RENDICONTI del SEMINARIO MATEMATICO della UNIVERSITÀ DI PADOVA

RICHARD D. CARMICHAEL

Distributional boundary values in $\mathfrak{D}'_{L^p}$. III

Rendiconti del Seminario Matematico della Università di Padova, tome 48 (1972), p. 137-158

<http://www.numdam.org/item?id=RSMUP_1972__48__137_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 48 (1973)

DISTRIBUTIONAL BOUNDARY VALUES IN $\mathcal{D}'_{L^p}$. III

RICHARD D. CARMICHAEL *)

1. Introduction.

In Carmichael [5, 6] we have obtained results in which distributions in $\mathcal{D}'_{L^p}$ are related to and represented as boundary values of analytic functions. In the present paper we shall continue our investigation of this topic.

All terminology concerning cones $C \subset \boldsymbol{R}^n$ and compact subcones in this paper will be the same as that in Carmichael [4, p. 845] or [6, p. 252]. In particular we call the readers attention to the function $u_C(t)$, the indicatrix of the cone C, the number ρ_C, which characterizes the nonconvexity of C, and the tubular cone $T^C = \boldsymbol{R}^n + iC \subset \boldsymbol{C}^n$, the definition of which can be found in the above references.

Let C be an open cone; and let $f(z)$, $z = x + iy \in \boldsymbol{C}^n$, satisfy

$$(1) \quad |f(z)| \leq K(C')(1+|z|)^N \exp[2\pi(A+\sigma)|y|], \; z \in T^{C'} = \boldsymbol{R}^n + iC',$$

for all real numbers $\sigma > 0$, where C' is an arbitrary compact subcone of C, A is a nonegative real number, N is any real number, and $K(C')$ is a constant depending on C'. The functions which we have studied in [5, 6] in relation to the $\mathcal{D}'_{L^p}$ distributions have been analytic functions in the octants

$$0_\delta = \{z : \delta_j \operatorname{Im}(z_j) > 0, \; \delta = (\delta_1, \ldots, \delta_n), \; \delta_j = \pm 1, \; j = 1, \ldots, n\}$$

*) Indirizzo dell'A.: Dept. of Mathematics, Wake Forest University, Winston Salem, North Carolina 27109, U.S.A.

or in the general tubular cone $T^C = \mathbf{R}^n + iC$ which satisfy boundedness conditions similar to (1). In [4, 7, 8, 9] we have related analytic functions having a growth condition as in (1) to other spaces of distributions. Letting $\mathcal{S}'$ denote the Schwartz space of tempered distributions, we have obtained the following result which has importance in quantum field theory and which will be useful in this paper.

THEOREM 1. *Let C be an open connected cone. Let $f(z)$ be analytic in $T^C = \mathbf{R}^n + iC$ and satisfy* (1). *Let $f(z) \to U$ in the $\mathcal{S}'$ topology as $y \to 0$, $y \in C' \subset C$. Then $U \in \mathcal{S}'$; there exists an element $V \in \mathcal{S}'$ such that* $\operatorname{supp}(V) \subseteq \{t : u_C(t) \leq A\}$ *and $U = \widehat{V}$; and $f(z) = \langle V, e^{2\pi i \langle z, t \rangle} \rangle$, $z \in T^{C'}$, $C' \subset C$.*

PROOF. See [4, Theorem 2]. Here $\operatorname{supp}(V)$ is the support of V, $\widehat{V}$ denotes the Fourier transform of V, and C' is an arbitrary compact subcone of C.

Korányi [14] and Stein, Weiss, and Weiss [16] have defined the classical Hardy $H^p(T^C)$ spaces, $1 \leq p \leq \infty$, for functions analytic in tube T^C. We note that $H^p(T^C) \subset \mathcal{D}'_{L^p} \subset \mathcal{S}'$, $1 \leq p \leq \infty$.

In this paper we shall obtain distributional boundary value results concerning the space of functions $H^p(T^C)$, and the boundary values will be seen to be elements of $\mathcal{D}'_{L^p}$. As in Carmichael [5, 6], the topology which we shall use will be that of $\mathcal{S}'$. In section 2 we shall obtain results similar to Theorem 1 for functions $f(z) \in H^\infty(T^C)$ and for functions $f(z) \in H^p(T^C)$, $1 \leq p < \infty$, which satisfy (1). Under these assumptions more can be said about the function $f(z)$ than in Theorem 1; we shall see that the convergence of $f(z)$ to an element in $\mathcal{D}'_{L^p} \subset \mathcal{S}'$ can be proved and that $f(z)$ can be represented by the Poisson integral of its boundary value as well as the Fourier-Laplace transform $\langle V, e^{2\pi i \langle z, t \rangle} \rangle$ of $V \in \mathcal{S}'$. Further, for suitable choices of p, $f(z) \in H^p(T^C)$ can also be represented by the Cauchy integral of its boundary value. If $f(z) \in H^2(T^C)$ it is known that $f(z)$ has each of the above representations. In the results of this paper we extend the values of p for which $H^p(T^C)$ functions have each of these representations. Further, we prove a version of Fatou's theorem in which more is concluded about the $H^\infty(T^C)$ function and its boundary value than in the classical setting for tube domains. In section 3 we shall obtain converse results to those of section 2 in which an $H^p(T^C)$ function is manufactured from a distribution. In particular we are interested in obtaining a converse to the classical Fatou theorem. Section 4 will be devoted to obtaining generalizations to disconnected tubular cones.

In the remainder of this introductory section we shall introduce the n dimensional notation and definitions to be used throughout this paper. The n dimensional notation and the definition of the derivative D^α, α being an n-tuple of nonnegative integers, will be the same as in Carmichael [4]. T^C will always represent the subset of $\boldsymbol{C}^n$ defined by $T^C = \boldsymbol{R}^n + iC$, where C is a cone. If C is connected, T^C will be called a tubular radial domain; while if C is not connected, we shall refer to T^C as a tubular cone. The function spaces $\mathcal{E}$ and $\mathcal{S}$ and the distribution spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$ are defined in Schwartz [15]; and all definitions of terms concerning distributions, such as support and convolution, are those of Schwartz. The Fourier and inverse Fourier transforms of L^1 functions and $\mathcal{S}'$ distributions are defined in Carmichael [4]. The Fourier transform of a function $\varphi(t)$ will be denoted by $\mathfrak{F}[\varphi(t); x]$ or $\widehat{\varphi}(x)$; similarly we denote the inverse Fourier transform as as $\mathfrak{F}^{-1}[\varphi(t); x]$. The Fourier and inverse Fourier transforms of $V \in \mathcal{S}'$ are denoted $\widehat{V}$ and $\mathfrak{F}^{-1}(V)$, respectively.

A sequence $\{\varphi_\lambda\} \in \mathcal{S}$ converges to $\varphi \in \mathcal{S}$ in $\mathcal{S}$ as $\lambda \to \lambda_0$ if

$$\lim_{\lambda \to \lambda_0} \sup_x | x^\beta D^\alpha(\varphi_\lambda(x) - \varphi(x)) | = 0,$$

where α and β are arbitrary n-tuples of nonnegative integers. Let $z \in T^C$, C being an open connected cone. By $f(z) \to V$ in the topology of $\mathcal{S}'$ as $y = \mathrm{Im}(z) \to 0$, $y \in C$, we mean that $\langle f(z), \varphi(x) \rangle \to \langle V, \varphi(x) \rangle$ as $y \to 0$, $y \in C$, where φ is any element of $\mathcal{S}$. We note that the boundary value V is obtained on the distinguished boundary of T^C, $\{z = x + iy : x \in \boldsymbol{R}^n, y = 0\}$, which is not necessarily the topological boundary unless $n = 1$.

