Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 47 (1972), p. 23-28

<http://www.numdam.org/item?id=RSMUP_1972__47__23_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 47 (1972)

A NOTE ON GEOMETRIC MAPPINGS

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Introduction.

The purpose of this note is to give a universal algebra interpretation of the following theorem proved by the author in [6, p. 31].

THEOREM. Let L be a geometric lattice of length ≥ 4 in which each hyperplane h_i is a maximum element mapped onto a point under a geometric mapping f_i. If in addition

(a) every two hyperplanes have a common complement or

(b) at least one line in L contains at least three points,

then L is the lattice of flats of an affine geometry.

Preliminaries.

Let L be a complete lattice in which elements is a join of atoms. We shall say that a set of atoms (points) is *closed* if it consists of precisely the set of atoms within some element of L. The closure of a set of points is the smallest closed set containing the given set. Closure will then be a closure operator in the usual sense. If we have two such lattices L_1 and L_2, and if f is a mapping from the points of L_1 *onto* the points of L_2 which sends closed sets onto closed sets and in which the inverse image of every closed set is closed, then we shall say that f is a *geometric mapping*. (In [6] it was required that both lattices be geometric lattices).

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We shall mainly be concerned with the following situation: A is an abstract algebra with finitary operations, Γ is the lattice of congruence relations on A, and L is the lattice of congruence classes corresponding to the various congruence relations on A plus the empty set φ ordered under inclusion. Then L is a complete lattice in which every element is a join of atoms. If A is a set with the identity function as an operation, then Γ is the lattice of equivalence relations on A, and L is the lattice of subsets. When A is a vector space, then Γ is isomorphic to the lattice of subspaces of A and is therefore a projective geometry, while L is the lattice of cosets (corresponding to subspaces) and is thus an affine geometry.

DEFINITION 1. An algebra A is an *S-algebra* if its congruence relations are determined by a single congruence class, that is, two congruence relations are identical if they have a single congruence class in common.

Evidently groups, rings, and modues are S-algebras; in fact, so are section-complemented lattices.

PROPOSITION 1. If A is an S-algebra, then the interval $[p, I]$ in L, where p is a point, is isomorphic to Γ.

PROOF. Let s be any element of A. To each congruence relation in Γ we associate the congruence class containing s. Because A is an S-algebra, this mapping f is $1-1$ from Γ onto the interval $[[s], I]$, and obviously preserves the partial ordering in the forward direction. If $f(x) \leq f(y)$, then x and $x \wedge y$ have a common class which implies that $x = x \wedge y$ or $x \leq y$. Thus the mapping f is an isomorphism.

If A is an algebra and A' is a homomorphic image of A, then A' is isomorphic to A/θ where θ is a congruence relation on A, and it is well-known [1, 2, 3] that the congruence relations in A containing θ stand in a natural $1-1$ correspondence with the congruence relations of A'. The homomorphism from A onto A' induces a natural mapping f from the points of L onto the points of L' (the lattices of congruence classes) in which the inverse image of every closed set is closed. In the case of groups and rings the mapping also sends closed sets onto closed sets because normal subgroups or ideals are mapped onto normal subgroups or ideals, respectively, under a homomorphism. But in general this is not the case as we shall later see. We discuss this situation further. Recall that two congruence relations θ, φ are *permutable* if $\theta\varphi=\varphi\theta=\theta\vee\varphi$. In other words if $x\equiv y(\theta)$ and $y\equiv z(\varphi)$ for some y, then there exists w such that $x\equiv w(\varphi)$ and $w\equiv z(\theta)$, and conversely. The reader should consult [7] where very similar propositions are proved about *geomorphisms.*

PROPOSITION 2. Let A be an algebra with permutable congruence relations, and L be its corresponding lattice of congruence classes. For every closed set $\mathfrak{C}$ in L, there exists a lattice L' which is a lattice of congruence classes of an algebra A', and a geometric mapping f from the points of L onto the point of L' such that $\mathfrak{C}$ is sent onto a point.

