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di Navier-Stokes**

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SU UN PROBLEMA ESTERNO
RELATIVO ALLE EQUAZIONI DI NAVIER-STOKES

PAOLO MURATORI *)

Introduzione.

Sia Ω il complementare di un aperto limitato Ω_0 dello spazio euclideo \mathbf{R}^3 con frontiera $\partial\Omega$ sufficientemente regolare. Fissato $T \in \mathbf{R}^+$ (insieme dei reali positivi), sia

$$Q = \Omega \times [-T, 0].$$

Consideriamo il sistema di Navier-Stokes ($u = (u_1, u_2, u_3)$)

$$(\alpha) \quad \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u = -\nabla p + F$$

$$(\beta) \quad \nabla \cdot u = 0$$

con $\nabla \times F = 0$, e consideriamo il problema dell'unicità di una soluzione u, p di (α) - (β) in Q con assegnati valori su $\partial\Omega \times [-T, 0]$ e $\Omega \times \{0\}$. I risultati conseguiti, espressi dai teoremi 1 e 2 del § 3, sono da porre a confronto con analoghi risultati di D. E. Edmunds [1] (vedi anche [2]); le ipotesi in cui mi sono posto sono meno restrittive di quelle del citato Autore. Precisamente se u, q e $u+v, q+p$ sono due soluzioni di (α) - (β) , l'ipotesi qui fatta sul comportamento asintotico di p è meno forte di quella fatta in [1]; inoltre non faccio ipotesi di limita-

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tezza delle derivate di u e v rispetto al tempo e di sommabilità su Ω dei quadrati delle derivate spaziali di v .

1. Sia Ω il complementare di un aperto limitato Ω_0 dello spazio euclideo \mathbf{R}^3 con frontiera $\partial\Omega$ sufficientemente regolare. Fissato $T \in \mathbf{R}^+$ (insieme dei reali positivi), sia

$$Q = \Omega \times [-T, 0].$$

Se $R \in \mathbf{R}^+$ poniamo

$$\omega'(R) = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2 + x_2^2 + x_3^2 \leq R^2\}$$

$$\sigma(R) = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2 + x_2^2 + x_3^2 = R^2\}$$

$\forall R \geq$ di un certo R_0 che supponiamo inoltre > 1 per cui risulta $\overline{\mathbf{R}^3 - \omega'(R_0)} \subset \Omega$

$$\omega_{R_1}(R) = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; R_1^2 \leq x_1^2 + x_2^2 + x_3^2 \leq R^2\}$$

$\forall R, R_1$ con $R_0 \leq R_1 \leq R$,

$$\omega_R = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; R^2 \leq x_1^2 + x_2^2 + x_3^2\}.$$

Useremo costantemente nel seguito le seguenti notazioni

$$L_1 = \left\{ g : Q \rightarrow \mathbf{R}; g \text{ continua in } Q, \right. \\ \left. \frac{\partial g}{\partial x_i} \text{ continua in Int}(Q) \text{ per } 1 \leq i \leq 3 \right\}$$

$$L_2 = \left\{ g : Q \rightarrow \mathbf{R}; g, \frac{\partial g}{\partial x_i} \text{ continue in } Q \text{ per } 1 \leq i \leq 3, \right. \\ \left. \frac{\partial^2 g}{\partial x_i \partial x_j} \text{ continue in Int}(Q) \text{ per } 1 \leq i, j \leq 3 \right\}$$

$$L_3 = \left\{ g : Q \rightarrow \mathbf{R}; g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x_i}, \frac{\partial^2 g}{\partial x_i \partial x_j}, \right. \\ \left. \frac{\partial^2 g}{\partial t \partial x_i} \text{ continue in } Q \text{ per } 1 \leq i, j \leq 3, \right. \\ \left. \frac{\partial^3 g}{\partial x_k \partial x_i \partial x_j} \text{ continue in } \text{Int}(Q) \text{ per } 1 \leq i, j, k \leq 3 \right\};$$

inoltre se $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ con $u_i, v_i \in L_2$ e $f \in L_2$ poniamo

$$(u, v)_{\omega_{R_1}(R)} = \int_{\omega_{R_1}(R)} \left(\sum_{i,j=1}^3 u_i v_j \right) dx, \quad \|u\|_{\omega_{R_1}(R)}^2 = (u, u)_{\omega_{R_1}(R)}$$

$$(u, v)_{\sigma(R)} = \int_{\sigma(R)} \left(\sum_{i,j=1}^3 u_i v_j \right) d\sigma, \quad \|n\|_{\sigma(R)}^2 = (u, u)_{\sigma(R)}$$

$$\| \| u \| \|_{\omega_{R_1}(R)}^2 = \int_{\omega_{R_1}(R)} \left(\sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right) dx, \quad \| \| u \| \|_{\sigma(R)}^2 = \int_{\sigma(R)} \left(\sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right) d\sigma$$

$$\| \| f, u \| \|_{\omega_{R_1}(R)}^2 = \int_{\omega_{R_1}(R)} f^2 \left(\sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right) dx,$$

$$\| \| f, u \| \|_{\sigma(R)}^2 = \int_{\sigma(R)} f^2 \left(\sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right) d\sigma \quad \forall R \geq R_1 \geq R_0.$$

Analogo significato per

$$\| \| u \| \|_{\omega(R)}, \| \| u \| \|_{\omega(R)}, \| \| u \| \|_{\Omega}^2, \| \| u \| \|_{\Omega}^2.$$

Gli stessi simboli saranno anche usati, col significato usuale, se al posto del vettore u si pone una funzione.

Riportiamo per comodità le seguenti note formule

(a) $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3, \quad u^2 = u \cdot u, \quad |u| = (u \cdot u)^{1/2}$

(b) $u \times v = (u_2 v_3 - u_3 v_2, \quad u_3 v_1 - u_1 v_3, \quad u_1 v_2 - u_2 v_1)$

$$(c) \quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$(d) \quad \nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$(e) \quad \nabla \times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

$$(f) \quad (u \cdot \nabla)v = \left(\sum_{i=1}^3 \frac{\partial v_1}{\partial x_i} u_i, \sum_{i=1}^3 \frac{\partial v_2}{\partial x_i} u_i, \sum_{i=1}^3 \frac{\partial v_3}{\partial x_i} u_i \right)$$

$$(g) \quad \nabla^2 u = \Delta u = \left(\sum_{i=1}^3 \frac{\partial^2 u_1}{\partial x_i^2}, \sum_{i=1}^3 \frac{\partial^2 u_2}{\partial x_i^2}, \sum_{i=1}^3 \frac{\partial^2 u_3}{\partial x_i^2} \right)$$

$$(h) \quad \nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$(i) \quad (\nabla u)v = \left(\sum_{i=1}^3 \frac{\partial u_i}{\partial x_1} v_i, \sum_{i=1}^3 \frac{\partial u_i}{\partial x_2} v_i, \sum_{i=1}^3 \frac{\partial u_i}{\partial x_3} v_i \right)$$

in particolare

$$(i) \quad (\nabla u)u = \frac{1}{2} \nabla(u^2)$$

$$(l) \quad \nabla(u \cdot v) = u \times (\nabla \times v) + (u \cdot \nabla)v + v \times (\nabla \times u) + (v \cdot \nabla)u$$

$$(m) \quad \nabla \cdot (u \times v) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$$

$$(n) \quad \nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \nabla^2 u = \nabla(\nabla \cdot u) - \Delta u$$

$$(o) \quad \nabla \cdot (fu) = f \nabla \cdot u + u \cdot \nabla f$$

$$(p) \quad \nabla \times (\nabla f) = 0$$

$$(q) \quad \nabla \times (u \times v) = (v \cdot \nabla)u - (\nabla \cdot u)v + (\nabla \cdot v)u - (u \cdot \nabla)v$$

(r) $\nabla \cdot (\nabla \times u) = 0$

(s) $u \cdot \Delta u = - \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 + \frac{1}{2} \nabla \cdot (\nabla(u^2))$

(t) $u \cdot \Delta u = -(\nabla \times u)^2 - \nabla \cdot ((\nabla \times u) \times u)$

(u) $\frac{\partial u}{\partial t} \cdot \nabla u = - \frac{1}{2} \frac{\partial}{\partial t} \left(\sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right) + \nabla \cdot \left((\nabla u) \frac{\partial u}{\partial t} \right)$

(v) $\frac{\partial u}{\partial t} \cdot \nabla u = - \frac{1}{2} \frac{\partial}{\partial t} (\nabla \times u)^2 - \nabla \cdot \left((\nabla \times u) \times \frac{\partial u}{\partial t} \right).$

Nel seguito useremo una serie di costanti, che denoteremo con $C_1, C_2, \dots, C'_1, \dots, C_\alpha, C_\beta, \dots$, senza specificare volta per volta la dipendenza da certe funzioni limitate che intervengono nei calcoli.

2. LEMMA 1. Sia $a \in \mathbf{R}^+$ e sia $\Phi : [a, +\infty[\rightarrow \mathbf{R}^+$ di classe $C^{(1)}$ tale che $\frac{d\Phi(R)}{dR} = \Phi'(R) \geq \delta \quad \forall R \geq a$ per un certo $\delta \in \mathbf{R}^+$. Allora, se esistono $C \in \mathbf{R}^+$ e $m \in \mathbf{N}$ tali che

$$\Phi(R) \leq C(R^m + R^{m+1}(\Phi'(R))^{1/2}) \quad \forall R \geq a,$$

esiste di conseguenza $R_1 \in \mathbf{R}^+$ tale che

$$\Phi(R) \leq R^{3m+3} \quad \forall R \geq R_1.$$

Per la dimostrazione vedi [3].

LEMMA 2. Se $w_1, w_2, w_3 \in L_3$ e $f, p \in L_2$, f è costante in t , $\nabla \cdot w = 0$, allora, posto

$$Lw = \Delta w - \frac{\partial w}{\partial t}, \quad \lambda = \tau - t, \quad C_{\tau, \gamma, \delta} = \left(2\tau + \frac{1}{2} \gamma^{-1} + \delta + 4\tau^2 \right),$$

si ha

$$\begin{aligned}
& \left(\frac{\gamma\delta}{4\tau^2} + \gamma \right) \int_{-\tau}^0 \lambda^{-2\gamma} \|f\mathcal{W}(t-h)\|_{\omega_{R_1}(R)}^2 dt + \int_{-\tau}^0 \lambda^{-2\gamma} \| \|f, \mathcal{W}(t-h)\| \|_{\omega_{R_1}(R)}^2 dt + \\
& + C_{\tau, \gamma, \delta} \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \mathcal{W}(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt \leq \\
& \leq -\frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} (\nabla(\mathcal{W}^2(t-h)), \nabla(f^2))_{\omega_{R_1}(R)} dt - \\
& - 2C_{\tau, \gamma, \delta} \int_{-\tau}^0 \lambda^{-\gamma} ((\nabla \mathcal{W}(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} \mathcal{W}(t-h)), \nabla(f^2))_{\omega_{R_1}(R)} dt + \\
& + \int_{-\tau}^0 \lambda^{-2\gamma} (p(t-h)\mathcal{W}(t-h), \nabla(f^2))_{\omega_{R_1}(R)} dt + \\
& + 2C_{\tau, \gamma, \delta} \int_{-\tau}^0 \lambda^{-2\gamma} \|fL\mathcal{W}(t-h) - f\nabla p(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\
& + 2C_{\tau, \gamma, \delta} \int_{-\tau}^0 \lambda^{-\gamma} \left(\frac{\partial}{\partial t} (\lambda^{-\gamma} \mathcal{W}(t-h)), p(t-h)\nabla(f^2) \right)_{\omega_{R_1}(R)} dt + \\
& + 4 \int_{-\tau}^0 \lambda^{-2\gamma} (\|f\mathcal{W}(t-h)\|_{\sigma(R_1)}^2 + \| \|f, \mathcal{W}(t-h)\| \|_{\sigma(R_1)}^2) dt + \\
& + 4 \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \mathcal{W}(t-h)) \right\|_{\sigma(R_1)}^2 dt + \\
& + 4 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|fp(t-h)\|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \mathcal{W}(t-h)) \right\|_{\sigma(R_1)}^2 dt \right)^{1/2} + \\
& + 4 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|fp(t-h)\|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|f\mathcal{W}(t-h)\|_{\sigma(R_1)}^2 dt \right)^{1/2} +
\end{aligned}$$

