RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

C. J. HIMMELBERG
F. S. VAN VLECK

Lipschitzian generalized differential equations

Rendiconti del Seminario Matematico della Università di Padova, tome 48 (1972), p. 159-169
<http://www.numdam.org/item?id=RSMUP_1972__48__159_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 48 (1973)

LIPSCHITZIAN GENERALIZED DIFFERENTIAL EQUATIONS *)

C. J. HIMMELBERG and F. S. VAN VLECK

ABSTRACT - It is shown that the generalized differential equation $\dot{x} \in R(t, x)$, $x(t_0) = x_0$, where R has closed values, is measurable in t, and Lipschitzian in x, has a solution provided there is an admissible candidate. This extends a result of Filippov who assumed R is in addition continuous in (t, x) and, in particular, includes the case $\dot{x} \in R(t)$ for R measurable. Further, it is shown that if R is integrably bounded, then the contraction principle for multifunctions of Covitz and Nadler can be applied to obtain a global solution, thus generalizing a result of Hermes. The proofs are essentially those given by Filippov and Hermes, but depend on the following Scorza Dragoni type theorem for multifunctions: If a multifunction $R: T \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is measurable in $t \in T$, continuous in $x \in \boldsymbol{R}^n$, then for each $\varepsilon > 0$ there is a subset T' of T such that measure $(T - T') < \varepsilon$ and $R|_{T' \times \boldsymbol{R}^n}$ is lower semicontinuous.

1. Introduction.

If Y is a metric space with metric d, let $S(Y)$ be the nonempty subsets of Y, let $CL(Y)$ be the nonempty closed subsets of Y, let

$$N_\varepsilon(C) = \{y \in Y \mid d(y, c) < \varepsilon \text{ for some } c \in C\}, \ \varepsilon > 0, \ C \in S(Y),$$

and, for $A, B \in S(Y)$, let

$$h_d(A, B) = \begin{cases} \inf\{\varepsilon > 0 \mid A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}, & \text{if the infimum exists,} \\ \infty & \text{, otherwise.} \end{cases}$$

*) Indirizzo degli AA.: Dept. of Mathematics, University of Kansas, Lawrence, Kansas 66044 - Univ. of Colorado, Boulder, Colorado 80302.

The research in this paper was partially supported by University of Kansas General Research Fund Grant 3918-5038.

h_d is the (generalized) Hausdorff pseudometric on $S(Y)$. A multifunction F from a set T into Y is a function from T into $S(Y)$, i.e. $F(t) \in S(Y)$ for each $t \in T$. If T is a topological space and $F: T \to Y$ is a multifunction, F is upper semicontinuous (lower semicontinuous) iff the set $\{t \in T \mid F(t) \cap B \neq \emptyset\}$ is closed (open) for each closed (resp. open) subset B of Y. F has closed graph iff Graph $F = \{(t, x) \mid x \in F(t),\ t \in T\}$ is a closed subset of $T \times Y$. If $(T, \mathfrak{A})$ is a measurable space, $F: T \to Y$ is a measurable (weakly measurable) multifunction iff the set $\{t \in T \mid F(t) \cap \cap B \neq \emptyset\}$ belongs to the σ-algebra $\mathfrak{A}$ of measurable sets for each closed (resp. open) subset B of Y. If T is a compact Hausdorff space with positive Radon measure μ, then $\mathfrak{A}$ is the class of μ-measurable subsets of T. In particular, if T is an interval on the real line and μ is Lebesque measure, $F: T \to Y$ is measurable iff $\{t \in T \mid F(t) \cap B \neq \emptyset\}$ is Lebesgue measurable for each closed subset B of Y. If X is a metric space with metric ρ, we say a multifunction $F: X \to Y$ is Lipschitzian iff there is a constant k such that for $x, x' \in X$,

$$h_d(F(x),\ F(x')) \leq k\rho(x,\ x').$$

In this paper we consider the generalized differential equation

$$\dot{x} \in R(t,\ x),\ x(t_0) = x_0\ , \tag{1}$$

where R is a multifunction from a real interval $T = [t_0\ ,\ t_1]$ cross $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ which satisfies the following conditions:

