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ON THE STRUCTURE OF SOLVABLE nC-GROUPS

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A finite *nC-group* has each normal subgroup complemented. A *complementing expansion* of a group G is defined to be an expression of G in the form $G=A_1 \dots A_n$ where each A_i is an abelian group having the property that each chararcteristic subgroup is a direct factor, A_{i+r} is contained in the normalizer in G of A_i for $(n-i)\geq r\geq 0$, and $A_i \cap \prod_{j=i+1}^{n} A_j=1$ for $i=1, \dots, n-1$. A *derived expansion* of a group G is defined to be a complementing expansion $G=A_1 \dots A_n$ such that $A_1 \dots A_i$ is the $(n-i)$-derived subgroup of G for $i=1, \dots, n$. It was shown in [3] that all complements of the $(n-1)$-derived subgroup of a solvable nC-group having solvability length n are conjugate. Furthermore, if $G=A_1 \dots A_n=B_1 \dots B_n$ are any two derived expansions of a solvable nC-group G, then $A_i \dots A_n$ and $B_i \dots B_n$ are conjugate in G for $i=1, \dots, n$. The object here is to unify these results with other similar findings and hence broaden the base for a better understanding of the structure of solvable nC-groups. In so doing, it is found that the collection of all solvable nC-groups is a nonsaturated formation that is normal subgroup inherited but not subgroup inherited.

Only *finite* groups will be considered. The notation is that used in the standard references such as [7] with possibly one exception. The expression $G=[A]B$ will denote that B complements the normal subgroup A in the group G.

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1. A *formation* $\boldsymbol{F}$ is a class of groups satisfying the conditions (1) if $G \in \boldsymbol{F}$, then $f(G) \in \boldsymbol{F}$ for each epimorphism f of G and (2) if M and N are normal subgroups of G such that G/N, $G/M \in \boldsymbol{F}$, then $G/N \cap M \in \boldsymbol{F}$. The formations considered here contain a one element group.

DEFINITION 1.1. A formation $\boldsymbol{F}$ is a normal formation iff $G \in \boldsymbol{F}$ implies that each normal subgroup of G is in $\boldsymbol{F}$.

Each formation $\boldsymbol{F}$ contains a subcollection that is a normal formation (perhaps only the one element group). Just let $\boldsymbol{X}$ consist of all $G \in \boldsymbol{F}$ having each subnormal subgroup in $\boldsymbol{F}$. It is easily verified that $\boldsymbol{X}$ is a normal formation. (If restriction is made only to normal subgroups, then $\boldsymbol{X}$ is a formation, but not necessarily a normal formation.)

In this article, reference is made only to solvable nC-groups since F. Gross [6] and C. Christensen [3] have shown the equivalence of solvable K-groups and solvable nC-groups. They have also shown that

(1.2) each epimorphic image and each normal subgroup of an nC-group is an nC-group.

THEOREM 1.3. The class of solvable nC-groups is a normal formation.

PROOF. It is known [2], corrected proof [8], that a solvable group is an nC-group iff it is a subdirect product of solvable nC-groups H such that H has precisely one minimal normal subgroup. If G/N, G/M are solvable nC-groups, then $G/N \cap M$ is a subdirect product of G/N and G/M. Assume that $N \cap M = 1$. $G/M \times G/M$ is a solvable nC-group; see [3]. Since G/N and G/M are subdirect products of solvable nC-groups having precisely one minimal normal subgroup, then let D be the direct product of the associated direct products. Hence G is a subdirect product of D. So G is a solvable nC-group. The other conditions are satisfied by (1.2).

A Sylow 2-subgroup in the symmetric group of degree four indicates that the property of being a solvable nC-group is not necessarily subgroup inherited. Any nonabelian nilpotent group indicates that this property is not necessarily a saturated formation.

2. Each formation $\boldsymbol{F}$ gives rise to a unique characteristic subgroup $\boldsymbol{F}[G]$ in a group G. $\boldsymbol{F}[G]$ has the property, for $N \lhd G$, that $G/N \in \boldsymbol{F}$ iff $\boldsymbol{F}[G] \subseteq N$. It is known [4], for $N \lhd G$, that

$$\boldsymbol{F}[G/N] = N\boldsymbol{F}[G]/N. \tag{2.1}$$

In a group G, define a chain of subnormal subgroups $G = A_0 \supseteq A_1 \supseteq \supseteq \ldots \supseteq A_j \supseteq \ldots$, such that $A_{i-1}/A_i \in \boldsymbol{F}$, to be an *$\boldsymbol{F}$-chain*. An *$\boldsymbol{F}$-series* for G will have $A_n = 1$ for some integer n. If n is the least integer such that $A_n = 1$ and $A_i \subset A_{i-1}$ properly for $1 \leq i \leq n$, then G is said to have *$\boldsymbol{F}$-length n*.