On several occasions in this paper we shall make use of Theorem 4 in [4]. We note that this result holds for $A = 0$ as well as for $A > 0$; the proof for $A = 0$ is exactly the same. With this is mind, we shall assume henceforth that Theorem 4 in [4] holds for all real numbers $A \geq 0$. Unless otherwise specified, $g(x) \in L^p(f(z) \in H^p(T^C))$, $1 \leq p \leq \infty$, means throughout this paper $g(x) \in L^p(f(z) \in H^p(T^C))$ for some p,

$1 \leq p \leq \infty$. The definition of the $H^p(T^C)$ spaces, $1 \leq p \leq \infty$, which we shall use in this paper is given in [16].

2. Distributional boundary values of H^p functions.

Let C be an open connected cone, and let $0(C)$ denote the convex envelope (hull) of C. If $f(z)$ is analytic in T^C, then by Bochner's theorem on analytic extension [3, Chapter V], $f(z)$ has an analytic extension to $T^{0(C)}$. Further, if $f(z) \in H^p(T^C)$, then its extension is in $H^p(T^{0(C)})$ and

$$\sup_{y \in C} \int_{\mathbf{R}^n} | f(x+iy) |^p \, dx = \sup_{y \in 0(C)} \int_{\mathbf{R}^n} | f(x+iy) |^p \, dx.$$

(See [16, p. 1036]). Thus it suffices to assume that C is convex.

For $z \in T^C$, we define the Cauchy kernel $K(z-t)$ by

$$K(z-t) = \int_{C^*} e^{2\pi i \langle z-t, \eta \rangle} d\eta,$$

where $C^* = \{\eta : u_C(\eta) \leq 0\}$ is the dual cone of C. If $\overline{C}$ contains an entire straight line, then by a result of Vladimirov [18, Lemma 1, p. 222] the cone C^* lies in some $(n-1)$ dimensional plane; and $K(z-t)=0$. To avoid this triviality we assume throughout this section that the cone C is open, convex, and has the property that $\overline{C}$ contains no entire straight line.

From the Cauchy kernel we define the Poisson kernel corresponding to T^C by

$$Q(z;\ t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)}.$$

If T^C is the upper half plane in $\mathbf{C}^1$, then $K(z-t)$ and $Q(z;\ t)$ are $\frac{1}{2\pi i} \frac{1}{z-t}$ and $\frac{1}{\pi} \frac{y}{(t-x)^2+y^2}$, $z=x+iy$, respectively, which are the classical Cauchy and Poisson kernels.

Let $g\in L^p$, $1\leq p\leq\infty$. Then

$$\int_{R^n} g(t)K(z-t)dt \text{ and } \int_{R^n} g(t)Q(z;\ t)dt,\ z\in T^C,$$

are the Cauchy and Poisson integrals, respectively, of g. We can now prove

THEOREM 2. *Let $f(z)\in H^p(T^C)$, $1\leq p<\infty$; and let $f(z)$ satisfy* (1). *There exists a function $g(x)\in L^p$, $1\leq p<\infty$, such that $f(z)\to g(x)$ in the topology of $\mathcal{S}'$ (as well as in the L^p norm topology) as $y=\text{Im}(z)\to 0$, $y\in C$; and there exists an element $V\in\mathcal{S}'$ with* $\text{supp}(V)\subseteq S_A=\{t: u_C(t)\leq A\}$ *such that $g(x)=\widehat{V}$ and*

$$f(z)=\langle V,\ e^{2\pi i\langle z,t\rangle}\rangle=\int_{R^n} g(t)Q(z;\ t)dt,\ z\in T^C. \tag{2}$$

PROOF. Combining Propositions 4 and 3 (c) in Korányi [14], we obtain the existence of a function $g(x)\in L^p$, $1\leq p<\infty$, such that $f(z)\to g(x)$ in L^p as $y\to 0$, $y\in C$. Let $\varphi\in\mathcal{S}$. By Hölder's inequality,

$$|\langle f(z),\ \varphi(x)\rangle-\langle g(x),\ \varphi(x)\rangle|\leq\|f(z)-g(x)\|_{L^p}\|\varphi\|_{L^q}, \tag{3}$$

$\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$. If $p=1$ we have

$$|\langle f(z),\ \varphi(x)\rangle-\langle g(x),\ \varphi(x)\rangle|\leq K\int_{R^n}|f(z)-g(x)|\,dx, \tag{4}$$

where $|\varphi(x)|\leq K$. Since $\varphi\in\mathcal{S}\subset L^q$ for all q, $1\leq q\leq\infty$, then by (3), (4), and the fact that $f(z)\to g(x)$ in L^p, $1\leq p<\infty$, as $y\to 0$, $y\in C$, we have that $f(z)\to g(x)$ in $\mathcal{S}'$ as $y\to 0$, $y\in C$. Having obtained this $\mathcal{S}'$ boundery value, we now apply Theorem 1 and obtain an element $V\in\mathcal{S}'$ with $\text{supp}(V)\subseteq S_A$ such that $g(x)=\widehat{V}$ and $f(z)=\langle V,\ e^{2\pi i\langle z,t\rangle}\rangle$, $z\in T^{C'}$, $C'\subset C$. But under these conditions on V, we have by [4, Theorem 4] that $\langle V,\ e^{2\pi i\langle z,t\rangle}\rangle$ is analytic in T^C. Thus by the identity theorem for analytic functions, $f(z)=\langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z\in T^C$. Again applying Propositions 4 and 3 (c) of Korányi [14], we have

$$f(z)=\int_{\boldsymbol{R}^n} g(t)Q(z;\ t)dt,\ z\in T^C;$$

and (2) is obtained.

We now restrict p to $1\leq p\leq 2$ in Theorem 2 and obtain an interesting corollary. First, however, we prove the following lemma.

LEMMA 1. *Let* $f\in L^p$, $1\leq p\leq 2$. *Let* $g\in L^q, \frac{1}{p}+\frac{1}{q}=1$; *and assume that* $\mathfrak{F}^{-1}[g(t);\ x]$ *exists classically and belongs to* L^p, $1\leq p\leq 2$, $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\mathfrak{F}^{-1}(f * g)=\mathfrak{F}^{-1}(f)\mathfrak{F}^{-1}(g) \tag{5}$$

in $\mathcal{S}'$.

PROOF. Since $f\in L^p$, $1\leq p\leq 2$, then $\mathfrak{F}^{-1}[f(t);\ x]$ exists classically and is an element of $L^q, \frac{1}{p}+\frac{1}{q}=1$. By hypothesis $\mathfrak{F}^{-1}[g(t);\ x]\in L^p$, $1\leq p\leq 2$. Thus $\mathfrak{F}^{-1}[f(t);\ x]\ \mathfrak{F}^{-1}[g(t);\ x]\in L^1\subset\mathcal{S}'$. Further, it is known that $f * g$ exists as a classical convolution, is continuous, and is an element of $L^r, \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$, (i.e. L^∞). Thus $f * g\in\mathcal{S}'$, and $\mathfrak{F}^{-1}(f * g)\in\mathcal{S}'$. Since both sides of (5) are well defined as elements of $\mathcal{S}'$, (5) follows by a result of Schwartz [15, Chapter VII] which states that the inverse Fourier transform converts convolution into multiplication in $\mathcal{S}'$ when the algebraic operations are well defined in $\mathcal{S}'$.

