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PRIME DIVISORS OF q-BINOMIAL COEFFICIENTS

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Introduction.

1. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{q^{n-j+1}-1}{q^j-1}$$

for q an indeterminate and n a non-negative integer. It is known that the q-binomial coefficient is a polynomial in q and that for $q=1$ it reduces to the ordinary binomial coefficient. For additional properties and references see [2].

In this paper we generalize some recently proved results for ordinary binomial coefficients to q-binomial coefficients. In section 2 we consider the problem of determining if there are q-binomial coefficients divisible by a specified factor, and we generalize a theorem of Simmons [5], who considered this problem for ordinary binomial coefficients. In section 3 we find formulas for the number of q-binomial coefficients divisible by a fixed power of a prime, thus generalizing results of Carlitz [1] and the author [3], [4].

In section 2 we assume the following, which we call conditions (1.1).

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(1.1) Let $p_1, \ldots, p_k$ be prime numbers and let q be a rational number such that when q is reduced to its lowest terms, p_i does not divide the numerator or denominator for $i=1, \ldots, k$. Let $e(i)$ be the smallest positive integer such that $q^{e(i)} \equiv 1 \pmod{p_i}$ and let $p_i^{h(i)}$ be the highest power of p_i dividing $q^{e(i)}-1$. If $p_i^{h(i)}=2$, let $p_i^{t(i)}$ be the highest power of p_i dividing $q+1$.

In section 3 we assume:

(1.2) Assume (1.1) holds, with $k=1$. We use the notation $p=p_1$, $e=e(1)$, $h=h(1)$, $t=t(1)$.

Throughout this paper we shall use the following rule, which is due to Fray [2], for determining the highest power of a prime p dividing a q-binomial coefficient. Suppose conditions (1.2) hold. Then any positive integer n can be written uniquely as

$$n=a_0+e(a_1+a_2p+ \ldots +a_kp^{k-1}) \tag{1.3}$$

where

$$0 \leq a_0 < e, \quad 0 \leq a_i < p \quad (i=1, \ldots, k).$$

Similarly

$$\begin{aligned} r &= b_0+e(b_1+b_2p+ \ldots +b_kp^{k-1}) \\ n+r &= c_0+e(c_1+c_2p+ \ldots +c_kp^{k-1}). \end{aligned} \tag{1.4}$$

We can write

$$\begin{gathered} a_0+b_0=\varepsilon_0e+c_0 \\ \varepsilon_0+a_1+b_1=\varepsilon_1p+c_1 \\ \cdots \\ \varepsilon_{k-2}+a_{k-1}+b_{k-1}=\varepsilon_{k-1}p+c_{k-1} \\ \varepsilon_{k-1}+a_k+b_k=c_k \end{gathered}$$

where each ε_i is either zero or one. If $p^h>2$ then the highest power of p dividing $\begin{bmatrix} n+r \\ r \end{bmatrix}$ is p^s where

(1.5) $$s=\varepsilon_0 h+\varepsilon_1+\dots+\varepsilon_{k-1}\,.$$

If $p^h=2$ then the highest power of p dividing $\begin{bmatrix} n+r \\ r \end{bmatrix}$ is p^s where

(1.6) $$s=\varepsilon_1 t+\varepsilon_2+\dots+\varepsilon_{k-1}\,.$$

2. Specified divisors of q-binomial coefficients. Simmons [5] has shown that if r and N are any positive integers then there are infinitely many $m\geq r$ such that

$$\left(\binom{m}{r},\ N\right)=1.$$

This result can easily be generalized to q-binomial coefficients.

THEOREM 2.1. Let N and r be positive integers and let p_1, ..., p_k be the prime divisors of N. Assume conditions (1.1) hold. Then there are infinitely many $m\geq r$ such that

$$\left(\begin{bmatrix} m \\ r \end{bmatrix},\ N\right)=1.$$

PROOF. For each p_i we write

(2.1) $$r=b_0+e_i(b_1+b_2p_i+\dots+b_{f(i)}p_i^{f(i)-1})$$

where

$$0\leq b_0<e_i\,,\ 0\leq b_j<p_i\ \ (j=1,\ \dots,\ f(i)).$$

Let d be any positive integer and let

$$m=r+d\prod_{i=1}^{k} e_i p_i^{f(i)}\,.$$

By (1.5) and (1.6) it is clear that

$$\left(\begin{bmatrix} m \\ r \end{bmatrix},\ N\right)=1$$

Theorem 2.1 says that for arbitrary primes $p_1, \ldots, p_k$ there are an infinite number of positive integers m such that p_i^0 is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for $i=1, \ldots, k$, provided (1.1) holds. It seems natural to ask the following question: If $p_1, \ldots, p_k$ are arbitrary primes and $g(1), \ldots, g(k)$ are arbitrary non-negative integers, are there an infinite number of positive integers m such that $p_i^{g(i)}$ is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$? We shall prove that the answer is always yes for ordinary binomial coefficients. It is clear that the answer is not always yes for q-binomial coefficients, however. For example, if $p_1=3$, $q=8$, $g(1)=1$ and $r=1$, then $e(1)=2$ and since