Next denote in a group G, $\boldsymbol{F}_0[G] = G$, $\boldsymbol{F}_1[G] = \boldsymbol{F}[G]$, and $\boldsymbol{F}_j[G] = \boldsymbol{F}[\boldsymbol{F}_{j-1}[G]]$ for $j \geq 2$. The uniqueness of $\boldsymbol{F}[G]$ implies that $F_j[G] \lhd G$. Hence each group G has an *$\boldsymbol{F}$-derived chain* $G = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_j \supseteq \ldots$, where $F_j = \boldsymbol{F}_j[G]$. G will have have an *$\boldsymbol{F}$-derived series* iff for some least integer n, $F_n = 1$ and $F_j \subset F_{j-1}$ for $1 \leq j \leq n$. If G has such a series then G will be said to have an *$\boldsymbol{F}$-derived length n*.

An obvious example of the above is for $\boldsymbol{F}$ to be the class of abelian groups. Then $G^{(1)} = [G, G] = \boldsymbol{F}[G]$, $F_j = G^{(j)} = [G^{(j-1)}, G^{(j-1)}]$, and the $\boldsymbol{F}$-derived chain is the derived chain. The $\boldsymbol{F}$-derived length is the derived length (or solvability length). As the next several theorems will indicate, there are a number of structural properties in a solvable group that are not dependent on the fact that solvability is also a subgroup inherited property.

The expression « standard proof » appears in the proof of the next theorem to indicate that the method to be applied is one used in proving an analogous result for solvable or nilpotent groups.

THEOREM 2.2. Let $\boldsymbol{F}$ denote a normal formation.

(*a*) If $G = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_j \supseteq \ldots$ is an $\boldsymbol{F}$-chain, then $F_j \subseteq A_j$.

(*b*) $\boldsymbol{F}_i[G/N] = NF_i/N$ and $\boldsymbol{F}_i[G/F_j] = F_i/F_j$, for $i \leq j$.

(*c*) G has an $\boldsymbol{F}$-series iff G has an $\boldsymbol{F}$-derived series. The $\boldsymbol{F}$-derived length is the minimal length of any $\boldsymbol{F}$-series.

(*d*) If G has an $\boldsymbol{F}$-series of $\boldsymbol{F}$-derived length n, then each epimorphic image and each normal subgroup of G has an $\boldsymbol{F}$-derived series with $\boldsymbol{F}$-derived length $\leq n$.

(*e*) If G/N has $\boldsymbol{F}$-derived length j, then $F_j \subseteq N$. If $F_j \subseteq N$, then G/N has $\boldsymbol{F}$-derived length at most j.

(*f*) An extension G of a group A by a group B, such that A has $\boldsymbol{F}$-derived length n and B has $\boldsymbol{F}$-derived length m, has an $\boldsymbol{F}$-derived series of length $\leq m+n$.

(*g*) Each group G contains a unique characteristic subgroup that is maximal in G with respect to having an $\boldsymbol{F}$-series. This subgroup contains each normal subgroup of G that has an $\boldsymbol{F}$-series.

(*h*) If G has an $\boldsymbol{F}$-series, then each chief factor of G belongs *to* $\boldsymbol{F}$. Each normal subgroup of G is included in an $\boldsymbol{F}$-series for G.

Proof. Induction is used on the length of the series in (*a*). Assume $F_{j-1} \subseteq A_{j-1}$. Since $A_{j-1}/A_j \in \boldsymbol{F}$, then $A_j F_{j-1}/A_j \in \boldsymbol{F}$. Moreover $A_j F_{j-1}/A_j \cong F_{j-1}/A_j \cap F_{j-1} \in \boldsymbol{F}$. Hence $F_j \subseteq A_j \cap F_{j-1}$ implies that $F_j \subseteq A_j$.

(*b*) is proven by using (2.1), induction, and (*a*). For (*c*), use (*a*) and the definition of an $\boldsymbol{F}$-derived series. Apply (*b*) to prove the first part of (*d*) and a standard proof to the second part. (*e*) is an immediate consequence of (*b*). The remaining parts are standard proofs.