PROOF. Let $\mathfrak{C}$ be a closed set of L. It corresponds to a congruence relation θ (not necessarily unique) on A. Let L' be the lattice of congruence classes associated with A/θ, and let f be the natural induced mapping from the points of L onto the points of L'. Evidently $\mathfrak{C}$ is mapped onto a point. Let $\mathfrak{D}$ be a closed set of points that corresponds to congruence relation φ, and let $\mathfrak{E}$ be the closed set of points that contains $\mathfrak{D}$ and is a congruence class of $\theta\vee\varphi$. Since $\mathfrak{E}$ is mapped onto a closed set because $\theta\vee\varphi\geq\theta$, we shall complete the proof by showing that $f(\mathfrak{E})=f(\mathfrak{D})$. Let $y\in\mathfrak{E}$. There exists $x\in\mathfrak{D}$ such that $x\equiv y(\theta\vee\varphi)$. Since the congruence relations permute, there exists z such that $x\equiv z(\varphi)$ and $z\equiv y(\theta)$. Because $x\equiv z(\varphi)$, $z\in\mathfrak{D}$; and since $z\equiv y(\theta)$, $f(z)=f(y)$. Thus $f(\mathfrak{E})\subseteq f(\mathfrak{D})$, and since the reverse inclusion is obvious, $f(\mathfrak{E})=f(\mathfrak{D})$.

We see then that permutability of the congruence relations is a sufficient condition for the existence of geometric mappings. We now look into the necessity of this condition.

PROPOSITION 3. Let A be an S-algebra, and let L be its corresponding lattice of congruence classes. If L' is the lattice of congruence classes associated with A/θ and if the natural induced mapping f sends closed sets onto closed sets, then θ permutes with every congruence relation on A.

PROOF. Suppose that θ and φ do not permute. Then there exists a congruence class $\mathfrak{C}$ in θ and a congruence class $\mathfrak{D}$ in φ which do not overlap and yet lie within a common congruence class $\mathfrak{E}$ of $\theta\vee\varphi$. Since $\mathfrak{D}\subseteq\mathfrak{E}$, $f(\mathfrak{D})\subseteq f(\mathfrak{E})$, but $f(\mathfrak{D})\neq f(\mathfrak{E})$ because $\mathfrak{D}$ and $\mathfrak{C}$ do not overlap. If $f(\mathfrak{D})$ is closed, then it is a congruence class of a congruence relation Ψ', and the complete inverse image Ψ of Ψ' contains both θ and φ (since A is an S-algebra) and thus $\theta \vee \varphi$. But this implies that $\mathfrak{E} \subseteq f^{-1}(f(\mathfrak{D}))$ which is impossible since $f(\mathfrak{D}) \neq f(\mathfrak{E})$.

COROLLARY 1. *A* necessary and sufficient condition that the natural induced mapping of L upon L' be geometric, where an S-algebra A corresponds to L and A/θ to L', is that θ permute with every congruence relation on L.

COROLLARY 2. An S-algebra with permutable congruence relations has all possible geometric mappings induced by homomorpisms from its lattice of congruence classes onto another lattice of congruence classes, and they are essentially unique.

PROPOSITION 4. The homomorphic mapping of A upon A/θ, where θ is a congruence relation of A, sends the subsets of a partition φ corresponding to a congruence relation in A onto the subsest of a partition φ' corresponding to a congruence relation in A/θ if and only if θ and φ permute.

PROOF. If θ and φ permute, then the proof of Proposition 1 shows that the subsets of φ and $\theta \vee \varphi$ have the same images, and since, $\theta \vee \varphi \geq \theta$, the image of φ is a congruence relation. Suppose that θ and φ do not permute. If φ' corresponds to a congruence relation in A/θ, then its complete inverse image is a congruence relation Ψ in A. But Ψ contains both θ and φ and therefore $\theta \vee \varphi$. Thus we can apply the arguments of Proposition 3 to arrive at a contradiction.

COROLLARY 3. When θ and φ permute, the image of φ is the same as that of $\theta \vee \varphi$.