COROLLARY 1. *Let* $f(z)\in H^p(T^C)$, $1\leq p\leq 2$; *and let* $f(z)$ *satisfy* (1) *for* $A=0$. *There exists a function* $g(x)\in L^p$, $1\leq p\leq 2$, *such that* $f(z)\rightarrow g(x)$ *in the* $\mathcal{S}'$ *topology (as well as the* L^p *norm topology) as* $y\rightarrow 0$, $y\in C$; *and there exists a function* $h(t)\in L^q, \frac{1}{p}+\frac{1}{q}=1$, *with* $\mathrm{supp}\,(h)\subseteq C^*=\{t: u_C(t)\leq 0\}$ *almost everywhere such that* $g=\widehat{h}$ *in* $\mathcal{S}'$

and

$$(6)\qquad f(z)=\langle h(t),\ e^{2\pi i\langle z,t\rangle}\rangle=\int_{\mathbf{R}^n} g(t)K(z-t)dt=\int_{\mathbf{R}^n} g(t)Q(z;\ t)dt,$$

$z\in T^C$, *where the equality* (6) *is in* $\mathcal{S}'$.

PROOF. From Theorem 2 we obtain the function $g(x)\in L^p$, $1\le p\le 2$, and an element $V\in\mathcal{S}'$ with $\operatorname{supp}(V)\subseteq C^*$ and $g=\widehat{V}$. Thus $V=\mathfrak{F}^{-1}(g)$ in $\mathcal{S}'$, Since $g(x)\in L^p$, $1\le p\le 2$, $h(t)=\mathfrak{F}^{-1}[g(x);\ t]$ exists classically and is an element of $L^q, \frac{1}{p}+\frac{1}{q}=1$. Thus $V=h(t)$ in $\mathcal{S}'$, and $\operatorname{supp}(h)\subseteq C^*$ almost everywhere. Let $\varphi\in\mathcal{S}$. Performing a change of order of integration we obtain

$$(7)\qquad \langle\langle h(t),\ e^{2\pi i\langle z,t\rangle}\rangle,\ \varphi(x)\rangle=\langle h(t),\ e^{-2\pi\langle y,t\rangle}\widehat{\varphi}(t)\rangle= =\langle \mathfrak{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle}h(t)],\ \varphi(x)\rangle,$$

where $I_{C_*}(t)$ is the characteristic function of C^*. Now $I_{C^*}(t)e^{-2\pi\langle y,t\rangle}\in L^p$ for all p, $1\le p\le\infty$. In particular if $1\le p\le 2$, then

$$\mathfrak{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle};\ x]\in L^q \text{ for all } q, \frac{1}{p}+\frac{1}{q}+1.$$

We now apply Lemma 1 to obtain

$$\mathfrak{F}^{-1}(g*\mathfrak{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle};\ x])=h(t)I_{C^*}(t)e^{-2\pi\langle y,t\rangle}$$

in $\mathcal{S}'$. Thus

$$\mathfrak{F}(I_{C^*}(t)e^{-2\pi\langle y,t\rangle}h(t))=g*\mathfrak{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle};\ x]=g*\int_{C^*} e^{2\pi i\langle z,t\rangle}dt$$

in $\mathcal{S}'$. Returning to (7) we have

$$(8)\qquad \langle\langle h(t),\ e^{2\pi i\langle z,t\rangle}\rangle,\ \varphi(x)\rangle=\langle g*\int_{C^*} e^{2\pi i\langle z,t\rangle}d(t),\ \varphi(x)\rangle$$

$$=\langle\langle g(t),\ K(z-t)\rangle,\ \varphi(x)\rangle.$$

Combining (8) with (2) we thus obtain (6), and the proof is complete.

The results obtained in Theorem 2 and Corollary 1 are reminiscent of classical results for H^p spaces of functions analytic in a half plane in $\boldsymbol{C}^1$. Hille and Tamarkin [11, Theorem 2] have shown that if $f(z)$ is analytic in the half plane $\operatorname{Im}(z)>0$ and has a limit function $F(x)\in L^p$, and if $f(z)$ is represented by the Cauchy integral of $F(x)$, then it is also represented by the Poisson integral of $F(x)$ and vice versa. Hille and Tamarkin ([11, Theorem 3] and [12, Theorem]) have also obtained results relating analytic functions which have boundary values and which are represented by the Cauchy (Poisson) integral of their boundary values with a Fourier transform which vanishes on a half line. (For related results we also refer to [13]). Of course the Hille and Tamarkin theorems hold for the H^p spaces of functions analytic in a half plane. Stein and Weiss have shown that if $f(z)\in H^2(T^C)$, then equality (6) holds [17; Theorem 3.1, p. 101; Theorem 3.6, p. 103; Theorem 3.9, p. 106]. In Theorem 2 and Corollary 1 we have obtained conditions under which these classical results of Hille and Tamarkin are extended to the $H^p(T^C)$ spaces for other values of p.

We shall now obtain a result similar to Theorem 2 for $H^\infty(T^C)$. In this version of Fatou's theorem we are able to say more about the element of H^∞ and its boundary value than in the classical setting for tubular radial domains.

THEOREM 3. *Let $f(z)\in H^\infty(T^C)$. There exists a function $g(x)\in L^\infty$ such that $f(z)\to g(x)$ in the $\mathcal{S}'$ topology (as well as in the weak-star topology of L^∞) as $y\to 0$, $y\in C$; and there exists an element $V\in\mathcal{D}'_{L^2}$ with $g(x)=V$ and* $\operatorname{supp}(V)\subseteq C^*=\{t: u_C(t)\leq 0\}$ *such that*

$$f(z)=\langle V, e^{2\pi i\langle z,t\rangle}\rangle=\int_{\boldsymbol{R}^n} g(t)Q(z;t)dt, \quad z\in T^C.$$

PROOF. Combining Propositions 4 and 3 (d) in Koranyi [14], we obtain the existence of a function $g(x)\in L^\infty$ such that $f(z)\to g(x)$ in the weak-star topology of L^∞ as $y\to 0$, $y\in C$. This convergence and the Lebesgue dominated convergence theorem imply immediately that $f(z)\to g(x)$ in $\mathcal{S}'$ as $y\to 0$, $y\in C$. These same results of Korányi also imply

$$f(z)=\int_{R^n} g(t)Q(z;\ t)dt,\ z\in T^C.$$

We now put

$$F(z)=\frac{f(z)}{1+z_1^2\ \dots\ z_n^2}.$$

Since $f(z)\in H^\infty(T^C)$, then $F(z)\in H^2(T^C)$; and by a result of Bochner [2, section 3]. (See also Vladimirov [18, pp. 224-227]), there exists a function $\psi(t)\in L^2$ with $\operatorname{supp}(\psi)\subseteq C^*$ such that

$$F(z)=\int_{R^n}\psi(t)e^{2\pi i\langle z,t\rangle}dt,\ z\in T^C.$$

We now put $V=(1+D^{(2,\dots,2)})\psi(t)$. Then $\operatorname{supp}(V)=\operatorname{supp}(\psi)\subseteq C^*$; and by the Schwartz characterization theorem [15, Théorème XXV, p. 201], $V\in\mathfrak{D}'_{L^2}$. A straightforward calculation now gives

$$(9)\qquad \langle V,\ e^{2\pi i\langle z,t\rangle}\rangle=(1+z_1^2\ \dots\ z_n^2)\int_{R^n}\psi(t)e^{2\pi i\langle z,t\rangle}dt=f(z).$$