Theorem 2.3. Let $\boldsymbol{F}$ be a normal formation.

(*a*) The class $\boldsymbol{H}$ of all groups having an $\boldsymbol{F}$-series is a normal formation.

(*b*) The class $\boldsymbol{G}$ of all groups having an $\boldsymbol{F}$-series and of $\boldsymbol{F}$-derived length $r \leq n$ for an integer n, is a normal formation. For a group G, $\boldsymbol{G}[G] = \boldsymbol{F}_n[G]$.

Proof. Consider (*a*). If G/N, $G/M \in \boldsymbol{H}$, then for some least integer i, $F_i \subseteq N$, and for some least integer j, $F_j \subseteq M$. For $k = \max(i, j)$, $F_k \subseteq N \quad M$. Hence by Theorem 2.2 (*e*), $G/N \cap M \in \boldsymbol{H}$. By Theorem 2.2 (*d*), $\boldsymbol{H}$ is a normal formation.

The proof of the first part of (*b*) follows that of (*a*). The second part results from Theorem 2.2 (*b*).

Lemma 2.4. If a solvable group G having $\boldsymbol{F}$-derived length $n+1$ with respect to a normal formation $\boldsymbol{F}$ splits over F_n and F_n is a minimal normal subgroup of G, then the complements of F_n are conjugate in G.

PROOF. Suppose that $G=[F_n]A=[F_n]B$. Form $N=\cap A^g$, $\forall g \in G$. If $N \not\subseteq B$, then $G=NB$. Since B has $\boldsymbol{F}$-derived length n, by Theorem 2.2 (*b*), then $G/N \cong B/N \cap B$ has $\boldsymbol{F}$-derived length $\leq n$ by Theorem 2.2 (*d*). By Theorem 2.2 (*e*), $F_n \subseteq N$. Since $F_n \neq 1$, a contradiction arises. So $N \subseteq B$. As is known, G/N is isomorphic to a primitive permutation group (on the conjugate class of A). Moreover $G/N=[F_nN/N](A/N)=$ $=[F_nN/N](B/N)$. Since F_nN/N is a minimal normal subgroup of G/N, then A/N and B/N are conjugate in G/N (e.g. see Satz 3.2f, p. 159, [7]). Hence A and B are conjugate in G.

THEOREM 2.5. If a solvable group G having $\boldsymbol{F}$-derived length $n+1$ with respect to a normal formation $\boldsymbol{F}$ splits over F_n and F_n is a completely G-reducible abelian subgroup of G, then the complements of F_n are conjugate in G.

PROOF. Suppose that $F_n=\overset{k}{\underset{1}{\times}} M_j$ such that M_j is a minimal normal subgroup of G. Lemma 2.4 takes care of the case for $k=1$, so suppose that $k \geq 2$ and consider G/M_1. $\boldsymbol{F}_n[G/M_1]=F_n/M_1$ by Theorem 2.2 (*b*). Let $G=[F_n]A=[F_n]B$. It follows that $G/M_1=(F_n/M_1)(M_1B/M_1)$ and also that $F_n \cap M_1B=M_1(F_n \cap B)=M_1$. Hence $G/M_1=[F_n/M_1](M_1B/M_1)$. Similarly $G/M_1=[F_n/M_1](M_1A/M_1)$. Since $G=[M_1]((M_2 \times \ldots \times M_k)A)$, then $H=(M_2 \times \ldots \times M_k)A$ has $\boldsymbol{F}_n[H]=M_2 \times \ldots \times M_k$. Since F_n being completely G-reducible is equivalent to F_n being completely A-reducible, then $M_1(\overset{k}{\underset{2}{\times}} M_j)/M_1$ is completely G/M_1-reducible. Inductively, there exists $g \in G$ such that $(M_1B)^g=M_1B^g=M_1A$. By Theorem 2.2 (*e*), $\boldsymbol{F}_n[M_1A] \subseteq M_1$. If $\boldsymbol{F}_n[M_1A] \subset M_1$ properly, then, for $M^*=\boldsymbol{F}_n[M_1A]$, $\boldsymbol{F}_n[G] \subseteq M_2 \times \ldots \times M_k \times M^* \neq F_n$. Hence $\boldsymbol{F}_n[M_1A]=M_1$. Moreover, M_1 is a minimal normal subgroup of M_1A. By Lemma 2.4, there exists an element $h \in M_1A$ such that $A=(B^g)^h=B^{gh}$. So A and B are conjugate in G.