$$\begin{aligned}
 &+ 4 \int_{-\tau}^0 \lambda^{-2\gamma} (\|f w(t-h)\|_{\sigma(R)}^2 + \|f, w(t-h)\|_{\sigma(R)}^2) dt + \\
 &+ 4 \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \\
 &+ 4 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|f p(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt \right)^{1/2} + \\
 &+ 4 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|f p(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \|f w(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} + \\
 &+ C_{\tau, \gamma, \delta} (2\tau)^{-2\gamma} \|f, w(-t-h)\|_{\omega_{R_1}(R)}^2 + 4\gamma(\tau)^{-2\gamma-1} \|f w(-h)\|_{\omega_{R_1}(R)}^2 - \\
 &- \tau\gamma(2\tau)^{-2\gamma-1} \|f w(-\tau-h)\|_{\omega_{R_1}(R)}^2
 \end{aligned}$$

$$\forall \tau, R, R_1, \gamma, \delta \in \mathbf{R}^+, \forall h \geq 0 \text{ con } R \geq R_1 \geq R_0, \gamma \geq 1, \tau \leq \frac{1}{3},$$

$$0 \leq h + \tau \leq T, \quad 0 < \delta \leq \frac{1}{4}.$$

DIMOSTRAZIONE. Cominciamo con l'osservare che

$$\begin{aligned}
 (1) \quad & \int_{-\tau}^0 \lambda^{-2\gamma} \|f L w(t-h) - f \nabla p(t-h)\|_{\omega_{R_1}(R)}^2 dt = \\
 &= \int_{-\tau}^0 \left[\|f \lambda^{-\gamma} \Delta w(t-h) + f \gamma \lambda^{-\gamma-1} w(t-h)\|_{\omega_{R_1}(R)}^2 + \right. \\
 &+ \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 - \\
 &- 2 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)), f \lambda^{-\gamma} \Delta w(t-h) \right)_{\omega_{R_1}(R)} - \\
 &- 2\gamma \left(f \lambda^{-\gamma-1} w(t-h), f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right)_{\omega_{R_1}(R)} \left. \right] dt
 \end{aligned}$$

$$\begin{aligned}
& + 2 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)), f \lambda^{-\gamma} \nabla p(t-h) \right)_{\omega_{R_1(R)}} - \\
& - 2(f \lambda^{-\gamma} \Delta w(t-h) + f \gamma \lambda^{-1-\gamma} w(t-h), f \lambda^{-\gamma} \nabla p(t-h))_{\omega_{R_1(R)}} + \\
& + \lambda^{-2\gamma} \|\nabla p(t-h)\|_{\omega_{R_1(R)}}^2 \Big] dt.
\end{aligned}$$

Infatti

$$\begin{aligned}
& \lambda^{-2\gamma} \left\| f \Delta w(t-h) - f \frac{\partial}{\partial t} (w(t-h)) - f \nabla p(t-h) \right\|_{\omega_{R_1(R)}}^2 = \\
& = \lambda^{-2\gamma} \left[\left\| f \Delta w(t-h) \right\|_{\omega_{R_1(R)}}^2 - 2 \left(f \Delta w(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1(R)}} + \right. \\
& + \left\| f \frac{\partial}{\partial t} (w(t-h)) \right\|_{\omega_{R_1(R)}}^2 + \left\| f \nabla p(t-h) \right\|_{\omega_{R_1(R)}}^2 - \\
& - 2(f \Delta w(t-h), f \nabla p(t-h))_{\omega_{R_1(R)}} + \\
& \left. + 2 \left(f \frac{\partial}{\partial t} (w(t-h)), f \nabla p(t-h) \right)_{\omega_{R_1(R)}} \right], \\
& \left\| f \lambda^{-\gamma} \Delta w(t-h) + f \gamma \lambda^{-\gamma-1} w(t-h) \right\|_{\omega_{R_1(R)}}^2 = \lambda^{-2\gamma} \left[\left\| f \Delta w(t-h) \right\|_{\omega_{R_1(R)}}^2 + \right. \\
& \left. + \gamma^2 \lambda^{-2} \left\| f w(t-h) \right\|_{\omega_{R_1(R)}}^2 + 2\gamma \lambda^{-1} (f \Delta w(t-h), f w(t-h))_{\omega_{R_1(R)}} \right], \\
& \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1(R)}}^2 = \lambda^{-2\gamma} \left[\gamma^2 \lambda^{-2} \left\| f w(t-h) \right\|_{\omega_{R_1(R)}}^2 + \right. \\
& \left. + \left\| f \frac{\partial}{\partial t} (w(t-h)) \right\|_{\omega_{R_1(R)}}^2 + 2\gamma \lambda^{-1} \left(f w(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1(R)}} \right], \\
& - 2 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)), \lambda^{-\gamma} f \Delta w(t-h) \right)_{\omega_{R_1(R)}} = \\
& = \lambda^{-2\gamma} \left[-2\gamma \lambda^{-1} (f w(t-h), f \Delta w(t-h))_{\omega_{R_1(R)}} - \right. \\
& \left. - 2 \left(f \frac{\partial}{\partial t} (w(t-h)), f \Delta w(t-h) \right)_{\omega_{R_1(R)}} \right],
\end{aligned}$$

$$\begin{aligned}
 & -2\gamma \left(f\lambda^{-\gamma-1}w(t-h), f \frac{\partial}{\partial t} (\lambda^{-\gamma}w(t-h)) \right)_{\omega_{R_1}(R)} = \\
 & = \lambda^{-2\gamma} \left[-2\gamma^2\lambda^{-2} \| fw(t-h) \|_{\omega_{R_1}(R)}^2 - \right. \\
 & \left. -2\gamma\lambda^{-1} \left(fw(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1}(R)} \right], \\
 & 2 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma}w(t-h)), f\lambda^{-\gamma}\nabla p(t-h) \right)_{\omega_{R_1}(R)} = \\
 & = \lambda^{-2\gamma} \left[2\gamma\lambda^{-1}(fw(t-h), f\nabla p(t-h))_{\omega_{R_1}(R)} + \right. \\
 & \left. + 2 \left(f \frac{\partial}{\partial t} (w(t-h)), f\nabla p(t-h) \right)_{\omega_{R_1}(R)} \right], \\
 & -2(f\lambda^{-\gamma}\Delta w(t-h) + f\gamma\lambda^{-1-\gamma}w(t-h), f\lambda^{-\gamma}\nabla p(t-h))_{\omega_{R_1}(R)} = \\
 & = \lambda^{-2\gamma} \left[-2(f\Delta w(t-h), f\nabla p(t-h))_{\omega_{R_1}(R)} - \right. \\
 & \left. - \gamma\lambda^{-1}(fw(t-h), f\nabla p(t-h))_{\omega_{R_1}(R)} \right].
 \end{aligned}$$

Dalla (1) segue che

$$\begin{aligned}
 (2) \quad & \int_{-\tau}^0 \lambda^{-2\gamma} \| fLw(t-h) - f\nabla p(t-h) \|_{\omega_{R_1}(R)}^2 \geq \\
 & \geq \int_{-\tau}^0 \left[\left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma}(t-h)) \right\|_{\omega_{R_1}(R)}^2 + \right. \\
 & \left. + \gamma\lambda^{-2\gamma-2} \| fw(t-h) \|_{\omega_{R_1}(R)}^2 \right] dt - \\
 & - 2 \int_{-\tau}^0 dt \int_{\sigma(R)} f\lambda^{-\gamma} (\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma}w(t-h)) \cdot nd\sigma + \\
 & + 2 \int_{-\tau}^0 dt \int_{\sigma(R_1)} f\lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma}w(t-h)) \right) \cdot nd\sigma -
 \end{aligned}$$

$$\begin{aligned}
& -(2\tau)^{-2\gamma} \|\| f, w(-t-h) \|\|_{\omega_{R_1}(R)}^2 + \\
& + \gamma(2\tau)^{-2\gamma-1} \| fw(-t-h) \|_{\omega_{R_1}(R)}^2 + \\
& + 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla (f^2) dx + \\
& + (\tau)^{-2\gamma} \|\| f, w(-h) \|\|_{\omega_{R_1}(R)}^2 - \gamma(\tau)^{-2\gamma-1} \| fw(-h) \|_{\omega_{R_1}(R)}^2 + \\
& + 2 \int_{-\tau}^0 dt \int_{\sigma(R)} \lambda^{-\gamma} f p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot n d\sigma \\
& - 2 \int_{-\tau}^0 dt \int_{\sigma(R_1)} \lambda^{-\gamma} f^2 p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot n d\sigma - \\
& - 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot \nabla (f^2) dx
\end{aligned}$$

dove n è il vettore unitario normale alla superficie $\sigma(R)$ ($\sigma(R_1)$) diretto verso l'esterno; infatti

$$\begin{aligned}
& \| \lambda^{-\gamma} f \Delta w(t-h) + f \gamma \lambda^{-\gamma-1} w(t-h) \|_{\omega_{R_1}(R)}^2 + \lambda^{-2\gamma} \| f \nabla p(t-h) \|_{\omega_{R_1}(R)}^2 - \\
& - 2(f \lambda^{-\gamma} \Delta w(t-h) + f \gamma \lambda^{-\gamma-1} w(t-h), f \lambda^{-\gamma} \nabla p(t-h))_{\omega_{R_1}(R)} \geq 0,
\end{aligned}$$

$$\begin{aligned}
& 2 \int_{-\tau}^0 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)), \lambda^{-\gamma} f \nabla p(t-h) \right)_{\omega_{R_1}(R)} dt = \\
& = 2 \int_{-\tau}^0 dt \int_{\sigma(R)} f \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot n d\sigma - \\
& - 2 \int_{-\tau}^0 dt \int_{\sigma(R_1)} f^2 \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot n d\sigma - \\
& - 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot \nabla (f^2) dx,
\end{aligned}$$

$$\begin{aligned}
 & - \int_{-\tau}^0 2\gamma dt \int_{\omega_{R_1}(R)} f \lambda^{-\gamma-1} w(t-h) \cdot \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) dx = \\
 & = \int_{-\tau}^0 \gamma \lambda^{-2\gamma-2} \|fw(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\
 & + \gamma (2\tau)^{-2\gamma-1} \|fw(-\tau-h)\|_{\omega_{R_1}(R)}^2 - \gamma (\tau)^{-2\gamma-1} \|fw(-h)\|_{\omega_{R_1}(R)}^2, \\
 & -2 \int_{-\tau}^0 \left(f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)), f \lambda^{-\gamma} \Delta w(t-h) \right)_{\omega_{R_1}(R)} dt = \\
 & = -2 \int_{-\tau}^0 dt \int_{\sigma(R)} \lambda^{-\gamma} f (\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot nd\sigma + \\
 & + 2 \int_{-\tau}^0 dt \int_{\sigma(R_1)} \lambda^{-\gamma} f \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot nd\sigma - \\
 & - (2\tau)^{-2\gamma} \| \| f, w(-\tau-h) \| \|_{\omega_{R_1}(R)}^2 + \\
 & + 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla (f) dx + \\
 & + (\tau)^{-2\gamma} \| \| f, w(-h) \| \|_{\omega_{R_1}(R)}^2.
 \end{aligned}$$

D'altra parte

$$\begin{aligned}
 & 2 \int_{-\tau}^0 dt \int_{\sigma(R)} \lambda^{-\gamma} f \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot nd\sigma \leq \\
 & \leq \int_{-\tau}^0 \left\| \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)} \right\|^2 dt + \\
 & + \int_{-\tau}^0 \lambda^{-2\gamma} \| \| f, w(t-h) \| \|_{\sigma(R)}^2 dt,
 \end{aligned}$$

$$2 \left| \int_{-\tau}^0 dt \int_{\sigma(R)} f \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot n d\sigma \right| \leq \\ \leq 2 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|fp(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt \right)^{1/2}$$

e quindi dalla (2) segue che

$$(3) \quad \int_{-\tau}^0 \left[\left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 + \gamma \lambda^{-2\gamma-2} \|fw(t-h)\|_{\omega_{R_1}(R)}^2 \right] dt \leq \\ \leq \int_{-\tau}^0 \lambda^{-2\gamma} \|fLw(t-h) - f\nabla p(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\ + \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \int_{-\tau}^0 \lambda^{-2\gamma} \| \|f, w(t-h)\| \|_{\sigma(R)}^2 dt + \\ + \int_{-\tau}^0 \left[\left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 + \lambda^{-2\gamma} (\| \|f, w(t-h)\| \|_{\sigma(R_1)}^2) \right] dt + \\ + (2\tau)^{-2\gamma} \| \|f, w(-\tau-h)\| \|_{\omega_{R_1}(R)}^2 - \gamma (2\tau)^{-2\gamma-1} \|fw(-\tau-h)\|_{\omega_{R_1}(R)}^2 + \\ + \gamma (\tau)^{-2\gamma-1} \|fw(-h)\|_{\omega_{R_1}(R)}^2 - \\ - 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla (f) dx + \\ + 2 \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot \nabla (f) dx + \\ + 2 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|fp(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt \right)^{1/2} + \\ + 2 \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|fp(t-h)\|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 dt \right)^{1/2}.$$