Let $\xi(\eta)\in\mathcal{E}$, the space of infinitely differentiable functions, $\eta\in \boldsymbol{R}^1$, such that $\xi(\eta)=1$ for $\eta\geq 0$, $\xi(\eta)=0$ for $\eta\leq -\varepsilon$, $\varepsilon>0$; and $0\leq\xi(\eta)\leq 1$. Put $\gamma(t)=\xi(\langle t,\ y\rangle)$, $y\in C$. Let $\varphi\in\mathcal{S}$. Using (9) we obtain

$$(10)\qquad \langle f(z),\ \varphi(x)\rangle=\langle V,\ \gamma(t)e^{-2\pi\langle y,t\rangle}\check{\varphi}(t)\rangle,\ z\in T^C.$$

It is straightforward to show that $\gamma(t)e^{-2\pi\langle y,t\rangle}\check{\varphi}(t)\to\gamma(t)\check{\varphi}(t)$ in $\mathcal{S}$ as $y\to 0$, $y\in C$. Since $V\in\mathfrak{D}'_{L^2}\subset\mathcal{S}'$ (i.e. is continuous), then

$$(11)\qquad \langle V,\ \gamma(t)e^{-2\pi\langle y,t\rangle}\check{\varphi}(t)\rangle\to\langle V,\ \gamma(t)\check{\varphi}(t)\rangle=\langle\check{V},\ \varphi\rangle$$

as $y\to 0$, $y\in C$. (11) combined with (10) shows that $f(z)\to\check{V}$ in $\mathcal{S}'$ as $y\to 0$, $y\in C$. Since the limit in $\mathcal{S}'$ of $f(z)$ is unique, we thus have $g(x)=\widehat{V}$; and the proof is complete.

If $f(z)\in H^{\infty}(T^C)$, then by definition $f(z)$ is bounded for $z\in T^C$ and hence satisfies (1) for $A=0$. Thus once we obtained the boundary value $g(x)$ in Theorem 3, we could have immediately applied Theorem 1 to obtain an element $V\in\mathcal{S}'$ such that $g(x)=\widehat{V}$ and supp $(V)\subseteq C^*$. We see, however, from the proof of Theorem 3 that we can actually make the stronger statement that $V\in\mathcal{D}'_{L^2}(\mathcal{D}'_{L^2}\subset\mathcal{S}')$.

3. Converse results.

Throughout this section C will denote an open convex cone which has the property that $\overline{C}$ contains no entire straight line.

The following theorem and corollary can be viewed as connverses to the combination of Propositions 4 and 3 (c) of Korányi [14] and to Theorem 2 of the present paper for the corresponding values of p.

THEOREM 4. *Let* $g(t)\in L^p$, $1\leq p\leq 2$; *and let* $\operatorname{supp}(g)\subseteq C^*= =\{t: u_C(t)\leq 0\}$. *There exists a function* $f(z)\in H^q(T^C)$ *and a function* $h(x)=\mathfrak{F}[g(t);\ x]\in L^q, \frac{1}{p}+\frac{1}{q}=1$, $1\leq p\leq 2$, *such that* $f(z)\to h(x)$ *in the* $\mathcal{S}'$ *topology (as well as in the* L^q *norm topology,* $\frac{1}{p}+\frac{1}{q}=1$, $1<p\leq 2$, *or in the weak-star topology of* L^∞ *if* $p=1$) *as* $y\to 0$, $y\in C$.

PROOF. Let $I_{C^*}(t)$ denote the characteristic function of C^*, and let $\gamma(t)$ be defined as in the proof of Theorem 3. Put

$$f(z)=\int_{\mathbf{R}^n} g(t)e^{2\pi i\langle z,t\rangle}dt=\int_{\mathbf{R}^n} I_{C^*}(t)g(t)\gamma(t)e^{2\pi i\langle z,t\rangle}dt,\ z\in T^C.$$

Since $g(t)\in L^p\subset\mathcal{S}'$, $1\leq p\leq 2$, and $\operatorname{supp}(g)\subseteq C^*$, then by [4, Theorem 4] $f(z)$ is analytic in T^C. For the present we let $y=\operatorname{Im}(z)\in C$ be fixed. We have for $t\in\mathbf{R}^n$ that

$$(12)\qquad |I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}|\leq 1;$$

and $I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}\in L^q$ for all q, $1\leq q\leq\infty$. By Hölders inequality and (12), we have for $g(t)\in L^p$, $1\leq p\leq 2$, that $I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}g(t)\in$

$\in L^1 \cap L^p$. Now

$$f(z)=\int_{R^n} I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}g(t)e^{2\pi i\langle x,t\rangle}dt$$

$$=\mathfrak{F}[I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}g(t);\ x];$$

and the Fourier transform can be interpreted in the appropriate limit in the mean sense for $1<p\leq 2$. Thus by the Fourier transform theory, $f(z)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, $1\leq p\leq 2$, as a function of x for any fixed $y\in C$. If $p=1$, $q=\infty$; and using (12) we have

$$|f(z)|\leq\int_{R^n}|g(t)|\,dt<\infty. \tag{13}$$

For $1<p\leq 2$, we have again by the Fourier transform theory and (12) that

$$\int_{R^n}|f(x+iy)|^q\,dx\leq\|I_{C^*}(t)\gamma(t)e^{-2\pi\langle y,t\rangle}g(t)\|_{L^p}^q\leq\|g\|_{L^p}^q<\infty, \tag{14}$$

$\frac{1}{p}+\frac{1}{q}=1$. But the right hand sides of (13) and (14) are independent of $y\in C$. Thus the estimates in (13) and (14) hold for all $y\in C$; and it follows that $f(z)\in H^q(T^C)$, $\frac{1}{p}+\frac{1}{q}=1$, $1\leq p\leq 2$. Further, since $g(t)\in L^p$, $1\leq p\leq 2$, then $h(x)=\widehat{g}(x)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$; and using a proof similar to that in equations (10) and (11), we have $f(z)\to\widehat{g}(x)=h(x)$ in $\mathcal{S}'$ as $y\to 0$, $y\in C$.

Let $1<p\leq 2$. As in the proof of Theorem 2, we obtain the existence of a function $\psi(x)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, such that $f(z)\in H^q(T^C)$ converges in the L^q norm topology, and hence in the $\mathcal{S}'$ topology, to $\psi(x)$ as $y\to 0$, $y\in C$. Since the $\mathcal{S}'$ limit of $f(z)$ is unique, then $h(x)=\psi(x)$ almost

everywhere. Thus $f(z)\to h(x)$ in the L^q norm topology as $y\to 0$, $y\in C$. If $p=1$ and $q=\infty$, it similarly follows using the proof of Theorem 3 that $f(z)\in H^\infty(T^C)$ converges in the weak-star topology of L^∞ to $h(x)$ as $y\to 0$, $y\in C$; and the proof is complete.