COROLLARY 2.5.1. If a solvable nC-group G has $\boldsymbol{F}$-derived length $n+1$ and F_n is abelian, then the complements of F_n are conjugate in G.

PROOF. $\Phi(G)=1$ implies that F_n is completely G-reducible [5]. Then apply the Theorem.

COROLLORY 2.5.2. If a solvable nC-group G has $\boldsymbol{F}$-derived length

$n+1$ and F_n is abelian, then the complements of F_n are conjugate in each F_j for $j \leq n$.

PROOF. Use (1.2) and Theorem 2.2 (*b*).

A converse to Corollary 2.5.1 is not given since a much stronger result [9] is already known, namely: *A* solvable group *G* is an *nC*-group iff *G* splits over a completely *A*-reducible elementary abelian subgroup by an *nC*-group *A*.

The proof of Corollary 2.5.1 is dependent upon F_n being abelian. At times one is able to obtain some results whenever F_n is nonabelian.

LEMMA 2.6. For normal formations $\boldsymbol{F}$ and $\boldsymbol{G}$, let G be a solvable nC-group having $\boldsymbol{F}$-derived length $n+1$ and F_n of $\boldsymbol{G}$-derived length $k+1$. If $\boldsymbol{G}_k[F_n]$ is a minimal normal subgroup of G, then the complements of $\boldsymbol{G}_k[F_n]$ are conjugate in G.

PROOF. Denote $\boldsymbol{G}_k[F_n]$ by M. Suppose that $G=[M]A=[M]B$. Corollary 2.5.1 takes care of the case that $F_n=M$. So assume for $k \geq 1$ that $F_n=[M](F_n \cap A)=[M](F_n \cap B)$, such that $F_n \cap A \neq 1$, $F_n \cap B \neq 1$. Since F_n is a solvable nC-group, then, by Corollary 2.5.1, there exists an element $g \in F_n$ such that $N=F_n \cap A=(F_n \cap B)^g=F_n \cap B^g$. Consider $G=[M]A=[M]B^g$ and note that $N \lhd \langle A, B^g \rangle$. Since A and B^g are maximal in G, then either (1) $\langle A, B^g \rangle = A = B^g$ or (2) $\langle A, B^g \rangle = G$. If (1) occurs, then the result is valid. If (2) occurs, then $N \lhd G$ and $G/N=[N \times M/N](A/N)=[N \times M/N](B^g/N)$. Moreover $\boldsymbol{F}_n[G/N]= =F_n/N \cong M \neq 1$ is abelian. By Corollary 2.5.1, A/N and B^g/N are conjugate in G/N. So $A=(B^g)^h=B^{gh}$ for some element $h \in G$.

THEOREM 2.7. For normal formations $\boldsymbol{F}$ and $\boldsymbol{G}$, let G be a solvable nC-group having $\boldsymbol{F}$-derived length $n+1$ and F_n of $\boldsymbol{G}$-derived length $k+1$. If $\boldsymbol{G}_k[F_n]$ is an abelian subgroup of G, then the complements of $\boldsymbol{G}_k[F_n]$ are conjugate in G.

PROOF. By Corollary 2.5.1, the theorem is valid whenever $F_n= =\boldsymbol{G}_k[F_n]$. So consider the case for $k \geq 1$. Since G is a solvable nC-group, then $\boldsymbol{G}_k[F_n]$ is a direct product of abelian minimal normal subgroups of G. Induction will used on the number of the direct factors in $\boldsymbol{G}_k[F_n]$ since Lemma 2.6 shows that the result is valid for one such factor. From this point on the proof parallels the proof of Theorem 2.5 and it will be omitted.

It should be observed that if G is a solvable nC-group and it has $\boldsymbol{F}$-derived length $n+1$ such that F_n is nonabelian, then the obvious normal formaiton $\boldsymbol{G}$ at hand is that of abelian groups. The result of Theorem 2.7 will then be applicable to the abelian member of the derived series.

Proceeding in a manner as in [3], a group G has an *F-derived expansion* iff $G=G_1 \dots G_n$ such that G_i is contained in the normalizer of G_j for $i \geq j$, $G_i \cap (G_{i+1} \dots G_n)=1$, and $\boldsymbol{F}_j[G]=G_1 \dots G_{n-j}$. The G_j are called the *factors* of the expansion. Two derived expansions are said to be *conjugate* iff $m=n$ and G_i is conjugate to H_i for each i.

