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ENUMERATION OF SYMMETRIC ARRAYS WITH DIFFERENT ROW SUMS

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1. Introduction.

Let $S(r_1, ..., r_n)$ denote the number of $n \times n$ symmetric arrays $[a_{ij}]$ where the a_{ij} are nonnegative integers satisfying

$$\sum_{j=1}^{n} a_{ij} = r_i ; \quad i = 1, 2, ..., n. \tag{1.1}$$

It may be easily verified that $S(r)=1$ and $S(r_1, r_2) = \min(r_1, r_2)+1$. Professor L. Carlitz [1] found formulas for $S(r, r, r)$, $S(r, r, r, r)$ and $S(1, ..., 1)$ and in [2] he found related formulas. The author [4] derived formulas for $S(1, ..., 1, r)$ and $S(1, ..., 1, r, r)$. Professor H. Gupta [7] and the author [4] found formulas for $S(r_1, r_2, r_3)$.

Apparently, beyond these cases no other explicit formulas are known. Various recurrences and generating functions are known for other aspects of the problem; a bibliography on the problem is included in the list of references at the end of the paper. MacMahon [8] gives a treatment of similar problems.

Subject only to a condition based on the relative magnitude of the r's a formula for $S(r_1, r_2, r_3, r_4)$ is a result in this paper. See (3.9) below.

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2. Some Preliminaires.

Here we consider the sum

$$(2.1)\qquad \sigma_n = \sum_{i_1+i_2+\dots+i_q=n} (r_1-i_1)^{e_1}(r_2-i_2)^{e_2} \dots (r_t-i_t)^{e_t}$$

where, of course, $t \leq q$. This becomes

$$\sum_{k=0}^{n} \sum_{i_1+\dots+i_t=k} (r_1-i_1)^{e_1} \dots (r_t-i_t)^{e_t} \sum_{i_{t+1}+\dots+i_q=n-k} 1$$

$$= \sum_{k=0}^{n} \sum_{i_1=0}^{k} \sum_{i_2=0}^{k-i_1} \dots \sum_{i_{t-1}=0}^{k-i_1-\dots-i_{t-2}} (r_1-i_1)^{e_1}$$

$$\dots (r_{t-1}-i_{t-1})^{e_{t-1}}(r_t-k+i_1+\dots+i_{t-1})^{e_t}\binom{q-t+n-k-1}{n-k}.$$

By multiplying this last expression by x^n and then summing on n from 0 to ∞, and by a little manipulation we find

$$\sum_{n=0}^{\infty} \sigma_n x^n$$

$$= \sum_{n,k,i_1,\dots,i_{t-1}=0}^{\infty} (r_1-i_1)^{e_1} \dots (r_{t-1}-i_1)^{e_{t-1}}(r_t-k)^{e_t}\binom{q-t+n-1}{n} \cdot x^{n+k+i_1+\dots+i_{t-1}}$$

$$(2.2)\qquad = (1-x)^{t-q} \sum_{i_1=0}^{\infty} (r_1-i_1)^{e_1} x^{i_1} \dots \sum_{i_t=0}^{\infty} (r_t-i_t)^{e_t} x^{i_t}$$

where, for consistency of appearence, we replaced k by i_t in the last step.

The expression (2.2) my again be obtained, rather heurestically, as follows: From (2.1),

$$\sum_{n=0}^{\infty} \sigma_n x^m = \sum_{n=0}^{\infty} \sum_{i_1+\dots+i_q=n} (r_1-i_1)^{e_1}(r_2-i_2)^{e_2} \dots (r_t-i_t)^{e_t} x^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1+\dots+i_t=k} (r_1-i_1)^{e_1} \dots (r_t-i_t)^{e_t}\binom{q-t+n-k-1}{n-k} x^n$$

$$= \sum_{n,k=0}^{\infty} \sum_{i_1+\ldots+i_t=k} (r_1-i_1)^{e_1} \ldots (r_t-i_t)^{e_t} \binom{q-t+n-1}{n} n^{n+k}$$

$$=(1-x)^{t-q} \sum_{k=0}^{\infty} \sum_{i_1+\ldots+i_t=k} (r_1-i_1)^{e_1} \ldots (r_t-i_t)^{e_t} x^{i_1+\cdot+i_t}.$$

At this point is must be observed that all of the values of $i_1, \ldots, i_t$ for which $i_1+\ldots+i_t=k$ as k ranges from 0 to ∞ do not exclude any combination of values for the i's. Therefore we may rewrite the last expression as

$$(1-x)^{t-q} \sum_{i_1+\ldots+i_t=0}^{\infty} (r_1-i_1)^{e_1} \ldots (r_t-i_t)^{e_t} x^{i_1+\cdot+i_t},$$

which is (2.2).