COROLLARY 2. *Let $g(x)\in L^2$, and let $V\in\mathcal{S}'$ such that* $\operatorname{supp}(V)\subseteq \subseteq C^*=\{t: u_C(t)\le 0\}$ *and $g(x)=\widehat{V}$ in $\mathcal{S}'$. There exists an element $f(z)\in H^2(T^C)$ such that $f(z)\to g(x)$ in the $\mathcal{S}'$ topology (as well as in the L^2 norm topology) as $y\to 0$, $y\in C$.*

PROOF. Since $g(x)\in L^2$, there exists an element $h(t)\in L^2$ such that $g(x)=\mathfrak{F}[h(t); x]$ and $h(t)=\mathfrak{F}^{-1}[g(x); t]$. For $\varphi\in\mathcal{S}$ we have

$$\langle V, \varphi\rangle=\langle \mathfrak{F}^{-1}(g), \varphi\rangle=\langle h, \varphi\rangle;$$

so that $V=h(t)$ in $\mathcal{S}'$ and $\operatorname{supp}(h)\subset C^*$ almost everywhere. We now put

$$f(z)=\int_{\mathbf{R}^n} h(t)e^{2\pi i\langle z, t\rangle}dt=\int_{\mathbf{R}^n} I_{C^*}(t)h(t)\gamma(t)e^{2\pi i\langle z, t\rangle}dt,\quad z\in T^C,$$

where $I_{C^*}(t)$ and $\gamma(t)$ are as in the proof of Theorem 4; and the conclusions follow from Theorem 4 for this $f(z)$.

We note that the functions $f(z)\in H^q(T^C)$ constructed in Theorem 4 and Corollary 2 satisfy the following boundedness condition:

$$|f(z)|\le K(C')(1+|z|)^N(1+|y|^{-M}),\quad z\in T^{C'},$$

where C' is an arbitrary compact subcone of C, $K(C')$ is a constant depending on C', and M and N are nonnegative integers which do not depend on C'. This result follows from Theorem 4 of Carmichael [4].

In Theorem 4 and Corollary 2 the manufactured function $f(z)$ has belonged to certain specified $H^q(T^C)$ spaces, namely those values of q satisfy $\frac{1}{p}+\frac{1}{q}=1$, $1\le p\le 2$. In the following theorem we obtain conditions under which the function $f(z)$ is in $H^q(T^C)$ for all q, $1\le q\le\infty$. This result generalizes a theorem of Carmichael [10, Theorem II.4].

THEOREM 5. *Let* $\varphi(t)\in\mathcal{S}$; *and let* $\operatorname{supp}(\varphi)\subseteq S_A=\{t: u_C(t)\le A\}$, $A\ge 0$. *There exists a function* $f(z)\in H^p(T^C)$ *for all* p, $1\le p\le\infty$, *such that* $f(z)\to\hat{\varphi}(x)\in\mathcal{S}$ *in the topology of* $\mathcal{S}'$ (*as well as pointwise*) *as* $y\to 0$, $y\in C$.

PROOF. Let $I_{S_A}(t)$ denote the characteristic function of S_A. Let $\xi(\eta)\in\mathcal{E}$, $\eta\in\boldsymbol{R}^1$, such that $\xi(\eta)=1$ for $\eta\ge -A$, $\xi(\eta)=0$ for $\eta\le -A-\varepsilon$, $\varepsilon>0$; and $0\le\xi(\eta)\le 1$. Put $\gamma(t)=\xi(\langle t, y\rangle)$, $y\in C$. Since $\varphi(t)\in\mathcal{S}\subset\mathcal{S}'$ and $\operatorname{supp}(\varphi)\subseteq S_A$, then by [4, Theorem 4],

$$f(z)=\int_{\boldsymbol{R}^n}\varphi(t)e^{2\pi i\langle z, t\rangle}dt=\int_{\boldsymbol{R}^n}I_{S_A}(t)\varphi(t)\gamma(t)e^{2\pi i\langle z, t\rangle}dt$$

is analytic in T^C. For $z\in T^C$ and $t\in\boldsymbol{R}^n$ we have

$$(15)\qquad |I_{S_A}(t)\varphi(t)\gamma(t)e^{2\pi i\langle z, t\rangle}|=|I_{S_A}(t)\gamma(t)e^{-2\pi\langle y, t\rangle}\varphi(t)|\le e^{2\pi A}|\varphi(t)|.$$

Since $\varphi(t)\in\mathcal{S}$ we may apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\substack{y\to 0\\ y\in C}} f(z)=\int_{\boldsymbol{R}^n}\varphi(t)e^{2\pi i\langle x, t\rangle}dt=\hat{\varphi}(x);$$

and $\hat{\varphi}(x)\in\mathcal{S}$. Further, using (15) we have for all $z\in T^C$ that

$$(16)\qquad |f(z)|\le e^{2\pi A}\int_{\boldsymbol{R}^n}|\varphi(t)|\,dt\le K<\infty,$$

K being a constant. Thus $f(z)\in H^\infty(T^C)$; and from another application of the Lebesgue dominated convergence theorem, we obtain that $f(z)\to\hat{\varphi}(x)$ in the $\mathcal{S}'$ topology as $y\to 0$, $y\in C$.

Now let $\alpha=(\alpha_1, \ldots, \alpha_n)$ be an arbitrary n-tuple of nonnegative integers. Using the facts that $D^\alpha\varphi(t)\in\mathcal{S}$ and $\operatorname{supp}(D^\alpha\varphi(t))\subseteq S_A$ for any α, we integrate by parts in the integral defining $f(z)$ (i.e. $\int_{\boldsymbol{R}^n}\varphi(t)e^{2\pi i\langle z, t\rangle}dt$) and obtain

$$f(z)=(-2\pi iz_1)^{-\alpha_1}\ldots(-2\pi iz_n)^{-\alpha_n}\int_{\boldsymbol{R}^n}D^\alpha\varphi(t)e^{2\pi i\langle z, t\rangle}dt;$$

so that

$$(17)\quad |z_1|^{\alpha_1} \dots |z_n|^{\alpha_n} \, |f(z)| \leq \frac{e^{2\pi A}}{(2\pi)^{\alpha_1+\dots+\alpha_n}} \int_{\mathbf{R}^n} |D^{\alpha}\varphi(t)| \, dt \leq K_{\alpha} < \infty .$$

From (16) and (17) we obtain

$$|f(z)| \leq (K+K_{\alpha})(1+|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{-1} \leq$$
$$\leq (K+K_{\alpha})(1+|x_1|^{\alpha_1} \dots |x_n|^{\alpha_n})^{-1},$$

and this inequality holds for all n-tuples α of nonnegative integers. We now choose $\alpha=(2, \dots, 2)$. For any p, $1 \leq p < \infty$ we thus have

$$(18)\quad \int_{\mathbf{R}^n} |f(x+iy)|^p \, dx \leq (K+K_{(2,\dots,2)})^p \int_{\mathbf{R}^n} (1+|x_1|^2 \dots |x_n|^2)^{-p} dx.$$

The right hand side of (18) is finite for any p, $1 \leq p < \infty$; and for each fixed p, the value of the right hand side of (18) is independent of $y \in C$. Thus $f(z) \in H^p(T^C)$ for all p, $1 \leq p < \infty$; and we have already seen that $f(z) \in H^{\infty}(T^C)$. The proof is complete.

Korányi [14, Propositions 4 and 3 (d)] has proved the classical Fatou theorem for functions $f(z) \in H^{\infty}(T^C)$. The following theorem is a converse to this result and to Theorem 3 of the present paper.