$$\begin{bmatrix} m \\ 1 \end{bmatrix} = (8^m-1)/7$$

it is clear that $3^{g(1)}$ is not the highest power of 3 dividing $\begin{bmatrix} m \\ 1 \end{bmatrix}$ for any m. In fact, by (1.5), if $p_1^{h(1)}>2$, $r<e(1)$ and $0<g(1)<h(1)$, then $p_1^{g(1)}$ will not be the highest power of p_1 dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for any m. By (1.6), if $p_1^{h(1)}=2$, $r=1$, and $0<g(1)<t(1)$ then $p_1^{g(1)}$ will not be the highest power of p_1 dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for any m.

THEOREM 2.2. Let r be a positive integer, $p_1, \ldots, p_k$ prime numbers and $g(1), \ldots, g(k)$ non-negative integers. Assume conditions (1.1) hold. If $p_i^{h(i)}>2$, assume $r\geq e(i)$ and/or $g(i)\geq h(i)$. If $p_i^{h(i)}=2$, assume $r>1$ and/or $g(i)\geq t(i)$. Then if $(e(i), e(j))=1$ for $i\neq j$ there are infinitely many positive integers m such that the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ is $p_i^{g(i)}$ $(i=1, \ldots, k)$.

PROOF. We again use expansions (2.1), assuming $b_{f(i)}\neq 0$. If $r=b_0$ we say that $f(i)=0$. Let

$$\begin{aligned} S_i &= r \text{ if } g(i)=0 \\ &= b_0+e_i(b_1+ \ldots + b_{f(i)-1}p_i^{f(i)-2}+p_i^{f(i)+g(i)-1}) \\ &\qquad \text{if } f(i)\geq 2,\ g(i)>0, \end{aligned}$$

$$=b_0+e_ip_i^{g(i)} \text{ if } f(i)=1,\ g(i)>0,\ p_i^{h(i)}>2,$$
$$=e_ip_i^{g(i)-h(i)} \text{ if } f(i)=0,\ g(i)\geq h(i),\ p_i^{h(i)}>2,$$
$$=p_i^{g(i)-t(i)+1} \text{ if } f(i)=1,\ g(i)\geq t(i),\ p_i^{h(i)}=2.$$

By the Chinese Remainder Theorem, the system of congruences

$$x\equiv S_i \pmod{e_ip_i^{f(i)+g(i)+1}} \tag{2.2}$$

has an infinite number of positive simultaneous solutions. If m is such a solution, it is clear by (1.5) and (1.6) that $p_i^{g(i)}$ is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$.

Theorem 2.2 could be stated more generally by replacing the condition that $(e(i), e(j))=1$ for $i\neq j$ by the condition that congruences (2.2) have a simultaneous solution.

COROLLARY. Let $N=p_1^{g(1)} \dots p_k^{g(k)}$ be any positive integer and let r be a positive integer. If the hypotheses of Theorem 2.2 are satisfied, then there are an infinite number of positive integers m such that

$$\begin{bmatrix} m \\ r \end{bmatrix}=NM,\ M\not\equiv 0 \pmod{p_i}\ (i=1, \dots, k).$$

We note that the conclusions of Theorem 2.2 and its corollary always hold for ordinary binomial coefficients.

3. The number of q-binomial coefficients divisible by a fixed power of a prime. L. Carlitz [1] has defined $\theta_j(n)$ as the number of binomial coefficients

$$\binom{n}{s} \quad (s=0, 1, \dots, n)$$

divisible by exactly p^j, where p is a prime number, and he has found formulas for $\theta_j(n)$ for certain values of j and n. The writer [3], [4] has also considered this problem. In particular, if we write

$$n=c_0+c_1p+\ \dots\ +c_rp^r(0\leq c_i<p)$$

then we have the formulas

$$\theta_0(n)=(c_0+1)(c_1+1)\ \dots\ (c_r+1) \tag{3.1}$$

$$\theta_1(n)=\sum_{i=0}^{r-1}(c_0+1)\ \dots\ (c_{i-1}+1)(p-c_i-1)c_{i+1}(c_{i+2}+1)\ \dots\ (c_r+1). \tag{3.2}$$

Assume that we have conditions (1.2) and let $\alpha_j(n)$ denote the number of q-binomial coefficients

$$\begin{bmatrix} n \\ s \end{bmatrix} \quad (s=0,\ 1,\ \dots,\ n)$$

divisible by exactly p^j. Fray [2] has proved that if n has expansion (1.3) then

$$\alpha_0(n)=(a_0+1)(a_1+1)\ \dots\ (a_k+1)$$

which is a special case of our next theorem.