Each nC-group of $\boldsymbol{F}$-derived length $n+1$ has such an expansion. If F_j denotes $G^{(j)}$, $G^{(j)}$ the j-derived subgroup of G, then the $\boldsymbol{F}$-derived expansion coincides with the derived expansion in [3] and the next theorem generalizes Theorem 5.7 in [3].

THEOREM 2.6. If $G=G_1 \dots G_n=H_1 \dots H_m$ are two $\boldsymbol{F}$-derived expansions for a normal formation $\boldsymbol{F}$ of a solvable nC-group G have abelian factors, then there exists an element $x \in G$ such that $G_i^x=H_i$ for $i=1, \dots, n$, i.e. the two expansions are conjugate.

PROOF. By Theorem 2.2, $n=m$. Moreover $G_1=\boldsymbol{F}_{n-1}[G]=H_1$. All complements are conjugate by Corollary 2.5.1. Hence there exists $y \in G$ such that $(G_2 \dots G_n)^y=H_2 \dots H_n$. Since $H_2=\boldsymbol{F}_{n-2}[H_2 \dots H_n]$, then $G_2^y=\boldsymbol{F}_{n-2}[G_2 \dots G_n]^y=\boldsymbol{F}_{n-2}[(G_2 \dots G_n)^y]=H_2$. Assume that there is a $y_i \in G_1 \dots G_{i-1}$ such that $G_j^{y_i}=H_j$ for $j \leq (i-1)$ and $(G_{i-1} \dots G_n)^{y_i}= =H_{i-1} \dots H_n \cdot H_{i-1}=G_{i-1}^{y_i}$. Then there exists $w \in G_{i-1}^{y_i}$ such that $(G_i \dots G_n)^{y_i w}=H_1 \dots H_n$. Hence $G_i^{y_i w}=H_i$. Moreover, w normalizes $G_j^{y_i}=H_j$ for $j \leq (i-1)$. Consequently, for $y_{i+1}=y_i w$ and $j \leq i$, $G_j^{y_{i+1}}=H_j$. Inductively, there exists $x \in G$ satisfying the conclusion of the theorem.

3. The notions of $\boldsymbol{F}$-chains and $\boldsymbol{F}$-series can also be generalized. Consider two normal formations $\boldsymbol{F}$ and $\boldsymbol{G}$ and a group G having an $\boldsymbol{F}$-chain $G=H_0 \supseteq H_1 \supseteq \dots \supseteq H_j \supseteq \dots$. The $\boldsymbol{F}$-chain $\{H_j\}$ is said to be *refined by a* **G**-*chain* iff each H_i/H_{i+1} admits a $\boldsymbol{G}$-chain for $i=0, 1, \dots$. Consequently one obtains a $\boldsymbol{G}$-chain $G=H_0=K_{0,0} \supseteq K_{0,1} \supseteq \dots \supseteq K_{0,j_0} \supseteq$ $\supseteq \dots H_1=K_{1,0} \supseteq \dots \supseteq K_{i,j} \supseteq \dots$ such that $K_{i,j}/K_{i,j+1} \in \boldsymbol{G}$. An $\boldsymbol{F}$-derived chain $\{F_j\}$ is said to be *refined by a* **G**-*derived chain* iff F_i/F_{i+1} admits

a $\boldsymbol{G}$-derived chain for $i=0, 1, \ldots$. This will also be called a $\boldsymbol{G}$-chain *derivation* of an $\boldsymbol{F}$-derived chain. The next theorem is an analogue to Theorem 2.2 (*a*).

THEOREM 3.1. Let $G=H_0=K_{0,0}\supseteq K_{0,1}\supseteq \ldots \supseteq H_1=K_{1,0}\supseteq \ldots \supseteq \supseteq H_j\supseteq \ldots$ be an $\boldsymbol{F}$-chain refined by a $\boldsymbol{G}$-chain and $G=G_{0,0}\supseteq G_{0,1}\supseteq \supseteq \ldots \supseteq F_1=G_{1,0}\supseteq \ldots \supseteq F_i\supseteq \ldots$ be a $\boldsymbol{G}$-chain derivation of an $\boldsymbol{F}$-derived chain. Then $G_{i,j}\subseteq K_{i,j}$ for each pair of integers i, j.