THEOREM 6. *Let the cone C be contained in $\{y : y_j > 0, j=1, \dots, n\}$. Let $g(x) \in L^{\infty}$ such that $g(x)=\widehat{V}$ in $\mathcal{S}'$ where $V \in \mathcal{S}'$ and* supp $(V) \subseteq C^* = \{t : u_C(t) \leq 0\}$. *There exists a function $f(z) \in H^{\infty}(T^C)$ such that $f(z) \to g(x)$ in the $\mathcal{S}'$ topology (as well as in the weak-star topology of L^{∞}) as $y \to 0$, $y \in C' \subset C$, where C' is an arbitrary compact subcone of C.*

PROOF. Put

$$h(x) = \frac{g(x)}{1+x_1^2 \dots x_n^2} .$$

Since $g(x) \in L^{\infty}$, $h(x) \in L^2$. By hypothesis $g(x)=\widehat{V}$; so that

$$V = \mathfrak{F}^{-1}(g) = \mathfrak{F}^{-1}[(1+x_1^2 \dots x_n^2) h(x)] \text{ in } \mathcal{S}'.$$

We thus have for $\varphi \in \mathcal{S}$ that

$$\langle V, \varphi\rangle = \langle (1+x_1^2 \ldots x_n^2)h(x), \mathfrak{F}^{-1}[\varphi(t); x]\rangle.$$

Since $h(x) \in L^2$, there exists a function $k(t) \in L^2$ such that $h(x) = \mathfrak{F}[k(t); x]$; and

$$\langle V, \varphi\rangle = \langle \mathfrak{F}[k(t); x], (1+x_1^2 \ldots x_n^2)\mathfrak{F}^{-1}[\varphi(t); x]\rangle =$$
$$= \langle (1+D^{(2,\ldots,2)})k(t), \varphi\rangle.$$

Thus $V = (1+D^{(2,\ldots,2)})k(t)$, and it follows that $\operatorname{supp}(k) \subseteq C^*$ almost everywhere. We now put

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle = \langle V, \gamma(t)e^{2\pi i\langle z,t\rangle}\rangle, \; z \in T^C,$$

where $\gamma(t)$ is defined as in the proof of Theorem 3. By [4, Theorem 4], $f(z)$ is analytic in T^C; and $f(z) \to \widehat{V} = g(x)$ in $\mathcal{S}'$ as $y \to 0$, $y \in C' \subset C$.

We now prove that $f(z)$ is bounded for $z \in T^C$. By a straightforward calculation we have

$$f(z) = \int_{\boldsymbol{R}^n} k(t)e^{2\pi i\langle z,t\rangle}dt + z_1^2 \ldots z_n^2 \int_{\boldsymbol{R}^n} k(t)e^{2\pi i\langle z,t\rangle}dt. \tag{19}$$

We put

$$P(z) = \int_{\boldsymbol{R}^n} k(t)e^{2\pi i\langle z,t\rangle}dt.$$

It is easily seen that $P(z)$ is bounded for $z \in T^C$; and again applying [4, Theorem 4], we have that $P(z)$ is analytic in T^C. To show that $z_1^2 \ldots z_n^2 P(z)$ is bounded for $z \in T^C$ we consider the function

$$F(\varepsilon, z) = \exp[i\varepsilon(z_1^\sigma + \ldots + z_n^\sigma)](z_1^2 \ldots z_n^2)P(z), \; z \in T^C,$$

where $0 < \sigma < 1$ and $\varepsilon > 0$ is fixed for the present. Since $P(z)$ is analytic in T^C, then $F(\varepsilon, z)$ is also. By our assumption on the cone C, we have that T^C is contained in the octant $\{z : \operatorname{Im}(z_j)>0,\ j=1, \ldots, n\}$. Thus for

$$z=(z_1, \ldots, z_n)\in T^C,\ z_j=r_j e^{i\theta_j},\ 0<\theta_j<\pi,\ j=1, \ldots n;$$

and

$$|F(\varepsilon, z)|\leq M(r_1^2 \ldots r_n^2)\exp[-\varepsilon(r_1^\sigma \sin\sigma\theta_1+ \ldots +r_n^\sigma \sin\sigma\theta_n)], \tag{20}$$

where M is the bound on $P(z)$. Now $0<\theta_j<\pi$, $j=1, \ldots, n$, and $0<\sigma<1$ imply $\sin\sigma\theta_j>0$, $j=1, \ldots, n$; and it follows from (20) that $F(\varepsilon, z)$ is bounded for each fixed $\varepsilon>0$ and for $z\in T^C$. Further, as $y\to 0$, $y\in C$,

$$F(\varepsilon, z)\to\exp[i\varepsilon(|x_1|^\sigma+ \ldots +|x_n|^\sigma)(\cos\sigma\pi+i\sin\sigma\pi)](x_1^2 \ldots x_n^2)h(x)$$

in the weak-star topology of L^∞. Since $0<\sigma<1$, then $\sin\sigma\pi>0$; and we have from the definition of $h(x)$ that

$$\begin{aligned}&|\exp[i\varepsilon(|x_1|^\sigma+ \ldots +|x_n|^\sigma)(\cos\sigma\pi+i\sin\sigma\pi)](x_1^2 \ldots x_n^2)h(x)|\leq\\ &\leq\frac{(x_1^2 \ldots x_n^2)|g(x)|}{1+x_1^2 \ldots x_n^2}\leq B,\end{aligned} \tag{21}$$

where B is the bound on $g(x)\in L^\infty$, and this bound in (21) is independent of ε. Thus for each fixed $\varepsilon>0$, $F(\varepsilon, z)\in H^\infty(T^C)$; and $F(\varepsilon, z)$ converges in the weak-star topology of L^∞ to a bounded measurable function. It follows from Propositions 4 and 3 (d) of Korányi [14] that

$$\begin{aligned}&F(\varepsilon, z)=\\ &=\int_{R^n}\exp[i\varepsilon(|t_1|^\sigma+ \ldots +|t_n|^\sigma)(\cos\sigma\pi+i\sin\sigma\pi)](t_1^2 \ldots t_n^2)h(t)Q(z; t)dt.\end{aligned} \tag{22}$$

Thus by (22), (21) and Proposition 2 (b) of Korányi [14], we have

$$|F(\varepsilon, z)|\leq B\int_{R^n} Q(z; t)dt=B; \tag{23}$$

and this bound is independent of $\varepsilon>0$. Returning to the definition of $F(\varepsilon, z)$ and using (23), we obtain

$$(24) \qquad |z_1^2 \dots z_n^2 P(z)| \leq B \exp [\varepsilon(r_1^\sigma \sin \sigma\theta_1 + \dots + r_n^\sigma \sin \sigma\theta_n)],$$

$\varepsilon>0$. Since $z_1^2 \dots z_n^2 P(z)$ and B are independent of ε, we let $\varepsilon \to 0$ in (24) and obtain that $z_1^2 \dots z_n^2 P(z)$ is bounded by B, the bound on $g(x) \in L^\infty$, for all $z \in T^C$. We now conclude from (19) that $f(z) \in H^\infty(T^C)$. Using this fact and exactly the same method used in the last paragraph of the proof of Theorem 4, we obtain that $f(z) \to g(x)$ in the weak-star topology of L^∞ as $y \to 0$, $y \in C' \subset C$; and the proof is complete.

Results similar to Theorem 6 can be proved using the same methods for the cone C being contained in any of the 2^n domains $\{y : \delta_j y_j > 0, \delta_j = \pm 1, j=1, \dots, n\}$; the choice of $\{y : y_j > 0, j=1, \dots, n\}$ was purely a matter of convenience. A special case of Theorem 6 has been obtained by Beltrami and Wohlers [1, Theorem 3] for one dimension and functions analytic in a half plane. We now obtain a corollary to Theorem 6.