We note that if $p>2$ and $j>h+k+1$, or if $p=2$ and $j>y+k-2$, where y is t if $p^h=2$ and $y=1$ if $p=2$, $h>1$, then

$$\alpha_j(n)=0.$$

In the next theorem we assume $j\leq h+k-1$ if $p>2$ and $j\leq y+k-2$ if $p=2$.

THEOREM 3.1. Assume (1.2) holds and n is a positive integer having expansion (1.3). For $m=1,\ \dots,\ k$ define

$$n_m=a_m+a_{m+1}p+\ \dots\ +a_kp^{k-m}.$$

If $p>2$ then

$$\alpha_j(n)=(a_0+1)\theta_j(n_1)+(e-a_0-1)a_1\theta_{j-h}(n_2)+$$
$$+(e-a_0-1)\sum_{m=1}^{j-h}(p-a_1)\ \dots\ (p-a_m)a_{m+1}\theta_{j-h-m}(n_{m+2}).$$

If $p=2$, $h>1$, then

$$\alpha_j(n)=\theta_j(n).$$

If $p^h=2$, then

$$\alpha_j(n)=(a_1+1)\theta_j(n_2)+(1-a_1)a_2\theta_{j-t}(n_3)+$$
$$+\sum_{m=2}^{j-t+1}(1-a_1)(2-a_2)\ \dots\ (2-a_m)a_{m+1}\theta_{j-t-m+1}(n_{m+2}).$$

PROOF. If $p>2$ we use (1.5). Let r have expansion (1.4). *If* $\begin{bmatrix} n \\ r \end{bmatrix}$ is to be divisible by exactly p^j then we consider the possibilities for ε_i. If $\varepsilon_0=0$ there are a_0+1 choices for b_0, namely

$$b_0=0,\ 1,\ \dots,\ a_0\,,$$

and clearly, by (1.5), there are

$$(a_0+1)\theta_j(n_1)$$

ways of writing r. If $\varepsilon_0=\dots=\varepsilon_m=1$, $\varepsilon_{m+1}\neq 1$, then there are $e-a_0-1$ choices for b_0,

$$b_0=a_0+1,\ a_0+2,\ \dots,\ e-1,$$

there are $p-a_i$ choices for b_i, $i=1,\ \dots,\ m$,

$$b_i=a_i\,,\ a_i+1,\ \dots,\ p-1,$$

and there are a_{m+1} choices for b_{m+1},

$$b_{m+1}=0,\ \dots,\ a_{m+1}-1.$$

By (1.5) it is clear that the number of choices for r is

$$(e-a_0-1)(p-a_1)\ \dots\ (p-a_m)a_{m+1}\theta_{j-h-m}(n_{m+2}).$$

Note that we let $\theta_{j-h-k+1}(n_{k+1})=1$ if $j-h-k+1\geq 0$. The proof is similar for the case $p=2$. Note that for this case $e=1$, $n=n_1$.

For example, if $p>2$ and either $h>j$ or $a_1=\dots=a_{j-h+1}=0$ then

$$\alpha_j(n)=(a_0+1)\theta_j(n_1).$$

Thus if $p>2$ and $s>j$

$$\alpha_j(ep^s)=\theta_j(p^s)=p^{j-1}(p-1),$$

$$\alpha_j(eap^s)=\theta_j(ap^s)=ap^{j-1}(p-1) \qquad (0\leq a<p).$$

Also

$$\begin{aligned}\alpha_1(n)&=(a_0+1)\theta_1(n_1) \text{ if } h>1,\ p>2,\\ &=(a_0+1)\theta_1(n_1)+(e-a_0-1)a_1\theta_0(n_2) \text{ if } h=1,\ p>2,\\ &=(a_1+1)\theta_1(n_2) \text{ if } p^h=2,\\ &=\theta_1(n) \text{ if } p=2,\ h>1.\end{aligned}$$

Thus by (3.1) and (3.2) we have for $0\leq a_0<e$, $0\leq a_i<p$ $(i=1, 2)$,

$$\begin{aligned}&\alpha_1(a_0+e(a_1+a_2p))\\ &\quad=(a_0+1)(p-a_1-1)a_2 \text{ if } p>2,\ h>1,\\ &\quad=(a_0+1)(p-a_1-1)a_2+(e-a_0-1)a_1(a_2+1) \text{ if } p>2,\ h=1,\\ &\quad=0 \text{ if } p^h=2,\\ &\quad=(1-a_1)a_2 \text{ if } p=2,\ h>1.\end{aligned}$$

By using Theorem 3.1 and the formulas for $\theta_j(n)$ found in [1], [3] and [4], we could write out many more formulas for $\alpha_j(n)$.

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