,(A;\ \Omega_1(A))$

and that

$$\mathrm{Aut}\,(A;\ 1;\ \langle x^p\rangle)=\mathrm{Aut}\,(A)>\mathrm{Aut}\,(A;\ \Omega_1(A)).$$

EXAMPLE 2.12. Let G be the group of order 3^5, with generators x, y and z subject to the relations:

$$x^9=y^9=z^3=1,\ z^x=z,\ y^x=y^2,\ y^z=yx^3.$$

Then $Z(G)=G'=\mho_1(G)=\langle x^3,\ y^3\rangle$. In particular G has class 2. Choose A to be the subgroup generated by x. $t_G(A)=3$ and $G=\langle E \mid E/F\in[A]\rangle$. A does not have the TI-property since $\mho_1(A)\cap G'=\langle x^3\rangle$. However Aut (A) is generated by the element θ defined by $(x)\theta=x^2$ and $\theta=\alpha(c)$, where c is the projection chain:

$$c=\left(A\to\frac{\langle x,\ z\rangle}{\langle z\rangle}\to\langle xz\rangle\to\frac{\langle xz,\ z^{-1}xy^{-1}\rangle}{\langle z^{-1}xy^{-1}\rangle}\to\langle x^2y^{-1}\rangle\to\frac{\langle x^2,\ y\rangle}{\langle y\rangle}\to A\right).$$

Thus

$$\pi_G(A)=\mathrm{Aut}\,(A)=\mathrm{Aut}\,(A;\ 1;\ \langle x^3\rangle)>\mathrm{Aut}\,(A;\ \Omega_1(A)).$$

3. The Projectivity Groups of Non-abelian Quotients.

THEOREM 3.1. Suppose A/B is a quotient in the p-group G. If $s_G(A/B)=p^\alpha$ then

$$\pi_G(A/B)\subseteq\mathrm{Aut}\,(A/B;\ \alpha;\ K/B)$$

where $K=\langle\langle B,\ H'\rangle\cap\mho_\alpha(A),\ B\rangle$ and $H=\langle E \mid E/F\in[A/B]\rangle$.

The proof of theorem 3.1 is similar to that of theorem 2.7.

COROLLARY 3.2. Suppose that A/B is a quotient in the p-group G. If

(i) $s_G(A/B)=p^\alpha$ and $\langle B,\ H'\rangle\cap\mho_\alpha(A)\leqslant B$,

or

(ii) $H=\langle E \mid E/F\in[A/B]\rangle$ is a regular p-group and

$$p^\alpha=\max\{s_G(A/B),\ \exp H' \bmod B\},$$

then $\pi_G(A/B)\leqslant\mathrm{Aut}\,(A/B,\ \Omega_\alpha(A/B))$.

4. The Subgroups $\pi_G^{(n)}(A/B)$.

DEFINITION 4.1. The number of projections in a projection chain is called the length of the chain. Suppose A/B and C/D are quotients in the group G and that c_1 and c_2 are loops beginning at A/B and C/D respectively. c_1 is said to be π-conjugate to c_2 if there exists a projection chain d between A/B and C/D such that $c_1 = dc_2d^{-1}$.

If n is any positive integer, define

$$\pi_G^{(n)}(A/B) = \langle \theta \in \pi_G(A/B) \mid \theta = \alpha(c),$$

where c is π-conjugate to a loop of length $n\rangle$.

Clearly $\pi_G^{(n)}(A/B)$ is always a normal subgroup of $\pi_G(A/B)$. Moreover, $\pi_G^{(1)}(A/B)$, $\pi_G^{(2)}(A/B)$ and $\pi_G^{(3)}(A/B)$ are trivial.

LEMMA 4.2. Suppose A/B is a quotient in the group G. If C/D is any quotient of G projective to A/B then $\pi_G^{(n)}(A/B)$ is isomorphic to $\pi_G^{(n)}(C/D)$.

PROOF. If d is a projection chain between A/B and C/D the map $\alpha(c) \to \alpha(d^{-1}cd)$, for all $\alpha(c)$ of $\pi_G^{(n)}(A/B)$ is an isomorphism between $\pi_G^{(n)}(A/B)$ and $\pi_G^{(n)}(C/D)$.

In the remainder of this section we shall be mainly concerned with $\pi_G^{(4)}(A/B)$, where A/B is an abelian quotient in the p-group G.