\dot{E}

$$\begin{aligned}
 (4) \quad & - \int_{-\tau}^0 (\lambda^{-\gamma-1} f w(t-h), \lambda^{-\gamma+1} f [Lw(t-h) - \nabla p(t-h)])_{\omega_{R_1}(R)} dt = \\
 & = \int_{-\tau}^0 \lambda^{-2\gamma} (\|f, w(t-h)\|_{\omega_{R_1}(R)}^2 - \gamma \lambda^{-1} \|fw(t-h)\|_{\omega_{R_1}(R)}^2) dt - \\
 & - \int_{-\tau}^0 dt \int_{\sigma(R)} \lambda^{-2\gamma} f ((\nabla w(t-h))w(t-h)) \cdot n d\sigma - \\
 & - \frac{1}{2} (2\tau)^{-2\gamma} \|fw(-\tau-h)\|_{\omega_{R_1}(R)}^2 + \int_{-\tau}^0 dt \int_{\sigma(R_1)} \lambda^{-2\gamma} f ((\nabla w(t-h))w(t-h)) \cdot n d\sigma + \\
 & + \frac{1}{2} (\tau)^{-2\gamma} \|fw(-h)\|_{\omega_{R_1}(R)}^2 + \\
 & + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} ((\nabla w(t-h))w(t-h)) \cdot \nabla (f^2) dx - \\
 & - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} p(t-h)w(t-h) \cdot \nabla (f^2) dx + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\sigma(R)} f^2 p(t-h)w(t-h) \cdot n d\sigma - \\
 & - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\sigma(R_1)} f^2 p(t-h)w(t-h) \cdot n d\sigma;
 \end{aligned}$$

infatti

$$\begin{aligned}
 & - \int_{-\tau}^0 \left(f \lambda^{-\gamma-1} w(t-h), -f \lambda^{-\gamma+1} \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1}(R)} dt = \\
 & = - \frac{1}{2} (2\tau)^{-2\gamma} \|fw(-\tau-h)\|_{\omega_{R_1}(R)}^2 + \\
 & + \frac{1}{2} (\tau)^{-2\gamma} \|fw(-h)\|_{\omega_{R_1}(R)}^2 - \gamma \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} f \lambda^{-2\gamma-1} w^2(t-h) dx,
 \end{aligned}$$

$$\begin{aligned}
& - \int_{-\tau}^0 (\lambda^{-\gamma-1} f w(t-h), \lambda^{-\gamma+1} f \Delta w(t-h))_{\omega_{R_1}(R)} dt = \\
& = - \int_{-\tau}^0 dt \int_{\sigma(R)} \lambda^{-2\gamma} f^2 ((\nabla w(t-h)) w(t-h)) \cdot n d\sigma + \\
& + \int_{-\tau}^0 dt \int_{\sigma(R_1)} \lambda^{-2\gamma} f^2 ((\nabla w(t-h)) w(t-h)) \cdot n d\sigma + \\
& + \int_{-\tau}^0 \lambda^{-2\gamma} \| \| f, w(t-h) \| \|_{\omega_{R_1}(R)}^2 dt + \\
& + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} ((\nabla w(t-h)) w(t-h)) \cdot \nabla (f^2) dx, \\
& \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} (\lambda^{-\gamma-1} f w(t-h), f \lambda^{-\gamma+1} f \nabla p(t-h))_{\omega_{R_1}(R)} dt = \\
& = - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} p(t-h) w(t-h) \cdot \nabla (f^2) dx + \\
& + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\sigma(R)} f^2 p(t-h) w(t-h) \cdot n d\sigma - \\
& - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\sigma(R_1)} f^2 p(t-h) w(t-h) \cdot n d\sigma.
\end{aligned}$$

Dalla (4) segue che

$$\begin{aligned}
(5) \quad & \int_{-\tau}^0 \lambda^{-2\gamma} \| \| f, w(t-h) \| \|_{\omega_{R_1}(R)}^2 dt \leq \\
& \leq 2\tau^2 \int_{-\tau}^0 \lambda^{-2\gamma} \| \| f L w(t-h) - f \nabla p(t-h) \| \|_{\omega_{R_1}(R)}^2 dt +
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} [\| f w(t-h) \|_{\sigma(R)}^2 + \| \| f, w(t-h) \| \|_{\sigma(R)}^2] dt + \\
 & + \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} [\| f w(t-h) \|_{\sigma(R_1)}^2 + \| \| f, w(t-h) \| \|_{\sigma(R_1)}^2] dt + \\
 & + \left(2\gamma\tau + \frac{1}{2} \right) \int_{-\tau}^0 \lambda^{-2-2\gamma} \| f w(t-h) \|_{\omega_{R_1}(R)}^2 dt + \\
 & + \frac{1}{2} (2\tau)^{-2\gamma} \| f w(-\tau-h) \|_{\omega_{R_1}(R)}^2 - \\
 & - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} ((\nabla w(-\tau-h)) w(-\tau-h)) \cdot \nabla (f^2) dx + \\
 & + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} p(t-h) w(t-h) \cdot \nabla (f^2) dx + \\
 & + \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f w(t-h) \|_{\sigma(R)}^2 dt \right)^{1/2} + \\
 & + \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f w(t-h) \|_{\sigma(R_1)}^2 dt \right)^{1/2},
 \end{aligned}$$

perchè

$$\begin{aligned}
 & \left| \int_{-\tau}^0 (\lambda^{-\gamma-1} f w(t-h), f \lambda^{-\gamma+1} [Lw(t-h) - \nabla p(t-h)])_{\omega_{R_1}(R)} dt \right| \leq \\
 & \leq \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma-2} \| f w(t-h) \|_{\omega_{R_1}(R)}^2 dt + \\
 & + 2\tau^2 \int_{-\tau}^0 \lambda^{-2\gamma} \| f Lw(t-h) - f \nabla p(t-h) \|_{\omega_{R_1}(R)}^2 dt,
 \end{aligned}$$

$$\gamma \int_{-\tau}^0 \lambda^{-2\gamma-1} \|f w(t-h)\|_{\omega_{R_1}(R)}^2 dt \leq 2\tau\gamma \int_{-\tau}^0 \lambda^{-2\gamma-2} \|f w(t-h)\|_{\omega_{R_1}(R)}^2 dt.$$

Sommando alla (5) la (3) moltiplicata per $2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2$ si ottiene

$$\begin{aligned} & (\delta\gamma + 4\tau^2\gamma) \int_{-\tau}^0 \lambda^{-2\gamma-2} \|f w(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\ & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2\right) \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma}(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt + \\ & + \int_{-\tau}^0 \lambda^{-2\gamma} \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2 dt \leq \\ & \leq \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 6\tau^2\right) \int_{-\tau}^0 \lambda^{-2\gamma} \|f L w(t-h) - f \nabla p(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\ & + \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} \|f w(t-h)\|_{\sigma(R)}^2 dt + \\ & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 + \frac{1}{2}\right) \int_{-\tau}^0 \lambda^{-2\gamma} \| \|f, w(t-h)\| \|_{\sigma(R)}^2 dt + \\ & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2\right) \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \\ & + \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} \|f w(t-h)\|_{\sigma(R_1)}^2 dt + \\ & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 + \frac{1}{2}\right) \int_{-\tau}^0 \lambda^{-2} \| \|f, w(t-h)\| \|_{\sigma(R_1)}^2 dt + \\ & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2\right) \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 dt + \end{aligned}$$

$$\begin{aligned}
 & + \left[\frac{1}{2}(2\tau)^{-2\gamma} - \gamma(2\tau)^{-2\gamma-1} \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \right] \| f w(-\tau-h) \|_{\omega_{R_1}(R)}^2 + \\
 & + \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) (2\tau)^{-2\gamma} \| \| f, w(-\tau-h) \| \|_{\omega_{R_1}(R)}^2 - \\
 & - \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} ((\nabla w(t-h))w(t-h)) \cdot \nabla (f^2) dx - \\
 & - 2 \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla (f^2) dx + \\
 & + \gamma(2\tau)^{-2\gamma-1} \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \| f w(-h) \|_{\omega_{R_1}(R)}^2 + \\
 & + \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} p(t-h) w(t-h) \cdot \nabla (f^2) dx + \\
 & + \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f w(t-h) \|_{\sigma(R)}^2 dt \right)^{1/2} + \\
 & + \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f w(t-h) \|_{\sigma(R_1)}^2 dt \right)^{1/2} + \\
 & + 2 \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} p(t-h) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \cdot \nabla (f^2) dx + \\
 & + 2 \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R)}^2 dt \right)^{1/2} \cdot \\
 & \cdot \left(\int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt \right)^{1/2} + 2 \left(2\tau + \frac{1}{2}\gamma^{-1} + \delta + 4\tau^2 \right) \cdot \\
 & \cdot \left(\int_{-\tau}^0 \lambda^{-2\gamma} \| f p(t-h) \|_{\sigma(R_1)}^2 dt \right)^{1/2} \left(\int_{-\tau}^0 \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 dt \right)^{1/2},
 \end{aligned}$$

da cui, tenendo presente che

$$\int_{-\tau}^0 \lambda^{-2\gamma} \|fw(t-h)\|_{\omega_{R_1}(R)}^2 dt \leq 4\tau^2 \int_{-\tau}^0 \lambda^{-2\gamma-2} \|fw(t-h)\|_{\omega_{R_1}(\bar{R})}^2 dt,$$

si ottiene il lemma.

OSSERVAZIONE. Nel caso in cui $w=0$ su $\partial\Omega \times [-T, 0]$, $f=1$ in Q e $|p(x, t)| \leq C/(1+|x|)^{1/2} \forall (x, t) \in Q$, se si integra su $\omega(R)$ anzichè su $\omega_{R_1}(R)$ si ottiene

$$\begin{aligned} & \int_{-\tau}^0 \lambda^{-2\gamma} \|w(t-h)\|_{\omega(R)}^2 dt + \int_{-\tau}^0 \lambda^{-2\gamma} \|w(t-h)\|_{\omega(R)}^2 dt + \\ & + C_{\tau, \gamma, \delta} \int_{-\tau}^0 \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega(R)}^2 dt \leq \\ & \leq 2C_{\tau, \gamma, \delta} \int_{-\tau}^0 \lambda^{-2\gamma} \|Lw(t-h) - \nabla p(t-h)\|_{\omega(R)}^2 dt + \\ & + 4 \int_{-\tau}^0 \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \\ & + 4 \int_{-\tau}^0 \lambda^{-2\gamma} (\|w(t-h)\|_{\sigma(R)}^2 + \|w(t-h)\|_{\sigma(R)}^2) dt + \\ & + C(\tau, \gamma) R^{1/2} \left(\int_{-\tau}^0 \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt \right)^{1/2} + \\ & + C(\tau, \gamma) R^{1/2} \left(\int_{-\tau}^0 \lambda^{-2\gamma} \|w(t-h)\|_{\sigma(R)}^2 dt \right)^{1/2} + \\ & + C_{\tau, \gamma, \delta} (2\tau)^{-2\gamma} \|w(-\tau-h)\|_{\omega(R)}^2 + \\ & + (4\gamma)(\tau)^{-2\gamma-1} \|w(-h)\|_{\omega(R)}^2 - \tau\gamma(2\tau)^{-2\gamma-1} \|w(-\tau-h)\|_{\omega(R)}^2 \end{aligned}$$

$$\forall \tau, R, R_1, \gamma, \delta \in \mathbf{R}^+, \forall h \geq 0 \text{ con } R \geq R_1 \geq R_0, \gamma \geq 1, \tau \leq \frac{1}{3},$$

$$0 \leq h + \tau \leq T, 0 < \delta \leq \frac{1}{4},$$

essendo $C(\tau, \gamma)$ una costante dipendente da τ e da γ .