(a) $R(t, x)$ is closed for each $(t, x) \in T \times \boldsymbol{R}^n$;

(b) $R(\cdot, x)$ is measurable for each $x \in \boldsymbol{R}^n$;

(c) $R(t, \cdot)$ is Lipschitzian with constant $k(t)$ such that

$$k \in L^1(T).$$

This problem has been studied by Filippov [F, Theorem 1], under the additional assumption that R is continuous in (t, x) with respect to the Hausdorff metric on $CL(\boldsymbol{R}^n)$. Also, Hermes [H 2] has shown that the contraction principle for multivalued functions yields this result under the conditions that k is a constant, and that R is compact valued and continuous in (t, x) with respect to the Hausdorff metric on the compact nonempty subsets of $\boldsymbol{R}^n$. We show that conditions (a), (b), (c) are sufficient to yield a solution of (1) provided there is an admissible candidate for a solution. Further, we show that if R is integrably bounded, then the result also follows from a contraction principle for multifunctions due to Covitz and Nadler [CN, Corollary 3]. The statements and proofs of these results are given in Section 3.

In Section 2 we establish a preliminary result which may be of interest by itself. In particular, we need to show that conditions (b) and (c) imply that the multifunction $t \to R(t, x(t))$ is measurable for each continuous (or measurable) function x on T. This is accomplished by establishing a Scorza Dragoni type theorem for multifunctions which do not necessarily have compact values.

2. A Scorza Dragoni Theorem for Multifunctions.

For functions Scorza Dragoni [SD] seems to have been the first to show that if $f(t, x)$ is measurable in t, $t \in T$, and uniformly continuous in x, $x \in X$, then for each $\varepsilon > 0$ there is a set $T_\varepsilon \subseteq T$ such that the measure of $T - T_\varepsilon$ is less than ε and $f\,|_{T_\varepsilon \times X}$ is continuous. It appears that the only results of this type for multifunctions were recently given by Kikuchi [K, Proposition 3] and Castaing [C, Remark 2] and require the multifunction to have compact values. Castaing's result follows from a result of his [C, Theorem] for functions by considering the multifunction as a function into the space of nonempty compact subsets with the Hausdorff metric and observing that this function satisfies the hypotheses of the result for functions. Our theorem is for multifunctions which do not necessarily have compact values or even closed values, and it is established by applying Castaing's result for functions to a function associated with the given multifunction. First we need a lemma.

LEMMA 1. $\boldsymbol{R}^n$ can be metrized with a metric ρ whose uniformity is totally bounded and weaker than the Euclidean uniformity. In fact, we can require $h_\rho \leq h$ if h_ρ is the Hausdorff pseudometric defined on $S(\boldsymbol{R}^n)$ by ρ and h is the (generalized) Hausdorff pseudometric defined by the Euclidean metric.

PROOF. Define a function φ from $\boldsymbol{R}^n$ to the open unit ball U by

$\varphi(x)=\dfrac{x}{1+|x|}$ for $x\in\boldsymbol{R}^n$. Then φ is a homeomorphism of $\boldsymbol{R}^n$ onto U $\left(\text{in fact } \varphi^{-1}(y)=\dfrac{y}{1-|y|}\right)$. Moreover, $|\varphi(x)-\varphi(y)|<|x-y|$ for $x\neq y$, so φ is uniformly continuous. Then the metric ρ defined on $\boldsymbol{R}^n$ by $\rho(x, y)=|\varphi(x)-\varphi(y)|$ is totally bounded and weaker than the Euclidean metric.