LEMMA 4.3. Suppose that A/B is a quotient in the group G. If c_1 is a loop beginning at A/B, which is π-conjugate to the loop c_2 of length four, then either

(i) $\alpha(c_1)$ is the identity automorphism

or

(ii) c_1 is i-homotopic to a loop c'_1 which is π-conjugate to an alternating loop c'_2 of the form

$$c'_2 = (C_1/D_1 \to E_1/F_1 \to C_2/D_2 \to E_2/F_2 \to C_1/D_1)$$

where $(C_1/D_1 \to E_1/F_1)$ is a projection up.

PROOF. Since c_1 is π-conjugate to c_2, there exists a projection chain d such that $c_1 = dc_2d^{-1}$.

Suppose

$$c_2=(A_1/B_1 \xrightarrow{\theta_1} A_2/B_2 \xrightarrow{\theta_2} A_3/B_3 \xrightarrow{\theta_3} A_4/B_4 \xrightarrow{\theta_4} A_1/B_1).$$

If c_2 is not an alternating loop then clearly (i) holds. If c_2 is an alternating loop and $(A_1/B_1 \rightarrow A_2/B_2)$ is a projection up then (ii) is satisfied with $c'_1=c_1$ and $c'_2=c_2$. Finally suppose $(A_1/B_1 \rightarrow A_2/B_2)$ is a projection down. Let c'_2 be the sequence of projections θ_2, θ_3, θ_4, θ_1 and c'_1 be the projection chain $d\theta_1 c_2' \theta_1^{-1} d^{-1}$. Then c_1 is i-homotopic to c'_1 and c'_1 is π-conjugate to c'_2, whence (ii) follows.

THEOREM 4.4. Suppose A/B is an abelian quotient in the p-group G and that $t_G(A/B)=p^\alpha$. Then

$$\pi_G^{(4)}(A/B)=\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B)).$$

PROOF. (*a*) We first show that $\pi_G^{(4)}(A/B)\leqslant \mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B))$. It is sufficient to show that all elements of $\pi_G^{(4)}(A/B)$ of the form $\alpha(c)$, where c is π-conjugate to a loop c_1 of length four, are contained in $\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B))$. By theorem 2.3, [2] and lemma 4.3 we can choose

$$c_1=\left(C/D \rightarrow \frac{\langle C,\ E\rangle}{[C/D,\ \theta_1]} \rightarrow E/F \rightarrow \frac{\langle E,\ C\rangle}{[E/F,\ \theta_2]} \rightarrow C/D\right),$$

for some quotient E/F and isomorphisms θ_1 and θ_2. By lemma 2.4 [2] if $xD\in C/D$ there exist elements z_1 and z_2 of $[C/D,\ \theta_1]$ and $[E/F,\ \theta_2]$ respectively, such that $(xD)\alpha(c_1)=(xD)(zD)$ where $z=z_1z_2$. Now $z^{p^\alpha}=z_1^{p^\alpha}z_2^{p^\alpha}\omega$ for some ω of $\langle E,\ C\rangle'$, whence

$$z^{p^\alpha}\in C\cap\langle\langle E,\ C\rangle',\ D\rangle=\langle D,\ C\cap\langle E,\ C\rangle'\rangle=D.$$

Thus $zD\in\Omega_\alpha(C/D)$ and so $\alpha(c_1)\in\mathrm{Aut}\,(C/D;\ \Omega_\alpha(C/D))$. Hence for c π-conjugate to c_1, $\alpha(c)$ is an element of $\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B))$.

(*b*) It remains to show that $\mathrm{Aut}\,(A/B;\ \Omega_\alpha(A/B))\leqslant\pi_G^{(4)}(A/B)$. Since $t_G(A/B)=p^\alpha$, there exists a quotient $C/D\in[A/B]$ and an element $F\in M_G(C/D)$ such that D is a normal subgroup of F, $\exp F/D=p^\alpha$ and E/D is abelian where $E=\langle F,\ C\rangle$. Without loss of generality it clearly suffices to show that

(1) $$\mathrm{Aut}\,(C/D;\ \Omega_\alpha(C/D)) \leqslant \pi_G^{(4)}(C/D).$$

Also by lemma 5.10, [2] we have

$$\pi_{E/D}^{(4)}(C/D) \leqslant \pi_G^{(4)}(C/D).$$

Moreover, $t_{E/D}(C/D) = t_G(C/D) = p^\alpha$ by lemma 2.1 ii). Thus, to prove (1) it suffices to assume that $E=G$ and $D=1$.