Corollary 3. *Let the cone C be contained in $\{y : y_j > 0, j = 1, \dots, n\}$. Let $f(z)$ be analytic in T^C and satisfy* (1) *for $A=0$. Let $f(z) \to g(x) \in L^\infty$ in the topology of $\mathcal{S}'$ as $y \to 0$, $y \in C$. Then $f(z) \in H^\infty(T^C)$.*

Proof. By Theorem 1, there exists an element $V \in \mathcal{S}'$ with $\operatorname{supp}(V) \subseteq C^*$ and $\hat{V} = g(x)$ such that $f(x) = \langle V, e^{2\pi i \langle z, t \rangle} \rangle$, $z \in T^{C'}$ $C' \subset C$. By hypothesis $f(z)$ is analytic in T^C; and by [4, Theorem 4], $\langle V, e^{2\pi i \langle z, t \rangle} \rangle$ is analytic in T^C. Thus by the identity theorem for analytic functions, $f(z) = \langle V, e^{2\pi i \langle z, t \rangle} \rangle$, $z \in T^C$; and the conclusion is immediate from Theorem 6.

4. Functions analytic in disconnected tubular cones.

Let C be an open cone which is the finite union of open cones C_j, $j=1, \dots, m$, each of which is convex and has the property that $\overline{C}_j$ contains no entire straight line. Throughout this section T^C will denote the tubular cone associated with the open (possibly disconnected) cone C which satisfies the above property; and we recall that $0(C)$ denotes the convex envelope (hull) of C.

Let $f(z)$ be analytic in $T^C=\boldsymbol{R}^n+iC$, $C=\bigcup_{j=1}^m C_j$, and satisfy (1). For each $j=1, \ldots, m$, suppose that $f(z)\in H^p(T^{C_j})$, $1\leq p<\infty$. By Theorem 2, there exist functions $g_j(x)\in L^p$, $1\leq p<\infty$, such that $f(z)\to g_j(x)$ in $\mathcal{S}'$ as $y\to 0$, $y\in C_j$, $j=1, \ldots, m$. We now prove the following generalization of Theorem 2.

THEOREM 7. *Let $f(z)$ be analytic in T^C and satisfy* (1). *For each $j=1, \ldots, m$, let $f(z)\in H^p(T^{C_j})$, $1\leq p<\infty$. Let the $\mathcal{S}'$ boundary values $g_j(x)\in L^p$ of $f(z)$, $z\in T^{C_j}$, be equal in $\mathcal{S}'$. Then $f(z)$ has an analytic extension (denoted $F(z)$) to $T^{0(C)}$; for any arbitrary compact subcone C' of $0(C)$, $D^\alpha F(z)$ satisfies*

$$(25)\qquad |D^\alpha F(z)|\leq K(C')(1+|z|)^N(1+|y|^{-M})\exp[2\pi A\rho_C|y|],\quad z\in T^{C'},$$

where α is an arbitrary n-tuple of nonnegative integers, $K(C')$ is a constant depending on C', and M and N are nonnegative integers which do not depend on C'; there exists a function $g(x)\in L^p$, $1\leq p<\infty$, such that $F(z)\to g(x)$ in the topology of $\mathcal{S}'$ as $y\to 0$, $y\in C'\subset 0(C)$; and if $p=2$, $F(z)\in H^2(T^{0(C)})$.

PROOF. By Theorem 2, there exist elements $V_j\in\mathcal{S}'$ with $\operatorname{supp}(V_j)\subseteq S_{A,j}=\{t: u_{C_j}(t)\leq A\}$ such that $g_j(x)=\hat{V}_j$ and

$$(26)\qquad f(z)=\langle V_j, e^{2\pi i\langle z,t\rangle}\rangle,\quad z\in T^{C_j},\ j=1, \ldots, m.$$

By assumption, $g_1(x)=\ldots=g_m(x)$ almost everywhere; and we call this common value $g(x)$. Since $V_j=\mathfrak{F}^{-1}(g_j)$, $j=1, \ldots, m$, it follows immediately that $V_1=\ldots=V_m$; and we call this common value V. Thus $g(x)\in L^p$, $1\leq p<\infty$, $g(x)=\hat{V}$, and $\operatorname{supp}(V)\subseteq\bigcup_{j=1}^m S_{A,j}$; so that V vanishes on $\bigcup_{j=1}^m\{t: u_{C_j}(t)>A\}$. Now

$$u_C(t)=\max_{j=1,\ldots,m} u_{C_j}(t);$$

and from the definition of ρ_C. (See [4, section II]) we have $u_{0(C)}(t)\leq$

$\leq \rho_C u_C(t)$. Thus

$$(27) \qquad u_{0(C)}(t) \leq \rho_C \max_{j=1,\dots,m} u_{C_j}(t);$$

and by a lemma of Vladimirov [18, Lemma 3, p. 220], $\rho_C < +\infty$. Now consider the set $J = \{t : u_{0(C)}(t) > A\rho_C\}$. If $t \in J$, then by (27), $t \in \{t : \max_{j=1,\dots,m} u_{C_j}(t) > A\}$. Hence $t \in \bigcup_{j=1}^{m} \{t : u_{C_j}(t) > A\}$, and on this set V vanishes. Thus V vanishes if $t \in J$ which implies that

$$\text{supp}\,(V) \subseteq \{t : u_{0(C)}(t) \leq A\rho_C\}.$$

Let $\xi(\eta) \in \mathcal{E}$, $\eta \in \boldsymbol{R}^1$, such that $\xi(\eta) = 1$ for $\eta \geq -A\rho_C$, $\xi(\eta) = 0$ for $\eta \leq -A\rho_C - \varepsilon$, $\varepsilon > 0$; and $0 \leq \xi(\eta) \leq 1$. Put $\gamma(t) = \xi(\langle t, y \rangle)$, $y \in 0(C)$. We now put

$$F(z) = \langle V, e^{2\pi i \langle z, t \rangle} \rangle = \langle V, \gamma(t) e^{2\pi i \langle z, t \rangle} \rangle, \quad z \in T^{0(C)}.$$

By [4, Theorem 4], $F(z)$ is analytic in $T^{0(C)}$, satisfies (25), and $F(z) \to \widehat{V} = g(x)$ in $\mathcal{S}'$ as $y \to 0$, $y \in C' \subset 0(C)$. Further since $V = V_j$, $j = 1, \dots, m$, then from (26) we have $f(z) = F(z)$, $z \in T^C$; and $F(z)$ is the analytic extension of $f(z)$ to $T^{0(C)}$.

If $p = 2$, then $g(x) \in L^2$; and there exists a function $h(t) \in L^2$ such that $g(x) = \mathfrak{F}[h(t); x]$. But then $\widehat{V} = \widehat{h}$ in $\mathcal{S}'$. Thus $V = h$ in $\mathcal{S}'$, and $\text{supp}\,(h) \subseteq \{t : u_{0(C)}(t) \leq A\rho_C\}$ almost everywhere. Letting $I(t)$ denote the characteristic function of this support set, we have as in (14) that

$$(28) \quad \int_{\boldsymbol{R}^n} |F(x+iy)|^2\, dx = \| I(t)\gamma(t) e^{-2\pi \langle y, t \rangle} h(t) \|_{L^2}^2 \leq \exp(4\pi A\rho_C) \| h \|_{L^2}^2 < \infty.$$

The estimate (28) holds for all $y \in 0(C)$. Thus $F(z) \in H^2(T^{0(C)})$; and the proof is complete.