THEOREM 1. Let T be a compact Hausdorff space with positive Radon measure μ, let X be a Polish space (= separable space metrizable with a complete metric), $F: T\times X\to\boldsymbol{R}^n$ a multifunction such that $F(t, x)$ is measurable in t for each x and continuous (with respect to the Hausdorff pseudometric on $S(\boldsymbol{R}^n)$) in x for each t. Then for any $\varepsilon>0$ there exists a closed subset T_ε of T such that $\mu(T-T_\varepsilon)<\varepsilon$ and $F|_{T_\varepsilon\times X}$ is lower semicontinuous. If, in addition, F is assumed to have closed values, then $F|_{T_\varepsilon\times X}$ has closed graph and is lower semicontinuous. (If F has compact values, then $F|_{T_\varepsilon\times X}$ is continuous [C, Remark 2]).

PROOF. Let E be $\boldsymbol{R}^n$ remetrized with the totally bounded metric of Lemma 1. Then $F: T\times X\to E$ is obviously still measurable in t. Moreover, $F: T\times X\to E$ is continuous in x, since $h_\rho\leq h$.

Let $\bar{E}$ be the completion of E and define $\bar{F}: T\times X\to\bar{E}$ by $\bar{F}(t, x)=\overline{F(t, x)}$, where here and throughout this proof all closures are with respect to $\bar{E}$.

Then $\bar{F}$ is weakly measurable (and hence measurable — see [HJV, Theorem 1]) in t for each x, since for each open $B\subseteq\bar{E}$ we have $\{t \mid \overline{F(t, x)}\cap B\neq\emptyset\}=\{t \mid F(t, x)\cap B\neq\emptyset\}$.

Also $\bar{F}$ is continuous with respect to the obviously defined Hausdorff metric $\bar{h}_\rho$ on $CL(\bar{E})$, since $\bar{h}_\rho(\overline{F(t, x)}, \overline{F(t, y)})=h_\rho(F(t, x), F(t, y))$.

Since $\bar{E}$ is compact, if follows that as a function from $T\times X$ into $CL(\bar{E})$, $\bar{F}$ is measurable in t and continuous in x. Thus by [C, Theorem], there exists, for each $\varepsilon>0$, a compact subset T_ε of T such that $\mu(T-T_\varepsilon)<\varepsilon$ and $\bar{F}|_{T_\varepsilon\times X}$ is continuous (with respect to $\bar{h}_\rho$) in t and x jointly, or, equivalently, $\bar{F}|_{T_\varepsilon\times X}: T_\varepsilon\times X\to\bar{E}$ is both upper and lower semicontinuous.

But lower semicontinuity for $\bar{F}|_{T_\varepsilon\times X}$ is equivalent to lower semicontinuity for $F|_{T_\varepsilon\times X}$. So $F|_{T_\varepsilon\times X}$ is lower semicontinuous.

Finally, if F has closed values, then Graph $F|_{T_\varepsilon \times X} = (T_\varepsilon \times X \times E) \cap$ $\cap$ Graph $\overline{F}|_{T_\varepsilon \times X}$ and the latter set is closed since $\overline{F}|_{T_\varepsilon \times X}$ is upper semicontinuous.

COROLLARY 1. Let F, T, X be as in Theorem 1. Let $x : T \to X$ be a measurable function and define a multifunction $G : T \to \boldsymbol{R}^n$ by $G(t) = F(t, x(t))$. Then F and G are weakly measurable. If F has closed values, then F and G are both measurable.

3. Lipschitzian Generalized Differential Equations.

We next establish the existence of a solution of

$$\dot{x} \in R(t, x),\ x(t_0) = x_0\,, \tag{1}$$

where R satisfies (a) - (c). By a solution we mean an absolutely continuous function $\varphi : [t_0, t_1] \to \boldsymbol{R}^n$ such that

$$\dot{\varphi}(t) \in R(t, \varphi(t)) \text{ a.e. and } \varphi(t_0) = x_0\,.$$

THEOREM 2. Suppose $R : T \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ satisfies hypotheses (a)-(c), where $T = [t_0, t_1]$ is a given compact interval. Suppose also that there exists an absolutely continuous function $y : T \to \boldsymbol{R}^n$ such that $\sup\{\operatorname{dist}(\dot{y}(t), R(t, y(t))) \mid t \in T\} = M < \infty$. Then equation (1) has a solution on T.