Now, note that

$$\text{(2.3)} \qquad \sum_{i=0}^{\infty} (r-i)^e x^i = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \binom{e}{j} r^{e-j}(-i)^j x^i$$

$$= \sum_{j=0}^{e} \binom{e}{j} r^{e-j}(-1)^j \sum_{i=0}^{\infty} i^j x^i$$

$$= \sum_{j=0}^{e} \binom{e}{j} r^{e-j}(-1)^j \frac{a_j(x)}{(1-x)^{j+1}}$$

where $a_j(x)$ is the $(j+1)^{st}$ Eulerian polynomial (cf. Riordan [9; problem 2, Chapter 2]). Consequently, combining (2.1), (2.2) and (2.3) we find

$$\text{(2.4)} \quad \sum_{n=0}^{\infty} \sum_{i_1+\ldots+i_q=n} (r_1-i_1)^{e_1} \ldots (r_t-i_t)^{e_t} x^n = (1-x)^{-q} \prod_{k=1}^{t} \left(r_k - \frac{a}{(1-x)} \right)^{e_k}$$

where it is understood that in the expansion of $(r_k-a/(1-x))^{e_k}$ we replace a^j by $a_j=a_j(x)$.

The generating function given by (2.4) will be useful in what follows.

Also useful are the first few Eulerian polynomials,

$$a_0(x)=1$$

$$a_1(x)=x$$

$$a_2(x)=x+x^2$$
$$a_3(x)=x+4x^2+x^3.$$

3. The 4×4 array. $S(r_1,\ldots,r_n)$ is symmetric in $r_1,\ldots,r_n$ so we may assume $r_1\leq\ldots\leq r_n$ without loss of generality. Thus taking $r\leq s\leq t$ the author [4] showed that

$$S(r,\ s,\ t)=\frac{3rs}{2}+\frac{r^2s}{2}-\frac{r^3}{6}+\frac{7r}{6}+s+1+F(r,\ s,\ t) \tag{3.1}$$

where $F(r, s, t)$ is a function given in [4] and where $F(r, s, t)=0$ if $t\geq r+s$. (Here we are assuming $r\leq s\leq t$ whereas in [4] we assumed $t\leq s\leq r$; this accounts for the different appearance.) Also, [4], we may easily establish the recurrence

$$S(r_1,\ \ldots,\ r_n)=\sum_{a_1+\ldots+a_{n-1}\leq r_1} S(r_2-a_2,\ \ldots,\ r_n-a_n). \tag{3.2}$$

Therefore, from (3.1) and (3.2)

$$S(r,\ s,\ t,\ u)=\sum_{i+j+k\leq r} S(s-i,\ t-j,\ u-k) \tag{3.3}$$

$$=\sum_{i+j+k\leq r}\left\{\frac{3}{2}(s-i)(t-j)+\frac{1}{2}(s-i)^2(t-j)-\frac{1}{6}(s-i)^3+\frac{6}{7}(s-i)+(t-j)+1\right\}.$$

Provided

$$u-k\geq s-i+t-j \tag{3.4}$$

and

$$s-i\leq t-j\leq u-k. \tag{3.5}$$

Inequality (3.4) is equivalent to

$$s-i\leq u-k-t+j. \tag{3.6}$$

Since $i+j+k\leq r\leq s$, the largest the left side of (3.6) can be is s. The smallest the right side of (3.6) can be is $u-r-t$. Therefore (3.4) holds provided $s\leq u-r-t$ or $u\geq r+s+t$.

Regarding (3.5), the inequality $u-k\geq t-j$ holds if (3.4) is true. On the other hand, $t-j\geq s-i$ provided $t-r\geq s$ since $i+j+k\leq r\leq s$. That is, $t\geq r+s$. Therefore (3.4) and (3.5) hold provided $u\geq r+s+t$ and $t\geq r+s$. Note that this implies $u\geq t\geq s$.

According to (2.4),

$$(3.7) \qquad \sum_{r=0} \sum_{i+j+k\leq r} (s-i)(t-j)x^r=(1-x)^{-4}\left(s-\frac{x}{1-x}\right)\left(t-\frac{x}{1-x}\right),$$

where, in this case, $a=a_1(x)=x$. (Note that the inequality $i+j+k\leq r$ may be replaced by the equation $i+j+k+p=r$; hence $q=4$.)