Let A be a subgroup of type $(\mu_1, \ldots, \mu_l)$ in the abelian p-group G, where G has type $(\lambda_1, \ldots, \lambda_k)$. If $P_G(A)=t$ then by lemma 1.2, $t_G(A) = p^{\lambda_t} = p^\alpha$. The theorem will be proved provided that we can show

(2) $$\mathrm{Aut}\,(A;\ \Omega_{\lambda_t}(A)) \leqslant \pi_G^{(4)}(A).$$

Suppose θ is an element of $\mathrm{Aut}\,(A;\ \Omega_{\lambda_t}(A))$. If $x \in A$ then there exists z of A such that $(x)\theta = xz$ where $|z| \leqslant p^{\lambda_t}$. Using arguments similar to those of Jacobson ([1], pp. 18-21), it is easy to show that every element of $\mathrm{Aut}\,(A;\ \Omega_{\lambda_t}(A))$ is the product of elementary automorphisms of the following two types:

(I) $$\{a_1, \ldots, a_l\} \to \{a_1, \ldots, a'_i, \ldots, a_l\}$$
where $a'_i = a_i a_j^\alpha$ and $|a_j^\alpha| \leqslant p^{\lambda_t}$.

(II) $$\{a, \ldots, a_l\} \to \{a, \ldots, a'_i, \ldots, a'_j, \ldots, a_l\}$$
where $a'_i = a_j$, $a'_j = a_i$ and $|a_i| = |a_j| \leqslant p^{\lambda_t}$.

Here $\{a_1, \ldots, a_l\}$ is a basis for A and the automorphisms are defined by taking elements from one basis to the corresponding elements in the other.

To complete the proof of the theorem we show that automorphisms of types (I) and (II) are projectivities in G of length four. By corollary 5.6, [2] we may assume the existence of a cyclic subgroup C of G of order p^{λ_t} which has trivial intersection with A. Suppose θ is an automorphism of type (I).

Choose an element z of C such that $|z| = |a_j^\alpha|$.

Put $H=\langle a_1, \ldots, a_i z, \ldots, a_l\rangle$,

$$\theta_1 : \{a_1, \ldots, a_l\} \to \{a_1, \ldots, a_i z, a_{i+1}, \ldots, a_l\}$$

$$\theta_2 : \{a_1, \ldots, a_i z, a_{i+1}, \ldots, a_l\} \to \{a_1, \ldots, a_i a_j^{\alpha}, a_{i+1}, \ldots, a_l\}.$$

Then

$$A \cap [A, \theta_1] = H \cap [A, \theta_1] = 1,$$

$$A \cap [C, \theta_2] = H \cap [H, \theta_2] = 1$$

and

$$c_1 = \left(A \to \frac{\langle A, H\rangle}{[A, \theta_1]} \to H \to \frac{\langle A, H\rangle}{[H, \theta_2]} \to A \right)$$

is a projection chain with $\alpha(c_1) = \theta_1\theta_2 = \theta$.

Suppose finally that θ is a projectivity of type (II).

Choose an element z of C whose order is equal to the order of a_i. Put

$$H = \langle a_1, \ldots, a_{i-1}, a_i a_j z, \ldots, a_{j-1}, z^{-1}, \ldots, a_l\rangle,$$

$$\varphi_1 : \{a_1, \ldots, a_l\} \to \{a_1, \ldots, a_{i-1}, a_i a_j z, \ldots, a_{j-1}, z^{-1}, \ldots, a_l\rangle$$

and

$$\varphi_2 : \{a_1, \ldots, a_{i-1}, a_i a_j z, \ldots, a_{j-1}, z^{-1}, \ldots, a_l\rangle \to$$
$$\to \{a_1, \ldots, a'_i, \ldots, a'_j, \ldots, a_l\rangle$$

where $a'_i = a_j$ and $a'_j = a_i$.

Then

$$A \cap [A, \varphi_1] = [A, \varphi_1] \cap H = 1,$$

$$H \cap [H, \varphi_2] = [H, \varphi_2] \cap A = 1$$

and

$$c_2 = \left(A \to \frac{\langle A, H\rangle}{[A, \varphi_1]} \to H \to \frac{\langle A, H\rangle}{[H, \varphi_2]} \to A \right)$$

is a projection chain with $\alpha(c_2) = \varphi_1\varphi_2 = \theta$.

Since θ may be any automorphism of type (I) or (II) we conclude that (2) holds and hence that the theorem is proved.

As a consequence of theorem 4.4 and theorem 2.10 we have

COROLLARY 4.5. Suppose that A/B is an abelian quotient in the p-group G. If A/B has the trivial intersection property then

$$\pi_G^{(4)}(A/B)=\pi_G(A/B).$$

COROLLARY 4.6. Suppose that A/B is an abelian quotient in the p-group G. Then $\pi_G^{(4)}(A/B)$ is a normal subgroup of Aut (A/B).

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[2] WEHRHAHN, K. H.: *Projectivities in Finite p-groups I*, Math. Zeitschr. (to appear).

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