PROOF. Clearly, $F_1\subseteq H_1\subseteq K_{0,j}$ for each integer j. Consider G/F_1. The chain $\{K_{0,j}/F_1\}$ is a $\boldsymbol{G}$-chain in G/F_1. By Theorem 2.2, $\boldsymbol{G}_j[G/F_1]\subseteq \subseteq K_{0,j}/F_1$ for all j. However by definition, $\boldsymbol{G}_j[G/F_1]=G_{0,j}/F_1$. Hence $G_{0,j}/F_1\subseteq K_{0,j}/F_1$ implies that $G_{0,j}\subseteq K_{0,j}$ for all j.

For an integer $i\geq 1$, assume that $G_{i,j}\subseteq K_{i,j}$. By Theorem 2.2, $F_i\subseteq H_i$ and $F_{i+1}\subseteq H_{i+1}$. Consider the quotient group $H_i/K_{i,1}$. If $F_i\nsubseteq K_{i,1}$, the one still has $F_iK_{i,1}/K_{i,1}$ normal in $H_i/K_{i,1}$. Since $H_i/K_{i,1}\in\boldsymbol{G}$, then $F_iK_{i,1}/K_{i,1}\cong F_i/F_i\cap K_{i,1}\in\boldsymbol{G}$. Moreover $F_i/F_i\cap K_{i,1}\cong F_i/F_{i+1}//F_i\cap K_{i,1}/ /F_{i+1}\in\boldsymbol{G}$ and $\boldsymbol{G}_i[F_i/F_{i+1}]=G_{i,1}/F_{i+1}$. So $G_{i,1}/F_{i+1}\subseteq F_i\cap K_{i,1}/F_{i+1}$. Therefore $G_{i,1}\subseteq K_{i,1}$.

Next assume that $G_{i,j}\subseteq K_{i,j}$ for an integer $j\geq 1$. $G_{i,j}K_{i,j+1}/K_{i,j+1}\lhd K_{i,j}/K_{i,j+1}$ implies that $G_{i,j}K_{i,j+1}/K_{i,j+1}\cong G_{i,j}/G_{i,j}\cap K_{i,j+1}\in\boldsymbol{G}$. Since $G_{i,j}/G_{i,j}\cap K_{i,j+1}\cong G_{i,j}/F_{i+1}//G_{i,j}\cap K_{i,j+1}/F_{i+1}\in\boldsymbol{G}$, then $G_{i,j+1}/F_{i+1}= =\boldsymbol{G}[G_{i,j}/F_{i+1}]\subseteq G_{i,j}\cap K_{i,j+1}/F_{i+1}$. Hence $G_{i,j+1}\subseteq G_{i,j}\cap K_{i,j+1}\subseteq K_{i,j+1}$, as was to be proven.

The extension of the definitions preceding Theorem 3.1 to refinemento of an $\boldsymbol{F}$-series by a $\boldsymbol{G}$-series and the refinement of an $\boldsymbol{F}$-derived series by a $\boldsymbol{G}$-derived series (a $\boldsymbol{G}$-series derivation of an $\boldsymbol{F}$-derived series) is evident. Of course each refinement of an $\boldsymbol{F}$-series by a $\boldsymbol{G}$-series is itself a $\boldsymbol{G}$-series. It is not being suggested that each $\boldsymbol{G}$-series is necessarily a refinement of an $\boldsymbol{F}$-series by a $\boldsymbol{G}$-series for some normal formation $\boldsymbol{F}$. Even though it may occur, it is not always expected that a group G having a $\boldsymbol{G}$-series derivation of an $\boldsymbol{F}$-derived series has this same series as an $\boldsymbol{F}$-series derivation of the $\boldsymbol{G}$-derived series. Examples of refinements arise naturally in solvable groups. If a solvable group G has an $\boldsymbol{F}$-derived series that differs from the derived series, then F_i/F_{i+1} always admits a derived series.

Some results analogous to those in Theorem 2.2 can now be established with the aid of Theorem 3.1. The proofs will be omitted since they so closely resemble the previous proofs.

THEOREM 3.2. Let $\boldsymbol{F}$ and $\boldsymbol{G}$ denote two normal formations.

(a) If a group G has an $\boldsymbol{F}$-series refined by a $\boldsymbol{G}$-series, then G has a $\boldsymbol{G}$-series derivation of an $\boldsymbol{F}$-derived series.