The following proposition, which is a slight generalization of lemmas used by Filippov [F] and Hermes [H 1] in their proofs, is needed in the proof of Theorem 2. A proof can be given along the lines of Hermes' proof using the results of [R, Corollary 1.3] and [KRN, Theorem 1] at appropriate places, so we delete the proof.

PROPOSITION 1. Let $(T, \mathfrak{A})$ be a measurable space, let $F : T \to \boldsymbol{R}^n$ be a measurable multifunction with closed values, and let $w : T \to \boldsymbol{R}^n$ be a measurable function. Then there is a measurable function $v : T \to \boldsymbol{R}^n$ such that $v(t) \in F(t)$ and $|v(t) - w(t)| = \operatorname{dist}(w(t), F(t))$ for $t \in T$.

PROOF OF THEOREM 2. The proof of this is the same as that given by Hermes [H 1] except we use Proposition 1 and Corollary 1 to establish the existence of appropriate iterates $v^i(t)$ and $x^{i+1}(t)=x_0+\int_{t_0}^{t} v^i(\tau)d\tau$ with $v^i(t)\in R(t, x^i(t))$ such that the sequence (x^i) converges uniformly on T to a function x while (v^i) converges in $L^\infty(T)$ to a function v. Since $x^{i+1}(t)=x_0+\int_{t_0}^{t} v^i(\tau)d\tau$, it is clear that $x(t)=x_0+\int_{t_0}^{t} v(\tau)d\tau$. The only other place where one must be careful is in showing that $\dot{x}(t)\in R(t, x(t))$, but that follows, for example, from the fact that a Lipschitzian multifunction with closed values has closed graph.

Note that this result unifies the treatment of the equation $\dot{x}\in R(t)$, R measurable, and that of $\dot{x}\in R(t, x)$.

Actually somewhat more can be proved using the above methods. As was shown by Filippov for R continuous, if R satisfies (a) - (c) in the region $t\in T$, $|x-y(t)|\le b$, where y is absolutely continuous, if $|y(t_0)-x_0|\le\delta<b$, and if there is a function $\rho\in L^1(T)$ such that $\operatorname{dist}(\dot{y}(t), R(t, y(t)))\le\rho(t)$ a.e., then there exists a solution $x(t)$ of (1) which satisfies

$$|x(t)-y(t)|\le\xi(t),\ |\dot{x}(t)-\dot{y}(t)|\le k(t)\xi(t)+\rho(t)\quad \text{a.e.},$$

where

$$\xi(t)=\delta e^{m(t)}+\int_{t_0}^{t} e^{m(t)-m(s)}\rho(s)ds$$

with

$$m(t)=\int_{t_0}^{t} k(\tau)d\tau,$$

as long as $\xi(t)\le b$.

We next show that a solution of (1) on all of T can be established by using Covitz and Nadler's [CN] contraction principle for multifunctions. In what follows we assume that the multifunction $R: T\times \boldsymbol{R}^n\to\boldsymbol{R}^n$ satisfies (a), (b), (c) and

(d) There is a function $r \in L^1(T)$ such that for $v \in R(t, x)$,

$$| v | \leq r(t) \quad \text{a.e.}$$

(It should be noted that (d) implies that there is an absolutely continuous function y such that $\sup \operatorname{dist}(\dot{y}(t), R(t, y(t))) \leq r(t)$ a.e.; simply take $y(t)=0$ for all t). Also we assume without loss of generality that $h(R(t, x), R(t, x')) < k(t) | x - x' |$ for $x \neq x'$ in (c).

We first renorm $L^1(T)$ with an equivalent norm by letting

$$\| x \| = \int_{T} e^{-LK(\tau)} | x(\tau) | d\tau \text{ for } x \in L^1(T),$$

where

$$K(\tau) = \int_{t_0}^{\tau} k(\tau) d\tau$$

and L is any constant greater than one. Also, we let H denote the Hausdorff pseudometric $h_{\| \|}$ on $L^1(T)$.