Expanding the right side of (3.7) and comparing the coefficient of x^r with that of the left side we find

$$(3.8) \qquad \sum_{i+j+k\leq r} (s-i)(t-j)$$

$$=st\binom{5+r}{5}-(s+t+2st)\binom{4+r}{5}+(1+s)(1+t)\binom{3+r}{5}.$$

By similarly applying (2.4) to each of the other terms of the summand of (3.3) we eventually arrive at the formula

$$(3.9) \qquad S(r,\ s,\ t,\ u)=\frac{1}{6}\binom{r+3}{3}(3st(s+5)+s(19-s^2)+18t+30)$$

$$-\binom{r+4}{4}(s+3)(t+2)$$

$$+\binom{r+5}{5}(t+2)$$

provided $u\geq t+s+r$, $t\geq s+r$ and $s\geq r$.

Note that the formula is independent of u. The number of solutions of

$$(3.10) \qquad \begin{aligned} a+b+c+d&=1\\ b+e+f+g&=2\\ c+f+h+i&=3\\ d+g+i+j&=99 \end{aligned}$$

is the same if we replace the number 99 by 6; inspection of the system (3.10) reveals that this is as it ought to be. The conditions of (3.9) are met and we have

$$S(1,\ 2,\ 3,\ 99)=S(1,\ 2,\ 3,\ 6)=65.$$

4. Related formulas and generating functions. Let $T(r_1, \ldots, r_n)$ be the number of arrays as described in section 1 expect that we replace (1.1) by

$$\sum_{j=1}^{n} a_{ij} \le r_i\ ; \quad i=1,\ 2,\ \ldots,\ n.$$

Without loss of generality we let $r_n = \max(r_1, \ldots, r_n)$. Then, if

$$r_n \ge r_1 + \ldots + r_{n-1}$$

it is apparent that

$$S(r_1, \ldots, r_n)=T(r_1, \ldots, r_{n-1}).$$

Therefore tne number of solutions of

$$\begin{aligned} a+b+c&\le r \\ b+d+e&\le s \\ c+e+f&\le t \end{aligned} \tag{4.1}$$

is given by (3.9) provided $t \ge r+s$.

The generating function for $T(r, s, t)$ is found as follows:

$$\sum_{r,s,t=0}^{\infty} T(r,\ s,\ t)x^r y^s z^t = \sum_{r,s,t=0}^{\infty} \sum_{I} x^r y^s z^t \tag{4.2}$$

where I is the system (4.1). We may replace the symbols $\le$ of (4.1) by $=$ provided we add new variables g, h, i respectively to the left sides of these equations. Then the right side of (4.2) becomes

$$\sum_{a,b,\ldots,i=0}^{\infty} x^{a+b+c+g} y^{b+d+e+h} z^{c+e+f+i}$$

$$=(1-x)^{-2}(1-y)^{-2}(1-z)^{-2}(1-xy)^{-1}(1-xz)^{-1}(1-yz)^{-1}.$$

In general, we find

$$\sum_{r_1,\ldots,r_n=0}^{\infty} T(r_1,\ \ldots,\ r_n)x_1^{r_1}\ \ldots\ x_n^{r_n}$$

$$=\prod_{i=1}^{n}(1-x_i)^{-2}\prod_{1\le j<k\le n}(1-x_jx_k)^{-1}.$$

In a similar fashion, and (see Roselle [10])

$$\sum_{r_1,\ldots,r_n=0}^{\infty} S(r_1,\ \ldots,\ r_n)x_1^{r_1}\ldots\ x_n^{r_n}$$

$$=\prod_{i=1}^{n}(1-x_i)^{-1}\prod_{1\le j<k\le n}(1-x_jx_k)^{-1}.$$

It is evident that $T(r)=r+1$. $T(r,\ s)$ is found as follows:

(4.3) $$T(r,\ s)=\sum_{\substack{a+b\le r\\ b+d\le s}} 1$$

$$=\sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{\substack{a+b=i\\ b+d=j}}$$

$$=\sum_{i=0}^{r}\sum_{j=0}^{s}\min(i,\ j)+\sum_{i=0}^{r}\sum_{j=0}^{s}1.$$

Now it is not difficult to show that

(4.4) $$\sum_{i=0}^{r}\sum_{j=0}^{s}\min(i,\ j)=\sum_{k\le\min(r,\ s)}(r-k)(s-k).$$

(Equation (4.4) is also a special case of (3.3) in [5]). Observe that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(r-k)(s-k)x^n$$

$$=(1-x)^{-1}\left(\sum_{k=0}^{\infty}rsx^k-(r+s)\sum_{k=0}^{\infty}kx^k+\sum_{k=0}^{\infty}k^2x^k\right)$$

$$=(1-x)^{-4}(rs(1-x)^2-(r+s)x(1-x)+x+x^2).$$

From this and (4.3) we readily find

$$T(r,\ s)=rs\binom{3+\min(r,\ s)}{3}-(2rs+r+s-1)\binom{2+\min(r,\ s)}{3}$$
$$+(rs+r+s+1)\binom{1+\min(r,\ s)}{3}+(r+1)(s+1).$$

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