Examples of S-algebra with permutable congruence relations are groups, rings, modules and section-complemented lattices [4]. If we consider a group G as an algebra with unary operators of the form $f_a(x) = ax$, for every $a \in G$, then the congruence relations are the left-coset decompositions of the various subgroups. We thus have examples of S-algebras which do not necessarily have permutable congruence relations, for the relations permute if and only if the subgroups do. We see that the group S_3 under left translations provides us with an example of a homomorphism which does not induce a geometric mapping if we apply Proposition 3. If A is a set with the identity function as an operation, then it possesses all possible geometric mappings, because every subset is closed. Thus permutability of the congruence relations is not necessary for the existence of geometric mappings when the algebra is not an S-algebra, and a closed set mapping onto a point does not determine the mapping. We see then, that in general, homomorphisms which determine geometric mappings may not map congruence relations onto congruence relations.

Main Result.

THEOREM 1. Let A be an S-algebra with permutable congruence relations and let L, its corresponding lattice of congruence classes, be geometric of length ≥ 4. Then L is the lattice of flats of an affine geometry where the points of the geometry correspond to the elements of the algebra A.

PROOF. A hyperplane h_i in A corresponds to a unique maximal congruence relation θ_i. The natural mapping of A upon A/θ_i induces a geometric mapping of L upon a lattice of length 3 wich is geometric and in which h_i is mapped upon a point. Hence, if one line in L has at least three points, then the result follows from the quoted theorem.

If every line in L has two points, then we must use a special argument. We observe that since A is an S-algebra, Γ is a geometric lattice, and since the congruence relations on A permute, Γ is a modular lattice. If P is a plane in L, it corresponds to a congruence relation θ_3 which is a line in Γ. We then choose θ_1, θ_2 such that θ_1 and θ_2 are maximal congruences and $\theta_1 \wedge \theta_2$ and θ_3 are complementary. Then

$$A \cong A/\theta_1 \wedge \theta_2 \times A/\theta_3 \text{ and } A/\theta_1 \wedge \theta_2 \cong A/\theta_1 \times A/\theta_2 .$$

Furthemore, P is isomorphic to $L_{\theta_1 \wedge \theta_2}$ since the congruence classes in θ_3 or any congruence relation contained in it are mapped $1-1$ under the mapping modulo $\theta_1 \wedge \theta_2$. Now L_{θ_1} is of length 3 and by [6] it is isomorphic to an interval sublattice [0, 1] of L where 1 is a line. Thus L_{θ_1} is a line with two points and A/θ_1 is an algebra of two elements. Hence $A_{\theta_1 \wedge \theta_2}$ is an algebra of four elements and $\Gamma_{\theta_1 \wedge \theta_2}$ is a sublattice of the partition lattice on four elements containing 0 and 1, where the partitions permute, one class determines the partition and $\Gamma_{\theta_1 \wedge \theta_2}$ is complemented. Thus $\Gamma_{\theta_1 \wedge \theta_2}$ contains at least two of the partitions [12] [34], [13] [24], [14] [23]. Suppose it contains, say, [12] [34], [13] [24]. Now θ_1 and θ_2 can be viewed as [1′2′] [3′4′], [1′3′] [2′4′] where the prime symbols represent the various pairs with the appropriate numeral as first element. Thus L has [1′2′], [3′4′], [1′3′], [2′4′] as hyperplanes and [1′], [2′], [3′], [4′] as subhyperplanes which determine a congruence relation and are convered by the hyperplanes. Since line in L has two points, [(1, a)(4, a] and [(2, a)(3, a)] are congruence classes, for some a. If L is a geometric lattice, then [1′] and [(1, a)(4, a)] must join to a hyperplane as they have (1, a) as a meet. Now [1′2′] and [1′3′] cannot be this hyperplane. As A is an S-algebra, the only possible congruence relation that can contain [1′] [2′] [3′] [4′] besides θ_1 and θ_2 is [1′4′] [2′3′]. Since [1′] and [(1, a)(4, a)] do join to a hyperplane, this relation must exist. But modulo $\theta_1 \wedge \theta_2$ the classes [1′4′], [2′3] map onto [14] and [23]. Thus $L_{\theta_1 \wedge \theta_2}$ also contains [14] and [23]; therefore it is the affine plane with two points on each line. But then L is a special (see [6]) geometric lattice in which Euclid's parallel axiom holds in any plane. Hence L is an affine geometry.

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Manoscritto pervenuto in redazione il 23 settembre 1971.