For $x \in L^1(T)$, define a multifunction R_x by $R_x(t) = R(t, x(t))$. Clearly R_x has closed values, and, by Corollary 1, R_x is measurable. Let $M_R : L^1(T) \to L^1(T)$ be a multifunction defined by

$$M_R(x) = \{ v \mid v \text{ is a measurable selector for } R_x \}.$$

(By [KRN, Theorem 1], $M_R(x)$ is nonempty for each $x \in L^1(T)$ and, by (d), $M_R(x) \subseteq L^1(T)$). Define $I : L^1(T) \to C(T) \subseteq L^1(T)$ by

$$(Ix)(t) = x_0 + \int_{t_0}^{t} x(\tau) d\tau, \quad x \in L^1(T).$$

Finally, let $F : L^1(T) \to L^1(T)$ be the multifunction $M_R \circ I$, i.e. $F(x) = M_R(Ix)$.

PROPOSITION 2. F has closed nonempty values and satisfies

$$H(F(x), F(x')) \leq \frac{1}{L} \| x - x' \|.$$

Thus F is a contractive multifunction and has a fixed point x.

PROOF. First $M_R(x)$ is closed in $L^1(T)$ for $x \in L^1(T)$. For suppose (x^n) is a sequence in $M_R(x)$ which converges to a function $y \in L^1(T)$. Then there is a subsequence (x^{n_k}) of (x^n) which converges almost everywhere to y. Thus, for almost all t, $(x^{n_k}(t)) \to y(t)$. But

$$x^{n_k}(t) \in R(t, x(t)) \text{ a.e.}$$

and $R(t, x(t))$ is closed. Hence

$$y(t) \in R(t, x(t)) \text{ a.e.}$$

and $M_R(x)$ is closed

It follows trivially that $M_R(Ix)$ is closed for each $x \in L^1(T)$.

Next we show that M_R satisfies

$$H(M_R(x), M_R(x')) \leq \int_T e^{-LK(\tau)} k(\tau) \mid x(\tau) - x'(\tau) \mid d\tau$$

for $x, x' \in L^1(T)$. To see this, let $v \in M_R(x)$. Then v is measurable and $v(t) \in R(t, x(t))$ a.e. Since $h(R(t, x(t)), R(t, x'(t))) < k(t) \mid x(t) - x'(t) \mid$ for $x(t) \neq x'(t)$, there is a $z \in R(t, x'(t))$ such that $\mid v(t) - z \mid \leq k(t) \mid x(t) - x'(t) \mid$. Thus the multifunction G defined by $G(t) = R_{x'}(t) \cap K(t)$, where

$$K(t) = \{ z \mid \mid v(t) - z \mid \leq k(t) \mid x(t) - x'(t) \mid \}$$

is nonempty. K is measurable and hence $G = R_{x'} \cap K$ is also measurable. Let w be a measurable selector for G. Then $w(t) \in R(t, x'(t))$ a.e. and $\mid v(t) - w(t) \mid \leq k(t) \mid x(t) - x'(t) \mid$. Thus

$$\| v - w \| \leq \int_T e^{-LK(\tau)} k(\tau) \mid x(\tau) - x'(\tau) \mid d\tau.$$

From this and the analogous inequality obtained by interchanging the roles of x and x' we get

$$H(M_R(x), M_R(x')) \leq \int_T e^{-LK(\tau)} k(\tau) \mid x(\tau) - x'(\tau) \mid d\tau.$$

Since $F=M_R\circ I$, we have

$$H(M_R(Ix),\ M_R(Ix'))\leq\int_T e^{-LK(\tau)}k(\tau)\,|\,(Ix)(\tau)-(Ix')(\tau)\,|\,d\tau$$

$$\leq\int_T e^{-LK(\tau)}k(\tau)\int_{t_0}^{\tau}|\,x(s)-x'(s)\,|\,ds d\tau.$$

By interchanging the order of integration, we obtain

$$H(M_R(Ix),\ M_R(Ix'))\leq\int_T|\,x(s)-x'(s)\,|\int_s^{t_1}k(\tau)e^{-LK(\tau)}d\tau ds=$$

$$=\int_T|\,x(s)-x'(s)\,|\,\frac{1}{L}[e^{-LK(s)}-e^{-LK(t_1)}]ds$$

$$\leq\frac{1}{L}\int_T|\,x(s)-x'(s)\,|\,e^{-LK(s)}ds=\frac{1}{L}\|\,x-x'\,\|\,.$$

Finally, we apply the contraction principle of Covitz and Nadler [CN, Corollary 3] to F to obtain a function $x\in L^1(T)$ such that $x\in M_R(Ix)$. This completes the proof of the proposition.