Since $f(z) = F(z)$, $z \in T^C$, then the conclusion of Theorem 7 states that $f(z)$ satisfies (25) for $z \in T^C$. Further, for $p = 2$, we proved in Theorem 7 that $F(z) \in H^2(T^{0(C)})$; and it follows as before that $F(z) \to g(x)$ in the L^2 norm topology as $y \to 0$, $y \in C' \subset 0(C)$, as well as in the $\mathcal{S}'$ topology.

Theorem 7 generalizes Theorem 2. In the same manner a generalization of Theorem 3 can be obtained for disconnected tubular cones, and we leave the formulation of such a result to the inteerested reader.

We now obtained a generalization of Theorem 6, the converse Fatou theorem

THEOREM 8. *Let the tubular cone* $T^C=\mathfrak{R}^n+iC$, $C=\bigcup_{j=1}^{m} C_j$, *have the property that* $T^{0(C)}\subseteq\{z : \operatorname{Im}(z_j)>0, j=1, \ldots, n\}$. *Let* $g_j(x)$, $j=1, \ldots, m$, *be* L^∞ *functions such that for each* $g_j(x)$ *there exists an element* $V_j\in\mathcal{S}'$ *with* supp $(V_j)\subseteq C^*_j=\{t : u_{C_j}(t)\leq 0\}$ *and* $g_j(x)=\widehat{V_j}$. *Let* $g_1(x)= \ldots =g_m(x)$ *in* $\mathcal{S}'$. *Then there exists a function* $F(z)\in H^\infty(T^{0(C)})$ *such that*

$$\lim_{\substack{y\to 0\\ y\in C'_j\subset C_j}} F(z)=g_j(x),\ j=1, \ldots, m,$$

in the topology of $\mathcal{S}'$ (*as well as in the weak-star topology of* L^∞) *where* C'_j *is an arbitrary compact subcone of* C_j, $j=1, \ldots, m$.

PROOF. As in the proof of Theorem 7, $g(x)=g_1(x)= \ldots =g_m(x)$ implies $V=V_1= \ldots =V_m$; and $g(x)=\widehat{V}$ where supp $(V)\subseteq\{t : u_{0(C)}(t)\leq 0\}$. Here $g(x)\in L^\infty$. We put

$$F(z)=\langle V, e^{2\pi i\langle z, t\rangle}\rangle=\langle V, \gamma(t)e^{2\pi i\langle z, t\rangle}\rangle,\ z\in T^{0(C)},$$

where $\gamma(t)$ is defined as in the proof of Theorem 7 for $A=0$. From the assumption on $T^{0(C)}$ and the proof of Theorem 6, we have $F(z)\in H^\infty(T^{0(C)})$. Since $V=V_j$, $j=1, \ldots, m$, then

$$F(z)=\langle V_j, e^{2\pi i\langle z, t\rangle}\rangle,\ z\in T^{C_j},\ j=1, \ldots, m.$$

By [4, Theorem 4], $F(z)\to\widehat{V}_j=g_j(x)\in L^\infty$ in the topology of $\mathcal{S}'$ as $y\to 0$, $y\in C'_j\subset C_j$, $j=1, \ldots, m$. But since $F(z)\in H^\infty(T^{0(C)})$, then $F(z)\in H^\infty(T^{C_j})$, $j=1, \ldots, m$; and arguing as in the last paragraph of the proof of Theorem 4, we have $F(z)\to g_j(x)\in L^\infty$ in the weak-star topology of L^∞ as $y\to 0$, $y\in C'_j\subset C_j$, $j=1, \ldots, m$.

We note that the function $F(z)\in H^\infty(T^{0(C)})$ constructed in Theorem 8 has the additional property that $F(z)\to g(x)$ in both the $\mathcal{S}'$ and weak-star L^∞ topologies as $y \to 0$, $y \in C' \subset O(C)$. This result follows immediately from Theoren 6. Generalizations of Theorem 4 and Corollary 2 can also be obtained for disconnected tubular cones. Their formulation and proof are similar in form to Theorem 8, and again we leave the details to the interested reader.

REFERENCES

[1] BELTRAMI, E. J., and WOHLERS, M. R.: *Distributional Boundary Values of Functions Holomorphic in a Half Plane*, J. Math. Mech. 15 (1966), 137-145.

[2] BOCHNER, S.: *Group Invarience of Cauchy's Formula in Several Variables*, Annals of Math. 45 (1944), 686-707.

[3] BOCHNER, S., and MARTIN, W. T.: *Several Complex Variables*, Princeton University Press, Princeton, N.J., 1948.

[4] CARMICHAEL, RICHARD D.: *Distributional Boundary Values of Functions Analytic in Tubular Radial Domains*, Indiana U. Math. J. (formerly J. Math. Mech.) 20 (1971), 843-853.

[5] CARMICHAEL, RICHARD D.: *Distributional Boundary Values in* $\mathfrak{D}'_{L^p}$, Rendiconti Sem. Mat. Università di Padova 43 (1970), 35-53.

[6] CARMICHAEL, RICHARD D.: *Distributional Boundary Values in* $\mathfrak{D}'_{L^p}$. II, Rendiconti Sem. Mat. Università di Padova 45 (1971), 249-277.

[7] CARMICHAEL, RICHARD D.: *Functions Analytic in an Octant and Boundary Values of Distributions*, J. Math. Analysis Appl. 33 (1971), 616-626.

[8] CARMICHAEL, RICHARD D.: *Distributions as the Boundary Values of Analitic Functions*, Proc. Japan Acad. 45 (1969), 861-865.

[9] CARMICHAEL, RICHARD D.: *Generalized Cauchy and Poisson Integrals and Distributional Boundary Values*, SIAM J. Math. Analysis (to appear).

[10] CARMICHAEL, RICHARD D.: *The Paley-Wiener-Schwartz Theorem for Functions Analytic in a Half Space*, Ph. D. Thesis, Duke University, Durham, N.C., 1968.

[11] HILLE, EINAR, and TAMARKIN, J. D.: *On a Theorem of Paley and Wiener*, Annals of Math. 34 (1933), 606-614.

[12] HILLE, EINAR, and TAMARKIN, J. D.: *A Remark on Fourier Transforms and Functions Analytic in a Half Plane*, Compositio Math. 1 (1934), 98-102.

[13] HILLE, EINAR, and TAMARKIN, J. D.: *On the Absolute Integrability of Fourier Transforms*, Fundamenta Math. 25 (1935), 329-352.

[14] KORÁNYI, ADAM: *A Poisson Integral for Homogeneous Wedge Domains*, J. D'Analyse Math. 14 (1965), 275-284.

[15] SCHWARTZ, L.: *Théorie Des Distributions*, Hermann, Paris, 1966.

[16] STEIN, E. M., WEISS, G., and WEISS, M.: *H^p Classes of Holomorphic Functions in Tube Domains*, Proc. Nat. Acad. Sciences U.S.A. 52 (1964), 1035-1039.

[17] STEIN, E. M., and WEISS, G.: *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, N.J., 1971.

[18] VLADIMIROV, V. S.: *Methods of the Theory of Functions of Several Complex Variables*, M.I.T. Press, Cambridge, Mass., 1966.

Manoscritto pervenuto in redazione il 17 aprile 1972.