LEMMA 3. Sia $a \in \mathbf{R}^+$ e sia $\Phi : [a, +\infty[\rightarrow \mathbf{R}^+$ di classe $C^{(1)}$ tale che $\Phi'(R) \geq 0 \quad \forall R \geq a$. Se esistono un $\gamma \in \mathbf{R}^+$ e una successione $(\beta_n)_{n \in \mathbf{N}}$ divergente positivamente tali che

$$\Phi(\beta_n) \leq \beta_n^\gamma \quad \forall n \in \mathbf{N},$$

allora, fissato $\delta, 0 < \delta \leq 1$, esiste di conseguenza una successione $(\alpha_n)_{n \in \mathbf{N}}$ divergente positivamente tale che

$$\Phi'(\alpha_n) \leq \alpha_n^{\gamma+\delta-1} \quad \forall n \in \mathbf{N}.$$

DIMOSTRAZIONE. Supponiamo per assurdo che non esista una tale successione; allora esiste $R_1 \geq a$ tale che

$$\Phi'(R) \geq R^{\gamma+\delta-1} \quad \forall R \geq R_1,$$

da cui, integrando sull'intervallo $[R_1, R]$ ($R > R_1$) si ottiene

$$\frac{R^{\gamma+\delta} - R_1^{\gamma+\delta}}{\gamma + \delta} \leq \Phi(R) - \Phi(R_1) \quad \forall R \geq R_1$$

e quindi

$$\beta_n^{\gamma+\delta} \frac{1}{\gamma + \delta} \leq R_1^{\gamma+\delta} \frac{1}{\gamma + \delta} + \Phi(\beta_n) \leq R_1^{\gamma+\delta} \frac{1}{\gamma + \delta} + \beta_n^\gamma$$

$\forall \beta_n \geq R_1$, che è un assurdo.

LEMMA 4. Sia u, p soluzione di (α) - (β) in Q con

- i) $u_i \in L_3, F_i \in L_1$ per $1 \leq i \leq 3, p \in L_2$
- ii) $u_i, \frac{\partial u_i}{\partial x_j}$ limitate in Q per $1 \leq i, j \leq 3$
- iii) u_1, u_2, u_3 nulle su $\Omega \times \{0\}$.

Allora esistono $h_0, h_1, h_2, \dots, h_s$ con $0 = h_0 < h_1 < h_2 < \dots < h_s = T$ tali che

$$\int_{h_j}^{h_{j+1}} d\tau \int_{-\tau}^{-h_j} dt \int_{\omega_{R_0}} w^2 dx < +\infty \quad \text{per } 0 \leq j \leq s.$$

DIMOSTRAZIONE. Per provare il lemma, procediamo per tappe.

I. Posto

$$(6) \quad w = \nabla \times u$$

$$(7) \quad \Phi(R) = \int_{-T}^0 dt \int_{R_0}^R ||| W |||_{\omega_{R_0}(r)}^2 dr$$

$$(8) \quad \Psi(R) = \int_{-T}^0 dt \int_t^0 d\xi \int_{R_0}^R d\rho \int_{R_0}^{\rho} \left\| \frac{\partial w}{\partial t} \right\|_{\omega_{R_0}(r)}^2 dr$$

proviamo che esiste $\bar{R}_0 > 0$ tale che

$$(9) \quad \int_{R_0}^R \Phi(\rho) d\rho \leq R^{16} \quad \forall R \geq \bar{R}_0$$

$$(10) \quad \Psi(R) \leq R^{51} \quad \forall R \geq \bar{R}_0.$$

Dalla (α) e dalla (6), ricordando che per ipotesi $\nabla \times F = 0$, si ottiene

$$(11) \quad \frac{\partial w}{\partial t} - \Delta w + (u \cdot \nabla)w - (w \cdot \nabla)u = 0.$$

Moltiplicando scalarmente la (11) per w si ottiene

$$(12) \quad \frac{1}{2} \frac{\partial(w^2)}{\partial t} - w \cdot \Delta w + ((u \cdot \nabla)w) \cdot w - ((w \cdot \nabla)u) \cdot w = 0$$

da cui, tenendo presente la (s) e che

$$((u \cdot \nabla)w) \cdot w = \nabla \cdot \left(\frac{1}{2} w^2 u \right)$$

$$|((w \cdot \nabla)u) \cdot w| \leq C_1 w^2$$

si ottiene

$$(13) \quad \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 \leq \frac{1}{2} \nabla \cdot (\nabla(w^2) - w^2 u) + C_1 w^2 - \frac{1}{2} \frac{\partial(w^2)}{\partial t}.$$

Dalla (13) eseguendo l'integrazione

$$\int_{-T}^0 dt \int_{R_0}^R dr \int_{\omega_{R_0}(r)} (...) dx$$

e tenendo presente la (ii) segue

$$\int_{-T}^0 dt \int_{R_0}^R ||| w |||_{\omega_{R_0}(r)}^2 dr \leq C_2 R^4 + C_2 R^5 \left(\int_{-T}^0 ||| w |||_{\omega_{R_0}(R)}^2 dt \right)^{1/2} \quad \forall R \geq R_0$$

e quindi tenendo presente la (7) si ottiene

$$\Phi(R) \leq C_2 R^4 + C_2 R^5 (\Phi'(R))^{1/2} \quad \forall R \geq R_0,$$

da cui, per il lemma 1, segue che

$$(14) \quad \Phi(R_1) \leq R^{15} \quad \forall R \geq R_1$$

per un certo $R_1 > 0$ da ciò segue la (9). Moltiplicando ora la (11) scalarmente per $\frac{\partial w}{\partial t}$ otteniamo

$$\left(\frac{\partial w}{\partial t} \right)^2 - \frac{\partial w}{\partial t} \cdot \Delta w + ((u \cdot \nabla)w) \cdot \frac{\partial w}{\partial t} - ((w \cdot \nabla)u) \cdot \frac{\partial w}{\partial t} = 0,$$

da cui, tenendo presente la (u) e che

$$\begin{aligned} \left| ((w \cdot \nabla)u) \cdot \frac{\partial w}{\partial t} \right| &\leq C_3 + \frac{1}{8} \left(\frac{\partial w}{\partial t} \right)^2 \\ \left| ((u \cdot \nabla)w) \cdot \frac{\partial w}{\partial t} \right| &\leq C_3 \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 + \frac{1}{8} \left(\frac{\partial w}{\partial t} \right)^2 \end{aligned}$$

si ottiene

$$\frac{1}{4} \left(\frac{\partial w}{\partial t} \right)^2 \leq \nabla \cdot \left((\nabla w) \frac{\partial w}{\partial t} \right) - \frac{1}{2} \frac{\partial}{\partial t} \left(\sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 \right) + C_3 \left(\sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 \right) + C_3,$$

da cui, ricordando (14) e (7),

$$\begin{aligned} \int_{R_0}^R d\rho \int_{-T}^0 dt \int_t^0 d\xi \int_{R_0}^{\rho} \left\| \frac{\partial w}{\partial t} \right\|_{\omega_{R_0}(\rho)}^2 dr &\leq \int_{R_0}^R d\rho \int_{-T}^0 dt \int_t^0 d\xi \int_{\omega_{R_0}(\rho)} (\nabla w) \frac{\partial w}{\partial t} dx + \\ &+ C_4 R^{16} \leq C_5 R^8 \left(\int_{-T}^0 dt \int_t^0 d\xi \int_{R_0}^R \left\| \frac{\partial w}{\partial t} \right\|_{\omega_{R_0}(\rho)}^2 dr \right)^{1/2} + C_5 R^{16}; \end{aligned}$$

quindi, ricordando la (8), si ottiene

$$\psi(R) \leq C_5 R^{16} + C_5 R^{17} (\psi'(R))^{1/2} \quad \forall R \geq R_1;$$

di qui, per il lemma 1, segue la (10).

II. Poniamo ora

$$\begin{aligned} (15) \quad \wedge_{R_1}(R, m, h) &= \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} (\|fw(t-h)\|_{\omega_{R_1}(R)}^2 + \\ &+ \tau^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2) dt + \\ &+ \delta \int_0^{\tau_0} \tau^2 (2\tau)^{R_1^{\alpha_m}} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} ((t-\tau)^{-R_1^{\alpha_m/2}} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt + \\ &+ \int_{-\tau_0}^0 t^2 \|fw(t-h)\|_{\omega_{R_1}(R)}^2 dt + \frac{1}{2} \int_{-\tau_0}^0 t^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2 dt \end{aligned}$$

dove $f(x, t) = \exp(-a(1+x_1^2+x_2^2+x_3^2)^{1/20})$ e $\alpha_m = 1 / \left(9 - \frac{1}{m}\right)$.

Proviamo che esistono $\delta, \tau_0, a, R_2, C_{10} \in \mathbf{R}^+$ tali che

$$(16) \quad \Lambda_{R_1}(R, m, h) \leq C_{10} \Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \|fw(-h)\|_{\omega_{R_1}}^2 \\ \forall R, R_1, h, m \text{ con } R \geq R_1 \geq R_2, 0 \leq h + \tau_0 \leq T, m \in N$$

dove $\Lambda'(R_1, m, h) = \left(\frac{d\Lambda_{R_1}(R, m, h)}{dR} \right)_{R=R_1}$.

Cominciamo con l'osservare che

$$\int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} (fLw(t-h))^2 dx = \\ = \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} [f(w(t-h) \cdot \nabla)u(t-h) - f(u(t-h) \cdot \nabla)w(t-h)]^2 dx \leq \\ \leq C_6 \int_{-\tau}^0 \lambda^{-2\gamma} (\|fw(t-h)\|_{\omega_{R_1}(R)}^2 + \|f, w(t-h)\|_{\omega_{R_1}(R)}^2) dt,$$

$$\left| \frac{1}{2} \int_{-\tau}^0 \lambda^{-2\gamma} dt \int_{\omega_{R_1}(R)} \nabla(w^2(t-h)) \cdot \nabla(f^2) dx \right| \leq \\ \leq aC_7 \int_{-\tau}^0 \lambda^{-2\gamma} (\|fw(t-h)\|_{\omega_{R_1}(R)}^2 + \|f, w(t-h)\|_{\omega_{R_1}(R)}^2) dt,$$

$$\left| \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla(f^2) dx \right| \leq \\ \leq aC_8 \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt + \\ + aC_8 \int_{-\tau}^0 \lambda^{-2\gamma} \|f, w(t-h)\|_{\omega_{R_1}(R)}^2 dt.$$

Allora, detta K la più grande delle costanti C_6, C_7, C_8 e scelti $a, \delta, \tau_0, \gamma_0 \in \mathbf{R}^+$ soddisfacenti le condizioni

$$aK < \frac{1}{4}, \quad \gamma_0 > 2 \left(a + 2 \left(2\tau_0 + \frac{1}{2\gamma_0} + \delta + 4\tau_0^2 \right) \right) K,$$

$$1 > 2K \left(a + 2(a+1) \left(2\tau_0 + \frac{1}{2\gamma_0} + \delta + 4\tau_0^2 \right) \right),$$

oltre alle condizioni già imposte $\gamma_0 \geq 1, 0 < \tau_0 \leq \frac{1}{3}, 0 \leq h + \tau_0 \leq T, 0 < \delta \leq \frac{1}{4}$, per il lemma 2 se si pone $p=0$ si ha

$$\begin{aligned} & \left(\gamma + \frac{\delta\gamma}{4\tau^2} \right) \int_{-\tau}^0 \lambda^{-2\gamma} \|f\omega(t-h)\|_{\omega_{R_1}(R)}^2 dt + \int_{-\tau}^0 \lambda^{-2\gamma} \| \|f, \omega(t-h)\|_{\omega_{R_1}(R)} \|^2 dt + \\ & + \delta \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \omega(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt \leq \\ & \leq 8 \int_{-\tau}^0 \lambda^{-2\gamma} (\|f\omega(t-h)\|_{\sigma(R)}^2 + \| \|f, \omega(t-h)\|_{\sigma(R)} \|^2) dt + \\ & + 8 \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \omega(t-h)) \right\|_{\sigma(R)}^2 dt + \\ & + 8 \int_{-\tau}^0 \lambda^{-2\gamma} (\|f\omega(t-h)\|_{\sigma(R_1)}^2 + \| \|f, \omega(t-h)\|_{\sigma(R_1)} \|^2) dt + \\ & + 8 \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} \omega(t-h)) \right\|_{\sigma(R_1)}^2 dt + \\ & + 2C_{\tau, \gamma, \delta} (2\tau)^{-2\gamma} \| \|f, \omega(-\tau-h)\|_{\omega_{R_1}(R)} \|^2 - \\ & - 2\tau\gamma(2\tau)^{-2\gamma-1} \|f\omega(-\tau-h)\|_{\omega_{R_1}(R)}^2 + 4\gamma(\tau)^{-2\gamma-1} \|f\omega(-h)\|_{\omega_{R_1}(R)}^2 \end{aligned}$$