The entire argument is now completed by noting that if we let $\varphi=Ix$, where x is the above fixed point, then $\varphi=Ix\in IM_R(Ix)=IM_R(\varphi)$. Thus, φ is the desired solution of (1) on the entire interval T.

It is perhaps worth remarking that if the functions $k(t)$ and $r(t)$ in hypotheses (c) and (d) are replaced by constants k and r, respectively, then $M_R\circ I$ maps $L^\infty(T)$ into $L^\infty(T)$. Further, if we renorm $L^\infty(T)$ by letting

$$\|\,x\,\|_\infty=\text{ess sup}\,\{e^{-kL(t-t_0)}\,|\,x(t)\,|\ \ |\,t\in T\},$$

then it is not difficult to show that

$$H_\infty(M_R(Ix),\ M_R(Ix'))\leq\frac{1}{L}\|\,x-x'\,\|_\infty\,,$$

where H_∞ is the Hausdorff pseudometric corresponding to $\| \ \|_\infty$. In particular, in this case we obtain a solution φ of (1) on T whose derivative $\dot{\varphi}$ is essentially bounded by r.

The idea of renorming certain function spaces in order to change a locally contracting operator into a global contraction is due to Bielicki [B]; we have shown above that this idea can equally well be applied to contracting multifunctions. Lasota and Opial [LO] have also indicated its usefulness in connection with solving generalized differential equations via the Fan fixed point theorem.

REFERENCES

[B] BIELECKI, A.: *Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinairies*, Bull. Polish Acad. Sci. 4 (1956), 261-264.

[C] CASTAING, A.: *Sur le graphe d'une multi-application Souslinienne (Mesurable)*, Faculté des Sciences de Montpellier, Mathématiques (1969).

[CN] COVITZ, H. - NALDER, S. B. Jr.: *Multi-valued contraction mappings in generalized metric space*, Israel J. Math. 8 (1970), 5-11.

[F] FILIPPOV, A. F.: *Classical solutions of differential equations with multivalued right-hand sides*, SIAM J. Control 5 (1967), 609-621 (English Transl.).

[H 1] HERMES H.: *The generalized differential equation* $\dot{x} \in R(t, x)$, Advanc. Math. 4 (1970), 149-169.

[H 2] HERMES, H.: *On the structure of attainable sets for generalized differential equations and control systems*, J. Diff. Eqs. 9 (1971), 141-154.

[HJV] HIMMELBERG, C. - JACOBS, M. - VANVLECK, F.: *Measurable multifunctions, selectors, and Filippov's implicit functions lemma*, J. Math. Anal. Appl. 25 (1969), 276-284.

[K] KIKUCHI, N.: *On contingent equations satisfying the Carathéodory type conditions*, RIMS. Kyoto Univ. Ser. A 3 (1968), 361-371.

[KRN] KURATOWSKI, K. - RYLL-NARDZEWSKI: *A general theorem on selectors*, Bull. Polish Acad. Sci. 13 (1965), 397-403.

[LO] LASOTA, A. - OPIAL, Z.: *An application of the Kakutani-Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. 13 (1965), 781-786.

[R] ROCKAFELLAR, R. T.: *Measurable dependence of convex sets and functions on parameters*, J. Math. Anal. Appl. 28 (1969), 4-25.

[SD] Scorza Dragoni, G.: *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile,* Rend. Sem. Mat. Univ. Padova 17 (1948), 102-106.

Manoscritto pervenuto in redazione il 18 aprile 1972.