(b) The number of terms, k, in a $\boldsymbol{G}$-series derivation of an $\boldsymbol{F}$-derived series is an invariant of the group. The length of any $\boldsymbol{F}$-series refined by a $\boldsymbol{G}$-series is $\geq k$.

(c) Each normal subgroup and each epimorphic image of a group having an $\boldsymbol{F}$-series refined by a $\boldsymbol{G}$-series also has an $\boldsymbol{F}$-series refined by a $\boldsymbol{G}$-series.

(d) The class of all groups having an $\boldsymbol{F}$-series refined by a $\boldsymbol{G}$-series is a normal formation.

Suppose that a group G has a $\boldsymbol{G}$-series derivation of an $\boldsymbol{F}$-derived series and that $G_{n,k+1}=F_{n+1}=1$, $F_n\neq 1$, and $G_{n,k}\neq 1$. Then $G_{n,k}$ may be properly contained in G_m of a $\boldsymbol{G}$-derived series of $\boldsymbol{G}$-derived length $m+1$. Of course if $G_{n,k}$ is also abelian in a solvable nC-group, then a special case of Theorem 2.7 can be stated.

It is evident that the concept of refinement could be extended to any finite collection of normal formations arising out of series and using an ordered arrangement of successive refinements. One can obtain the obvious analogues of Theorems 3.1 and 3.2.

EXAMPLE. The class of finite groups, $\boldsymbol{E}$, in which all Sylow subgroups are elementary abelian is a normal formation. Since each solvable group G admits at least one normal maximal subgroup, then $\boldsymbol{E}[G]\subset G$, properly. Consequently each finite solvable group G has an $\boldsymbol{E}$-derived series. The symmetric group of degree four is an nC-group that has $\boldsymbol{E}$-derived length two but derived length three. More generally, the $\boldsymbol{E}$-derived length does not exceed the derived length in a solvable nC-group as one can readily verify.

Consider the $\boldsymbol{E}$-derived series $G=E_0\supset E_1\supset E_2=1$. Suppose that E_1 is abelian and let P denote a Sylow p-subgroup of G. If G/E_1 is abelian, then $E_1P\lhd G$. Since $\boldsymbol{E}[G]$ is generated by $\Phi(S)$ for all Sylow subgroups S of G (see [1]), then $\Phi(P)\lhd E_1P$. Hence $\Phi(P)\subseteq\Phi(E_1P)\subseteq\Phi(G)=1$.

So P is elementary abelian. Consequently all Sylow subgroups of G are elementary abelian, i.e. $E_1 = 1$. The contradiction implies that G/E_1 must be nonabelian. So one concludes that no two consecutive factors of an $\boldsymbol{E}$-derived series are abelian. Moreover, in a solvable nC-group of $\boldsymbol{E}$-derived length $k+1$, if an abelian factor occurs, then it must be E_k. Therefore if $\boldsymbol{A}$ denotes the formation of abelian groups and a solvable nC-group has $\boldsymbol{E}$-length greater than one, then the $\boldsymbol{A}$-series derivation of the $\boldsymbol{E}$-derived series exists nontrivially.

BIBLIOGRAPHY

[1] Bechtell, H.: *Elementary groups,* Trans. Amer. Math. Soc., 114 (1965), 355-362.

[2] Bechtell, H.: *A note on finite solvable K-groups,* Proc. Amer. Math. Soc., 17 (1966), 1147-1450.

[3] Christensen, C.: *Groups with complemented normal subgroups,* J. London Math. Soc., 42 (1967), 208-216.

[4] Cline, E.: *On an embedding property of generalized Carter subgroups,* Pac. J. Math., 42 (1967), 491-519.

[5] Gaschütz, W.: *Uber die Φ-Untergruppe endlicher Gruppen,* Math. Z., 80 (1963), 300-305.

[6] Gross, F.: *Überauflösbare Gruppen mit elementar-abelschen Sylowgruppen,* Ph. D. Thesis, Christian-Albrechts Universitat, Kiel, 1962.

[7] Huppert, B.: *Endliche Gruppen I,* Springer, Berlin, 1967.

[8] Kotzen, M.: *On the structure of a finite solvable K-group,* Proc. Amer. Math. Soc., 27 (1971), 16-18.

[9] Napolitani, F.: *Sui gruppi risolubili complementati,* Rend. Sem. Mat. Univ. Padova, 38 (1967), 118-120.

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