$\forall \tau, \gamma, R, R_1, h$ con $0 < \tau \leq \tau_0, \gamma \geq \gamma_0, R \geq R_1 \geq R_0, 0 \leq h + \tau_0 \leq T,$

da cui, moltiplicando ambo i membri per $(2\tau)^{2\gamma+2}$ e integrando su $[0, \tau_0]$ si ottiene per $\gamma\delta \geq 1$

$$\begin{aligned}
 (17) \quad & \int_0^{\tau_0} (2\tau)^{2\gamma} d\tau \int_{-\tau}^0 \lambda^{-2\gamma} (\|fw(t-h)\|_{\omega_{R_1}(R)}^2 + \tau^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2) dt + \\
 & + \int_{-\tau_0}^0 t^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2 dt + \\
 & + \delta \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt + \\
 & + 4\gamma \int_{-\tau_0}^0 t^2 \| fw(t-h) \|_{\omega_{R_1}(R)}^2 dt \leq \\
 & \leq 8 \int_0^{\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^0 \lambda^{-2\gamma} (\|fw(t-h)\|_{\sigma(R_1)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R_1)}^2) dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^0 \lambda^{-2\gamma} (\|fw(t-h)\|_{\sigma(R)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R)}^2) dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \\
 & + \gamma 2^{2\gamma} \| fw(-h) \|_{\omega_{R_1}(R)}^2 + 5 \int_{-\tau_0}^0 t^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2 dt.
 \end{aligned}$$

Ora dalla (12) si ottiene

$$f^2 t^2 \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 (t-h) \leq -\frac{1}{2} t^2 \rho \frac{\partial}{\partial t} (w^2(t-h)) +$$

$$\begin{aligned}
& + \frac{1}{2} \nabla \cdot (t^2 f^2 (\nabla (w^2(t-h)) - w^2(t-h)u(t-h))) + \\
& + C_1 t^2 f^2 w^2(t-h) + \frac{1}{2} | (\nabla (w^2(t-h)) - w^2(t-h)u(t-h)) \cdot \nabla (f^2) | t^2
\end{aligned}$$

da cui

$$\begin{aligned}
(18) \quad & \int_{-\tau}^0 dt \int_{\omega_{R_1}(R)} t^2 f^2 \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 (t-h) dx \leq \\
& \leq \frac{1}{2} \int_{-\tau_0}^0 t^2 dt \int_{\sigma(R_1)} f^2 | \nabla (w^2(t-h)) - w^2(t-h)u(t-h) | d\sigma + \\
& + \frac{1}{2} \int_{-\tau_0}^0 t^2 dt \int_{\sigma(R)} f^2 | \nabla (w^2(t-h)) - w^2(t-h)u(t-h) | d\sigma + \\
& + \frac{1}{2} \int_{-\tau_0}^0 t^2 dt \int_{\omega_{R_1}(R)} | \nabla (w^2(t-h)) - w^2(t-h)u(t-h) | | \nabla (f^2) | dx + \\
& + C_1 \int_{-\tau_0}^0 t^2 dt \int_{\omega_{R_1}(R)} f^2 w^2(t-h) dx + \frac{1}{2} \tau_0^2 \int_{\omega_{R_1}(R)} f^2 w^2(-\tau_0-h) dx + \\
& + \int_{-\tau_0}^0 t \| f w(t-h) \|_{\omega_{R_1}(R)}^2 dt \leq C_9 \int_{-\tau_0}^0 t^2 (\| f w(t-h) \|_{\sigma(R_1)}^2 + \\
& + \| \| f, w(t-h) \| \|_{\sigma(R_1)}^2) dt + C_9 \int_{-\tau_0}^0 t^2 (\| f w(t-h) \|_{\sigma(R)}^2 + \\
& + \| \| f, w(t-h) \| \|_{\sigma(R)}^2) dt + (C_1 + aC_9) \int_{-\tau_0}^0 t^2 \| f w(t-h) \|_{\omega_{R_1}(R)}^2 dt + \\
& + aC_9 \int_{-\tau_0}^0 t^2 \| \| f, w(t-h) \| \|_{\omega_{R_1}(R)}^2 dt + \\
& + \frac{1}{2} \tau_0^2 \int_{\omega_{R_1}(R)} f^2 w^2(-\tau_0-h) dx + \int_{-\tau_0}^0 t \| f w(t-h) \|_{\omega_{R_1}(R)}^2 dt.
\end{aligned}$$

Maggiorando mediante la (18) l'ultimo addendo nel secondo membro della (17), si ottiene

$$\begin{aligned}
 (19) \quad & \int_0^{\tau_0} (2\tau)^{2\gamma} d\tau \int_{-\tau}^0 \lambda^{-2\gamma} (\|f w(t-h)\|_{\omega_{R_1}(R)}^2 + \tau^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2) dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2\gamma} \tau^2 d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\omega_{R_1}(R)}^2 dt + \int_{-\tau_0}^0 t^2 \|f w(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\
 & + \frac{1}{2} \int_{-\tau_0}^0 t^2 \| \|f, w(t-h)\| \|_{\omega_{R_1}(R)}^2 dt \leq \\
 & \leq 8 \int_0^{\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^0 \lambda^{-\gamma} (\|f w(t-h)\|_{\sigma(R_1)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R_1)}^2) dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R_1)}^2 dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^0 \lambda^{-2\gamma} (\|f w(t-h)\|_{\sigma(R)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R)}^2) dt + \\
 & + 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^0 \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|_{\sigma(R)}^2 dt + \\
 & + 5C_9 \int_{-\tau_0}^0 t^2 (\|f w(t-h)\|_{\sigma(R_1)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R_1)}^2) dt + \\
 & + 5C_9 \int_{-\tau_0}^0 t^2 (\|f w(t-h)\|_{\sigma(R)}^2 + \| \|f, w(t-h)\| \|_{\sigma(R)}^2) dt + \\
 & + \frac{5}{2} \tau_0^2 \int_{\omega_{R_1}(R)} f w^2(-\tau_0-h) dx + 5 \int_{-\tau_0}^0 t \|f w(t-h)\|_{\omega_{R_1}(R)}^2 dt + \\
 & + \gamma 2^{2\gamma} \|f w(-h)\|_{\omega_{R_1}(R)}^2 \\
 & \forall \gamma, h \text{ con } \gamma \geq \gamma_1 \geq \gamma_0, \quad 4\gamma_1 \geq 1 + 5(C_1 + aC_9), \quad 5aC_9 \leq \frac{1}{2} \text{ e } 0 \leq h + \tau_0 \leq T.
 \end{aligned}$$

Dalla (19) posto $2\gamma = R_1^{\alpha} m$ e tenendo presente la (ii) si ottiene

$$(20) \quad \Lambda_{R_1}(R, m, h) \leq C_{10} \Lambda'(R_1, m, h) + C_{10} \Lambda'(R, m, h) + \\ + C_{10} + R_1^{\alpha} m 2^{R_1^{\alpha} m} \|f\omega(-h)\|_{\omega_{R_1}}^2$$

$\forall R, R_1, h, m$ con $m \in \mathbb{N}$, $h + \tau_0 \leq T$, $R \geq R_1 \geq R_2$, per un certo $R_2 > 0$.

Fissato $R_1 \geq R_2$ supponiamo che esista $R_3 \geq R_1$ tale che sia

$$\Lambda_{R_1}(R_3, m, h) > C_{10} \Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha} m 2^{R_1^{\alpha} m} \|f\omega(-h)\|_{\omega_{R_1}}^2;$$

allora essendo

$$\Lambda_{R_1}(R_3, m, h) \leq \Lambda_{R_1}(R, m, h) \text{ per } R \geq R_3 \text{ perchè } \Lambda'(R, m, h) \geq 0,$$

si ha

$$0 < \beta = 1 - \frac{C_{10} \Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha} m 2^{R_1^{\alpha} m} \|f\omega(-h)\|_{\omega_{R_1}}^2}{\Lambda_{R_1}(R_3, m, h)} \leq \\ \leq 1 - \frac{C_{10} \Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha} m 2^{R_1^{\alpha} m} \|f\omega(-h)\|_{\omega_{R_1}}^2}{\Lambda_{R_1}(R, m, h)} \leq \\ \leq C_{10} \frac{\Lambda'(R, m, h)}{\Lambda_{R_1}(R, m, h)} \quad \forall R \geq R_3,$$

da cui, integrando sull'intervallo $[R_3, R]$ ($R > R_3$)

$$(21) \quad \beta(R - R_3) \leq C_{10} \log \left(\frac{\Lambda_{R_1}(R_3, m, h)}{\Lambda_{R_1}(R, m, h)} \right) \quad \forall R \geq R_3.$$

Inoltre dalla (15), per le (9), (10) e la (ii), segue che esiste R_4 tale che

$$\int_{R_1}^R d\rho \int_{R_1}^{\rho} \Lambda_{R_1}(r, m, h) dr \leq R^{52} \quad \forall R \geq R_4;$$

da ciò, applicando il lemma 3 due volte, discende l'esistenza di una successione $(\alpha_n)_{n \in N}$ divergente positivamente tale che

$$\Lambda_{R_1}(\alpha_n, m, h) \leq \alpha_n^{52} \quad \forall n \in N$$

quest'ultima contraddice la (21); da ciò segue l'affermazione (16).

III. Posto

$$(22) \quad \Gamma(R_1, m, h) = \lim_{R \rightarrow +\infty} \Lambda_{R_1}(R, m, h), \quad \Gamma'(R, m, h) = \frac{d\Gamma(R, m, h)}{dR}$$

proviamo che esiste R_5 tale che

$$(23) \quad \frac{1}{2} \Gamma(R_1, m, h) \leq -C_{10} \Gamma'(R_1, m, h) + C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \|f w(-h)\|_{\omega_{R_1}}^2, \\ \forall R_1, h, m \text{ con } R_1 \geq R_5, m \in N, h + \tau_0 \leq T.$$

Infatti, posto

$$A = \int_0^{\tau_0} d\tau \int_{-\tau}^0 \alpha_m R_1^{\alpha_m - 1} \log \left(\frac{2\tau}{\tau - t} \right) \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} (\|f w(t-h)\|_{\omega_{R_1}}^2 + \\ + \tau^2 \|f, w(t-h)\|_{\omega_{R_1}}^2) dt, \\ B = \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left[\frac{1}{2} \alpha_m R_1^{2\alpha_m - 1} \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{(\tau - t)^2} \|f w(t-h)\|_{\omega_{R_1}}^2 + \right. \\ \left. + \frac{\alpha_m}{4} R_1^{3\alpha_m - 1} \log \left(\frac{2\tau}{\tau - t} \right) \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{(\tau - t)^2} \|f w(t-h)\|_{\omega_{R_1}}^2 + \right. \\ \left. + \alpha_m R_1^{\alpha_m - 1} \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau - t} \left(f w(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right) \right]_{\omega_{R_1}} + \\ \left. + \alpha_m \tau^2 R_1^{\alpha_m - 1} \log \left(\frac{2\tau}{\tau - t} \right) \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} \left\| f \frac{\partial}{\partial t} (w(t-h)) \right\|_{\omega_{R_1}} + \right. \\ \left. + \alpha_m R_1^{2\alpha_m - 1} \log \left(\frac{2\tau}{\tau - t} \right) \left(\frac{2\tau}{\tau - t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau - t} \left(f w(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right) \right]_{\omega_{R_1}} dt,$$

tenendo presente che

$$(24) \quad \tau^2(2\tau)^{R_1^{\alpha_m}} \left[\frac{\partial}{\partial t} (\lambda^{-R_1^{\alpha_m/2}} w(t-h)) \right] = \frac{1}{4} R_1^{2\alpha_m} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{(\tau-t)^2} w^2(t-h) + \\ + \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \left(\frac{\partial}{\partial t} (w(t-h)) \right)^2 \tau^2 + \frac{1}{2} R_1^{\alpha_m} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau-t} \frac{\partial}{\partial t} (w^2(t-\tau)),$$

si ha

$$(25) \quad -\Gamma(R_1, m, h) = \Lambda'(R_1, m, h) - A - \delta B.$$

Dalla (24) si deduce

$$\tau^2 \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \left(\frac{\partial}{\partial t} (w(t-h)) \right)^2 \leq \\ \leq \frac{1}{2} R_1^{2\alpha_m} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau-t} (w(t-h))^2 + 2\tau^2(2\tau)^{R_1^{\alpha_m}} \left(\frac{\partial}{\partial t} (\lambda^{-R_1^{\alpha_m/2}} w(t-h)) \right)$$

e quindi

$$\alpha_m R_1^{\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau-t} \left(f w(t-h), \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1}}^2 \leq \\ \leq \alpha_m R_1^{\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 + \\ + \alpha_m R_1^{\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \tau^2 \left\| f \frac{\partial}{\partial t} (w(t-h)) \right\|_{\omega_{R_1}}^2 \leq \\ \leq \alpha_m R_1^{\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 + \\ + \frac{1}{2} \alpha_m R_1^{3\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 + \\ + 2\alpha_m R_1^{\alpha_m-1} (2\tau)^{R_1^{\alpha_m}} \tau^2 \left\| f \frac{\partial}{\partial t} (\lambda^{-R_1^{\alpha_m/2}} w(t-h)) \right\|_{\omega_{R_1}}^2,$$

$$\begin{aligned}
 R_1^{2\alpha_m-1} \alpha_m \log \left(\frac{2\tau}{\tau-t} \right) \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \frac{\tau^2}{\tau-t} \left(f w(t-h), f \frac{\partial}{\partial t} (w(t-h)) \right)_{\omega_{R_1}} &\leq \\
 &\leq 2\alpha_m R_1^{2\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 + \\
 &+ \alpha_m R_1^{4\alpha_m-1} \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 + \\
 &+ 4\alpha_m R_1^{2\alpha_m-1} (2\tau)^{R_1^{\alpha_m}} \tau^2 \left\| f \frac{\partial}{\partial t} (\lambda^{-R_1^{\alpha_m/2}} w(t-h)) \right\|_{\omega_{R_1}}^2 ;
 \end{aligned}$$

perciò si ha

$$\begin{aligned}
 (26) \quad \delta C_{10} B &\leq C_{11} R_1^{-\alpha_m} \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} \| f w(t-h) \|_{\omega_{R_1}}^2 dt + \\
 &+ C_{11} R_1^{-\alpha_m} \int_0^{\tau_0} d\tau \int_{-\tau}^0 \tau^2 (2\tau)^{R_1^{\alpha_m}} \left\| f \frac{\partial}{\partial t} (\lambda^{-R_1^{\alpha_m/2}} w(t-h)) \right\|_{\omega_{R_1}}^2 dt
 \end{aligned}$$

ed inoltre

$$\begin{aligned}
 (27) \quad A &\leq 2\alpha_m R_1^{-\alpha_m} \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left(\frac{2\tau}{\tau-t} \right)^{R_1^{\alpha_m}} (\| f w(t-h) \|_{\omega_{R_1}}^2 + \\
 &+ \tau^2 \| \| f, w(t-h) \| \|_{\omega_{R_1}}^2) dt.
 \end{aligned}$$

Dalla (25), per le (26) e (27), si deduce

$$-C_{10} \Gamma'(R_1, m, h) \geq C_{10} \Lambda'(R_1, m, h) - \frac{1}{2} \Gamma(R_1, m, h) \quad \forall R_1 \geq R_5$$

per un certo $R_5 > 0$, e combinando quest'ultima con la (16) si ottiene la (23).

IV. Dimostriamo ora il lemma. Dalla (23) preso $m=1$ e $h=0$ si ottiene

$$\frac{1}{2} \Gamma(R_1, 1, 0) \leq -C_{10} \Gamma'(R_1, 1, 0) + C_{10} \quad \forall R_1 \geq R_5$$

da cui tenendo presente che $\Gamma(R_1, m, h) \geq 0$ si ha

$$\Gamma'(R_1, 1, 0) \leq 1 \quad \forall R_1 \geq R_5$$

e quindi

$$(28) \quad \Gamma(R_1, 1, 0) \leq C_{12}R_1 \quad \forall R_1 \geq R_5.$$

Dalle (28), (22), (15) segue che

$$(29) \quad \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} f w^2(t) dx \leq C_{12}R_1 \left(\frac{4}{3}\right)^{-R_1^{\alpha_1}} \quad \forall R \geq R_1.$$

perchè

$$\begin{aligned} C_{12}R_1 &\geq \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f w^2(t) dx \geq \\ &\geq \int_0^{\tau_0} d\tau \int_{-\tau/2}^0 \left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f w^2(t) dx \geq \\ &\geq \int_0^{\tau_0} d\tau \int_{-\tau/2}^0 \left(\frac{4}{3}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f w^2(t) dx. \end{aligned}$$

Per la (29) esistono $R_6 \geq R_5$ e $\theta_1 > 0$ tali che

$$(30) \quad \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t) dx \leq e^{-\theta_1(R_6+K)^{\alpha_1}} \quad \forall K \in \mathbb{N}$$

infatti

$$\begin{aligned} &\int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6+K}(R_6+K+1)} f w^2(t) dx \geq \\ &\geq \exp(-2a(R_6+K+1)^{1/20}) \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t) dx. \end{aligned}$$

Proviamo ora che

$$(31) \quad \int_{R_6}^{+\infty} R_1^{\alpha_2} 2^{R_1^{\alpha_2}} dR_1 \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} w^2(t) dx < +\infty.$$

Infatti per la (30) si ha

$$\begin{aligned} & \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6}} |x|^{2+\alpha_2} 2^{|x|^{\alpha_2}} w^2(t) dx \leq \\ & \leq \sum_{K=0}^{\infty} (R_6 + K + 1)^{2\alpha_2} 2^{(R_6 + K + 1)^{\alpha_2}} \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6+K}} w^2(t) dx \leq \\ & \leq \sum_{K=0}^{\infty} (R_6 + K + 1)^{2\alpha_2} 2^{(R_6 + K + 1)^{\alpha_2}} e^{-\theta_1(R_6 + K)^{\alpha_1}} = C_{14} \end{aligned}$$

e perciò

$$(32) \quad \begin{aligned} R_1^{2+\alpha_2} 2^{R_1^{\alpha_2}} \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} w^2(t) dx & \leq \\ & \leq \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} |x|^{2+\alpha_2} 2^{|x|^{\alpha_2}} w^2(t) dx \leq C_{14} \end{aligned}$$

$\forall R_1 \geq R_6$. Dalla (32) segue subito la (31).

Dalla (31) per il teorema di Fubini discende l'esistenza di $h_1 > 0$ tale che $(\tau_0/2) \frac{5}{6} \leq h_1 \leq \tau_0/2$ e

$$(33) \quad \int_{R_6}^{+\infty} R_1^{\alpha_2} 2^{R_1^{\alpha_2}} \|fw(-h_1)\|_{\omega_{R_1}}^2 dR_1 < +\infty.$$

Inoltre dalla (32) si ottiene

$$\int_{h_0}^{h_1} d\tau \int_{-\tau}^{-h_0} dt \int_{\omega_{R_0}} w^2 dx < +\infty.$$

Dalla (23) preso $m=2$, $h=h_1$ si ottiene

$$C_{10}\Gamma'(R_1, 2, h_1) \leq C_{10} + R_1^{\alpha_2} 2^{R_1^{\alpha_2}} \|f w(-h_1)\|_{\omega_{R_1}}^2 \quad \forall R \geq R_6$$

da cui, per la (33), si ottiene

$$(34) \quad \Gamma(R_1, 2, h_1) \leq C_{15} R_1 \quad \forall R_1 \geq R_6.$$

Dalle (34), (22), (15) segue che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} f w^2(t-h_1) dx \leq C_{15} R_1 \left(\frac{4}{3}\right)^{-R_1^{\alpha_2}}$$

perchè

$$\begin{aligned} C_{15} R_1 &\geq \int_0^{\tau_0} d\tau \int_{-\tau}^0 \left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_2}} dt \int_{\omega_{R_1}} f w^2(t-h_1) dx \geq \\ &\geq \int_0^{\tau_0} d\tau \int_{-\tau/2}^0 \left(\frac{4}{3}\right)^{R_1^{\alpha_2}} dt \int_{\omega_{R_1}} f w^2(t-h_1) dx; \end{aligned}$$

perciò esistono $R_7 \geq R_6$ e $\theta_2 > 0$ tali che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_6+K(R_6+K+1)}} w^2(t-h_1) dx \leq e^{-\theta_2(R_6+K)^{\alpha_2}} \quad \forall K \in \mathbb{N};$$

e quindi come prima

$$\int_{R_7}^{+\infty} R_1^{\alpha_3} 2^{R_1^{\alpha_3}} dR_1 \int_0^{\tau_0/2} d\tau \int_{-\tau}^0 dt \int_{\omega_{R_1}} w^2(t-h_1) dx < +\infty;$$

di conseguenza per il teorema di Fubini esiste $h_2 > h_1$ tale che $h_1 + \frac{\tau_0}{2} \frac{5}{6} \leq h_2 \leq h_1 + \frac{\tau_0}{2}$ e

$$(35) \quad \int_{R_7}^{+\infty} R_1^{\alpha_3} 2^{R_1^{\alpha_3}} \|f w(-h_2)\|_{\omega_{R_1}}^2 dR_1 < +\infty.$$

Inoltre si ha

$$\int_{h_1}^{h_2} d\tau \int_{-\tau}^{-h_1} dt \int_{\omega_{R_0}} w^2 dx < +\infty.$$

Dalla (23) preso $m=3$ e $h=h_2$ si ottiene

$$C_{10}\Gamma'(R_1, 3, h_2) \leq C_{10} + R_1^{\alpha_3} 2^{R_1^{\alpha_3}} \|fw(-h_2)\|_{\omega_{R_1}}^2 \quad \forall R_1 \geq R_7$$

da cui, per la (35), si ottiene

$$\Gamma(R_1, 3, h_2) \leq C_{16}R_1 \quad \forall R_1 \geq R_7.$$

Ripetendo questo ragionamento un numero finito di volte si dimostra il lemma.

LEMMA 5. Siano u, q e $u+v, q+p$ due soluzioni di $(\alpha), (\beta)$ in Q , relative alla stessa F e con

(i) $u_i, v_i \in L_3, F_i \in L_1$ per $1 \leq i \leq 3, q, p \in L_2$

(ii) $u_i, \frac{\partial u_i}{\partial x_j}, v_i, \frac{\partial v_i}{\partial x_j}$ limitate in Q per $1 \leq i, j \leq 3$ e

$$|p(x, t)| \leq C/(1+|x|)^{1/2} \quad \forall (x, t) \in Q$$

(iii) u_i, v_i nulle su $\Omega \times \{0\}$ per $1 \leq i \leq 3$ e v_i nulle su $\partial\Omega \times \times [-T, 0]$ per $1 \leq i \leq 3$.

Allora esiste $\tau_2 \in]0, T]$ tale che $\left(\frac{\partial v_i}{\partial x_j}\right)^2$ è sommabile su Ω per quasi ogni $t \in [-\tau_2, 0]$ e per $1 \leq i, j \leq 3$.

DIMOSTRAZIONE. Per la (α) si ha

$$\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u = -\nabla q + F$$

$$\frac{\partial(u+v)}{\partial t} - \Delta(u+v) + ((u+v) \cdot \nabla)(u+v) = -\nabla(q+p) + F$$

da cui

$$(36) \quad \frac{\partial v}{\partial t} - \Delta v + (v \cdot \nabla)u + ((u+v) \cdot \nabla)v = -\nabla p.$$

Moltiplicando scalarmente la (36) per v e tenendo presente la (1) e che

$$((v \cdot \nabla)u) \cdot v + (((u+v) \cdot \nabla)v) \cdot v \leq C_\alpha v^2 + \frac{1}{2} \nabla \cdot (v^2(v+u)),$$

si ottiene

$$-\frac{1}{2} \frac{\partial(v^2)}{\partial t} \leq C_\alpha v^2 + (\nabla \times v^2) + \nabla \cdot \left(v \times (\nabla \times v) + \frac{1}{2} v^2(v+u) + pv \right)$$

da cui segue ($0 < \tau_1 \leq T$)

$$(37) \quad \begin{aligned} \frac{1}{2} \int_{-\tau_1}^0 dt \int_{\omega(R)} v^2 dx &\leq C_\beta \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} v^2 d\sigma + \\ &+ \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\omega(R)} (\nabla \times v)^2 dx + \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + \\ &+ \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} |pv| d\sigma + \tau_1 C_\alpha \int_{-\tau_1}^0 dt \int_{\omega(R)} v^2 dx. \end{aligned}$$

Poichè $|\nabla \times v| \leq |\nabla \times u| + |\nabla \times (u+v)|$ e poichè per il lemma 4 esiste C_γ tale che

$$\int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\omega(R)} (\nabla \times u)^2 dx \leq \frac{1}{2} C_\gamma,$$

$\forall R \geq R_0,$

$$\int_{-\tau}^{\tau_1} d\tau \int_0^{\tau_1} dt \int_{\omega(R)} (\nabla \times (u+v))^2 dx \leq \frac{1}{2} C_\gamma$$

si ha

$$(38) \quad \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\omega(R)} (\nabla \times v)^2 dx \leq C_\tau \quad \forall R \geq R_0, \quad \forall \tau_1 \leq h_1;$$

Posto

$$(39) \quad M(R) = \int_{-\tau_1}^0 d\tau \int_{\omega(R)} v^2 dx$$

dalla (37) per la (38) e la (ii) si ottiene, per $C_\alpha \tau_1 \leq \frac{1}{4}$ e $\tau_1 \leq h_1$,

$$(40) \quad \begin{aligned} M(R) &\leq C_\delta M'(R) + C_\delta (RM'(R))^{1/2} + C_\delta \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 dx + C_\delta \leq \\ &\leq \frac{1}{20} RM'(R) + C_\delta \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C_\lambda \quad \forall R \geq R_8. \end{aligned}$$

Proviamo che

$$(41) \quad M(R) \leq 2C_\lambda \quad \forall R \geq R_8.$$

Infatti se esiste $R_9 > R_8$ tale che $M(R_9) > 2C_\lambda$, tenendo presente che $M'(R) \geq 0$, si ha

$$M(R) \leq \frac{1}{10} RM'(R) + 2C_\delta \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 dx \quad \forall R \geq R_9$$

e quindi anche

$$\frac{1}{R} \leq \frac{1}{10} \frac{M'(R)}{M(R)} + C_\mu \int_0^{\tau_1} d\tau \int_{-\tau}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 dx,$$

da cui, integrando sull'intervallo $[R_9, R]$ ($R > R_9$) e tenendo presente la (38) si ottiene

$$10 \log \left(\frac{R}{R_9} \right) \leq \log \left(\frac{M(R)}{M(R_9)} \right) + C_\nu \quad \forall R \geq R_9,$$

da cui

$$R^{10}[e^{-c} R_2^{-10} M(R_2)] \leq M(R) \leq C_\xi R^3 \quad \forall R \geq R_9,$$

che è assurdo.

Moltiplicando scalarmente la (36) per v , tenendo presente la (s) e che

$$-((v \cdot \nabla)u) \cdot v - ((u+v) \cdot \nabla)v \cdot v \leq C_\alpha v^2 - \frac{1}{2} \nabla \cdot (v^2(v+u))$$

si ottiene

$$(42) \quad \frac{1}{2} \frac{\partial(v^2)}{\partial t} + \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2 \leq \nabla \cdot \left(\frac{1}{2} \nabla(v^2) - \frac{1}{2} v^2(v+u) - pv \right) + C_\alpha v^2$$

da cui, tenendo presente la (ii), si ottiene

$$\begin{aligned} & \frac{1}{2} \int_{-\tau_2}^0 dt \int_{\omega(R)} v^2 dx + \int_{-\tau_2}^0 dt \int_{-\tau_2}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx \leq C_\zeta \int_{-\tau_2}^0 dt \int_{\sigma(R)} v^2 d\sigma + \\ & + C_\zeta \int_{-\tau_2}^0 dt \int_{-\tau_2}^t d\xi \int_{\sigma(R)} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 d\sigma + C_\zeta R^{1/2} \left(\int_{-\tau_2}^0 dt \int_{\sigma(R)} v^2 d\sigma \right)^{1/2} + \\ & + \frac{\tau_2}{2} \int_{\omega(R)} v^2(-\tau_2) dx + C_\alpha \tau_2 \int_{-\tau_2}^0 dt \int_{\omega(R)} v^2 dx. \end{aligned}$$

Dalla (41) e dal teorema di Fubini segue che $v^2(\cdot, t)$ è sommabile su Ω per quasi $t \in [-\tau_1, 0]$ scegliamo $\tau_2 \in [0, \tau_1]$ vale che $v^2(\cdot, -\tau_2)$ sia sommabile su Ω e risulti $\tau_2 C_\alpha < 1/4$.

Allora posto

$$(43) \quad N(R) = \frac{1}{4} \int_{-\tau_2}^0 dt \int_{\omega(R)} v^2 dx + \int_{-\tau_2}^0 dt \int_{-\tau_2}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx,$$

si ha

$$\begin{aligned} N(R) & \leq 4C_\zeta N'(R) + 2C_\zeta (RN'(R))^{1/2} + C_\pi \leq \\ & \leq \frac{1}{20} RN'(R) + C_x \quad \forall R \geq R_{10}; \end{aligned}$$

di qui ragionando come prima risulta

$$N(R) \leq 2C_x \quad \forall R \geq R_{10}.$$

Da ciò per il teorema di Fubini segue il lemma.

3. TEOREMA 1. *Siano u, q e $u+v, q+p$ due soluzioni di (α) – (β) in Q , relative alla stessa F e con*

(i) $u_i, v_i \in L_3, F_i \in L_1$ per $1 \leq i \leq 3, p, q \in L_2$

(ii) $u_i, \frac{\partial u_i}{\partial x_j}, v_i, \frac{\partial v_i}{\partial x_j}$ limitate in Q per $1 \leq i, j \leq 3$ e

$$|p(x, t)| \leq C/(1+|x|)^{1/2} \quad \forall (x, t) \in Q$$

(iii) u_i, v_i nulle su $\Omega \times \{0\}$ per $1 \leq i \leq 3$ e v_i nulle su $\partial\Omega \times [-T, 0]$ per $1 \leq i \leq 3$.

Allora $p=0, v=0$.

DIMOSTRAZIONE. Cominciamo col provare che esiste una successione $(\alpha_n)_{n \in \mathbb{N}}$ divergente positivamente tale che

$$(44) \quad \int_{-T}^0 dt \int_{\omega_{R_0}(\alpha_n)} \left(\frac{\partial v}{\partial t} \right)^2 dx \leq \alpha_n^{15} \quad \forall n \in \mathbb{N}.$$

Allo scopo, moltiplicando scalarmente la (36) per $\frac{\partial v}{\partial t}$; si ottiene

$$(45) \quad \left(\frac{\partial v}{\partial t} \right)^2 - \Delta v \cdot \frac{\partial v}{\partial t} + \frac{\partial v}{\partial t} \cdot (v \cdot \nabla)u + \frac{\partial v}{\partial t} \cdot ((u+v) \cdot \nabla)v = - \frac{\partial v}{\partial t} \cdot \nabla p,$$

da cui, tenendo presente la (v) e che

$$(46) \quad \left| \frac{\partial v}{\partial t} \cdot [(v \cdot \nabla)u + ((u+v) \cdot \nabla)v] \right| \leq \frac{1}{4} \left(\frac{\partial v}{\partial t} \right)^2 + C'_1$$

si ha

$$\frac{1}{2} \left(\frac{\partial v}{\partial t} \right)^2 \leq C'_1 + \nabla \cdot \left(\frac{\partial v}{\partial t} \times (\nabla \times v) - p \frac{\partial v}{\partial t} \right) - \frac{1}{2} \frac{\partial}{\partial t} ((\nabla \times v)^2),$$

e quindi

$$\int_{-T}^0 dt \int_{R_0}^R dr \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t} \right)^2 dx \leq C'_2 R^4 + C'_2 R^2 \left(\int_{-T}^0 dt \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t} \right)^2 dx \right)^{1/2} \quad \forall R \geq R_0.$$

Perciò, posto

$$P(R) = \int_{-T}^0 dt \int_{R_0}^R dr \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t} \right)^2 dx$$

si ottiene

$$P(R) \leq C'_2 R^4 + C'_2 R^5 (P'(R))^{1/2} \quad \forall R \geq R_0;$$

di qui, per i lemmi 1 e 3, segue la (44).

Dalla (36) e dalla (ii) segue

$$(47) \quad (Lv - \nabla p)^2 = ((v \cdot \nabla)u + ((u+v) \cdot \nabla)v)^2 \leq C'_3 \left(v^2 + \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 \right).$$

Dall'osservazione al lemma 2 e tenendo presente il lemma 5, presi $h=0$, $w=v$ e R' , γ' , τ_3 , $\delta' > 0$ tali che

$$\begin{aligned} \|\| v(-\tau_3) \|\|_{\Omega}^2 < +\infty, \quad \|\| v(-\tau_3) \|\|_{\omega(R')}^2 > 0, \quad \tau_3 \leq \tau_2, \quad 2C_{\tau_3, \gamma', \delta'} C'_3 \leq \frac{1}{2} \\ \tau_3 \gamma' (2\tau_3)^{-2\gamma'-1} \|\| v(-\tau_3) \|\|_{\omega(R')}^2 \geq C_{\tau_3, \gamma', \delta'} (2\tau_3)^{-2\gamma'} \|\| v(-\tau_3) \|\|_{\Omega}^2 \end{aligned}$$

e posto

$$\begin{aligned} Q(R) &= \int_{-\tau_3}^0 \lambda^{-2\gamma'} \|\| v \|\|_{\omega(R)}^2 dt + \int_{-\tau_3}^0 \lambda^{-2\gamma'} \|\| v \|\|_{\omega(R)}^2 dt + \\ &+ C_{\tau_3, \gamma', \delta'} \int_{-\tau_3}^0 \left\| \frac{\partial}{\partial t} ((\lambda^{-\gamma'} v)) \right\|_{\omega(R)}^2 dt, \end{aligned}$$

tenendo presente la (47), si ottiene

$$(48) \quad Q(R) \leq C'_4 Q'(R) + C'_4 (RQ'(R))^{1/2} \quad \forall R \geq R' \geq R_0.$$

Proviamo che $Q(R)=0$ e quindi $v=0$.

Infatti, ragionando per assurdo, supponiamo che esista $R_{11} > R'$ tale che sia $Q(R_{11}) > 0$.

Dalla (48) si ottiene

$$Q(R_{11}) \leq C'_4 Q'(R) + C'_4 (RQ'(R))^{1/2} \quad \forall R \geq R_{11}$$

e quindi

$$(Q'(R))^{1/2} \geq \frac{2Q(R_{11})}{C'_4 R^{1/2} + (R + 4Q(R_{11})/C'_4)^{1/2}} \geq \frac{2Q(R_{11})}{C'_5 R^{1/2}}$$

$\forall R \geq R_{12} \geq R_{11}$ con R_{12} tale che $\frac{R}{Q(R)} \geq \frac{4}{C'_4}$ per $R \geq R_{12}$. Introducendo questa valutazione nella (48) si ha

$$Q(R) \leq C'_6 RQ'(R) \quad \forall R \geq R_{13} \geq R_{12},$$

da cui,

$$((R^{1/C'_6} Q(R_{13})) / R_{13}^{1/C'_6}) \leq Q(R) \quad \forall R \geq R_{13}.$$

Da questa segue che esiste $R_{14} \geq R_{13}$ tale che

$$(Q'(R))^{1/2} \geq \frac{2Q(R_{14})}{C'_4 R^{1/2} + (R + 4Q(R_{14})/C'_4)^{1/2}} \geq \frac{20C'_4}{R^{1/2}} \quad \forall R \geq R_{15} \geq R_{14}.$$

Introducendo questa valutazione nella (48) si ottiene

$$Q(R) \leq C'_4 Q'(R) + \frac{1}{20} RQ'(R) \leq \frac{R}{19} Q'(R)$$

$\forall R \geq R_{16} \geq R_{15}$, da cui

$$(49) \quad C'_7 R^{19} \leq Q(R) \quad \forall R \geq R_{16}.$$

Dalla (ii) e dalla (44) segue che

$$\int_{-\tau_3}^0 \lambda^{-2\gamma'} \|v\|_{\omega(\alpha_n)}^2 dt + \int_{-\tau_3}^0 \lambda^{-2\gamma'} \|v\|_{\omega(\alpha_n)}^2 dt \leq C'_8 \alpha_n^3 \quad \forall n \in N$$

$$\int_{-\tau_3}^0 \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} v) \right\|_{\omega(\alpha_n)}^2 dt = \int_{-\tau_3}^0 \left\| \gamma \lambda^{-\gamma'-1} v + \lambda^{-\gamma'} \frac{\partial v}{\partial t} \right\|_{\omega(\alpha_n)}^2 dt \leq C'_8 \alpha_n^{15} \quad \forall n \in \mathbb{N}$$

e quindi

$$Q(\alpha_n) \leq C'_9 \alpha_n^{15} \quad \forall n \in \mathbb{N}$$

che contraddice la (49).

Dunque $Q(R)=0$ e quindi $v=0, p=0$ su $\Omega \times [-\tau_3, 0]$.

Dopo di ciò riapplicando il lemma 5 e il ragionamento ora fatto si ottiene $v=0, p=0$ su $\Omega \times [-\tau_4, -\tau_3]$ con un τ_4 che si può ritenere prossimo tanto quanto si vuole a $2\tau_3$. Ecc.

Ciò conclude la dimostrazione.

TEOREMA 2. *Siano u, q e $u+v, q+p$ due soluzioni di (α) – (β) in Q relative alla stessa F e con*

$$(i) \quad u_i, v_i \in L_3, F_i \in L_1 \text{ per } 1 \leq i \leq 3, p, q \in L_2$$

$$(ii) \quad u_i, \frac{\partial u_i}{\partial x_j}, v_i, \frac{\partial v_i}{\partial x_j} \text{ limitate in } Q \text{ per } 1 \leq i, j \leq 3 \text{ e}$$

$$|p(x, t)| \leq C/(1+|x|)^{1/2} \quad \forall (x, t) \in Q$$

$$(iii) \quad v_i \text{ nulle su } \Omega \times \{0\} \text{ e su } \partial\Omega \times [-T, 0] \text{ per } 1 \leq i \leq 3$$

$$(iv) \quad \left(\frac{\partial v_i}{\partial x_j} \right)^2(-T) \text{ e } \left(\frac{\partial u_j}{\partial x_j} \right)^2(-T) \text{ sommabile su } \Omega \text{ per } 1 \leq i, j \leq 3.$$

Allora $v=0, p=0$.

DIMOSTRAZIONE. Cominciamo col provare che

$$(50) \quad \int_{-T}^0 dt \int_{\Omega} (\nabla \times v)^2 dx < +\infty.$$

Per $w = \nabla \times u$ oppure $w = \nabla \times (u+v)$ ricordiamo che

$$(13) \quad \frac{1}{2} \frac{\partial(w^2)}{\partial t} + \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 \leq \frac{1}{2} \nabla \cdot (\nabla(w^2) - w^2 u) + C_1 w^2.$$

Da essa si deduce per $0 < h \leq T$

$$\begin{aligned} & \frac{1}{2} \int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \int_{-T}^{-T+h} dt \int_{-T}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 dx \leq \\ & \leq C'_{10} \int_{-T}^{-T+h} dt \int_{-T}^t d\xi \int_{\sigma(R)} \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 dx + C'_{10} \int_{-T}^{-T+h} dt \int_{\sigma(R)} w^2 d\sigma + \\ & + hC_1 \int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \frac{h}{2} \int_{\omega(R)} w^2(-T) dx; \end{aligned}$$

ponendo

$$(51) \quad S(R) = \int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \int_{-T}^{-T+h} dt \int_{-T}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial w_i}{\partial x_j} \right)^2 dx,$$

se si prende $hC_1 \leq \frac{1}{4}$ ricordando la (iv), si ha

$$(52) \quad S(R) \leq C'_{11} S'(R) \leq C'_{11} \quad \forall R \geq R_0.$$

Proviamo che dalla (52) segue che

$$(53) \quad S(R) \leq 2C'_{11} \quad \forall R \geq R_0;$$

infatti in caso contrario, esisterebbe $R_{17} \geq R_0$ tale che $S(R_{17}) > 2C'_{11}$ e quindi dalla (52) seguirebbe

$$S(R) \leq 2C'_{11} S'(R) \quad \forall R \geq R_{17},$$

da cui

$$(54) \quad e^{R/(2C'_{11})} (e^{-R_{17}/(2C'_{11})} S(R_{17})) \leq S(R) \quad \forall R \geq R_{17}$$

Dalla (13), tenendo presente la (ii), si ottiene

$$\int_{-T}^0 dt \int_{R_0}^R ||| w |||_{\omega_{R_0}(r)}^2 dr \leq C'_{12} R^4 + C'_{12} R^5 \left(\int_{-T}^0 ||| w |||_{\omega_{R_0}(R)}^2 dt \right)^{1/2} \quad \forall R \geq R_0$$

e quindi ricordando la (7) si ottiene

$$\Phi(R) \leq C'_{12} R^4 + C'_{12} R^5 (\Phi'(R))^{1/2} \quad \forall R \geq R_0 ;$$

di qui, per il lemma 1, segue che

$$(55) \quad \Phi(R) \leq R^{15} \quad \forall R \geq R_1 .$$

Per la (ii) e le (55), (51) si ha

$$\int_{R_0}^R S(r) dr \leq R^{16} \quad \forall R \geq R_{18} ;$$

quindi, per il lemma 3, esiste una successione $(\alpha_n)_{n \in \mathbb{N}}$ divergente positivamente tale che

$$S(\alpha_n) \leq \alpha_n^{16}$$

Ciò contraddice la (54).

Dalla (53) in particolare discende che

$$\int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx \leq 2C'_{11} \quad \forall R \geq R_0 ,$$

perciò il teorema di Fubini segue che w^2 è sommabile su Ω per quasi ogni $t \in [-T, -T+h]$; non è restrittivo supporre che tra questi valori ci sia anche $-T+h$. Ripetendo allora il ragionamento fatto si vede che

$$\int_{-T+h}^{-T+2h} dt \int_{\omega(R)} w^2 dx \leq 2C'_{11} \quad \forall R \geq R_0$$

e che w^2 è sommabile per quasi ogni $t \in [-T+h, -T+2h]$. Così procedendo e tenendo che $|\nabla \times v| \leq |\nabla \times u| + |\nabla \times (u+v)|$ si prova la (50).

Proviamo ora che esiste h , $0 < h \leq T$, tale che

$$(56) \quad \int_{-h}^0 dt \int_{\Omega} v^2 dx < +\infty.$$

Allo scopo cominciamo con l'osservare che dalla (α) segue che

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \frac{1}{2} \nabla(u^2) - u \times (\nabla \times u) &= -\nabla q + F \\ \frac{\partial(u+v)}{\partial t} - \Delta(u+v) + \frac{1}{2} \nabla((u+v)^2) - (u+v) \times ((\nabla \times (u+v))) &= \\ &= -\nabla(q+p) + F \end{aligned}$$

e quindi

$$\frac{\partial v}{\partial t} - \Delta v + \frac{1}{2} \nabla(v^2 + 2u \cdot v) - (u+v) \times (\nabla \times v) - v \times (\nabla \times u) = -\nabla p.$$

Moltiplicando scalarmente quest'ultima per $\frac{\partial v}{\partial t}$ tenendo presente la (v) , e osservando che

$$\begin{aligned} \left| \frac{\partial v}{\partial t} \cdot [(u+v) \times (\nabla \times v) - v \times (\nabla \times u)] \right| &\leq \\ &\leq \frac{1}{2} \left(\frac{\partial v}{\partial t} \right)^2 + C'_{12} (\nabla \times v)^2 + C'_{12} v^2, \end{aligned}$$

si ottiene

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial v}{\partial t} \right)^2 + \frac{1}{2} \frac{\partial}{\partial t} (\nabla \times v)^2 &\leq -\nabla \cdot \left(\left(p + \frac{v^2}{2} + u \cdot v \right) \frac{\partial v}{\partial t} + (\nabla \times v) \times \frac{\partial v}{\partial t} \right) + \\ &+ C'_{12} (\nabla \times v)^2 + C'_{12} v^2. \end{aligned}$$

Ne segue che

$$(57) \quad \frac{1}{2} \int_{-h}^0 dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t} \right)^2 dx \leq$$

$$\begin{aligned}
&\leq \frac{1}{2} \int_{-h}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + \int_{-h}^0 dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t} \right)^2 d\sigma + C'_{13} \int_{-h}^0 dt \int_{\sigma(R)} v^2 d\sigma + \\
&+ C'_{13} R^{12} \left(\int_{-h}^0 dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t} \right)^2 d\sigma \right)^{1/2} + \frac{1}{2} \int_{\omega(R)} (\nabla \times v)^2 (-h) dx + \\
&+ C'_{12} \int_{-h}^0 dt \int_{\omega(R)} (\nabla \times v)^2 dx + C'_{12} \int_{-h}^0 dt \int_{\omega(R)} v^2 dx.
\end{aligned}$$

Proviamo che

$$(58) \quad \int_{-h}^0 dt \int_{\omega(R)} v^2 dx \leq h^2 \int_{-h}^0 dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t} \right)^2 dx \quad \forall R \geq R_0$$

$$(59) \quad \int_{-h}^0 dt \int_{\sigma(R)} v^2 d\sigma \leq h^2 \int_{-h}^0 dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t} \right)^2 d\sigma \quad \forall R \geq R_0.$$

Infatti

$$|v_i(t)| = \left| \int_0^t \left(\frac{\partial v_i}{\partial t} \right) d\xi \right| \leq |t|^{1/2} \left| \int_0^t \left(\frac{\partial v_i}{\partial t} \right)^2 d\xi \right|^{1/2} \quad \text{per } 1 \leq i \leq 3$$

e quindi

$$\int_{-h}^0 v^2(t) dt \leq h^2 \int_{-h}^0 \left(\frac{\partial v}{\partial t} \right)^2 d\xi.$$

Ciò prova la (58) e la (59).

Dalla (57) preso h tale che $(\nabla \times v)^2(-h)$ sia sommabile su Ω (il che è possibile per quasi ogni h , $0 \leq h \leq T$, per la (50) e per il teorema di Fubini) e tale che sia $c'_{12} h^2 < \frac{1}{4}$, ricordando ancora la (50) e le (58), (59), se si pone

$$(60) \quad T(R) = \int_{-h}^0 dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t} \right)^2 dx$$

si ottiene

$$\begin{aligned}
 T(R) &\leq C'_{14} T'(R) + C'_{14} R^{1/2} (T'(R))^{1/2} + C'_{14} \int_{-h}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C'_{14} \leq \\
 &\leq \frac{1}{40} R T'(R) + C'_{14} \int_{-h}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C'_{15} \quad \forall R \geq R_{19}.
 \end{aligned}$$

Proviamo ora che

$$(61) \quad T(R) \leq 2C'_{15} \quad \forall R \geq R_{19}.$$

Infatti se esiste $R_{20} \geq R_{19}$ tale che $T(R_{20}) > 2C'_{15}$, tenendo presente che $T'(R) \geq 0$, si ha

$$\frac{20}{R} \leq \frac{T(R)}{T'(R)} + C'_{16} \int_{-h}^0 dt \int_{\sigma(R)} (\nabla \times v)^2 dx, \quad \forall R \geq R_{20}$$

da cui, integrando sull'intervallo $[R_{20}, R]$ ($R > R_{20}$) e tenendo presente la (48) si ottiene

$$20 \log \left(\frac{R}{R_{20}} \right) \leq \log \left(\frac{T(R)}{T(R_{20})} \right) + C'_{17} \quad \forall R \geq R_{20},$$

e quindi

$$R^{20} [e^{-C'_{17}} R_{20}^{-20} T(R_{20})] \leq T(R) \quad \forall R \geq R_{20}$$

che è una contraddizione con la (44), che è valida anche nelle ipotesi del presente teorema. Dalla (60) e dalla (61) per la (58), segue la (56).

Proviamo ora che

$$(62) \quad \int_{-h}^0 dt \int_{-h}^t d\xi \int_{\Omega} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx < +\infty.$$

Allo scopo osserviamo che dalla (42) si ottiene

$$\frac{1}{2} \int_{-h}^0 dt \int_{\omega(R)} v^2 dx + \int_{-h}^0 dt \int_{-h}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx \leq$$

$$\begin{aligned} &\leq C'_{18} \int_{-h}^0 dt \int_{\sigma(R)} v^2 d\sigma + C'_{18} \int_{-h}^0 dt \int_{-h}^t d\xi \int_{\omega(R)} \sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx + \\ &+ C'_{18} R^{1/2} \left(\int_{-h}^0 dt \int_{\sigma(R)} v^2 d\sigma \right)^{1/2} + \frac{h}{2} \int_{\omega(R)} v^2(-h) dx + C'_{18} h \int_{-h}^0 dt \int_{\omega(R)} v^2 dx. \end{aligned}$$

Da quest'ultima chiamando ancora con $N(R)$ la (43) con h al posto di τ_2 e tenendo presente la (56) e il teorema di Fubini si ottiene

$$\begin{aligned} N(R) &\leq C'_{19} N'(R) + C'_{19} (RN(R))^{1/2} + C'_{19} \leq \\ &\leq \frac{1}{20} RN'(R) + C'_{20} \quad \forall R \geq R_{21}; \end{aligned}$$

di qui, ragionando come prima, segue

$$N(R) \leq 2C'_{20} \quad \forall R \geq R_0.$$

Quindi per il teorema di Fubini risulta che $\sum_{i,j=1}^3 \left(\frac{\partial v_i}{\partial x_j} \right)^2$ è sommabile per quasi ogni $t \in [-h, 0]$; perciò ragionando come nel teorema precedente si dimostra che $v=0, p=0$ in $[-h, 0]$; ripetendo poi i ragionamenti già fatti si prova che $v=0, p=0$ in $[-2h, 0]$; ecc. Con ciò si dimostra il teorema.

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