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On *k*-path hamiltonian graphs and line-graphs

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ON k-PATH HAMILTONIAN GRAPHS AND LINE-GRAPHS

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1. Throughout this paper, the word *graph* will be used for an undirected connected graph, without loops or multiple edges.

If G is a graph, $P(G)$ will denote the point-set of G and $E(G)$ its edge-set.

A graph is called:

— *the line-graph $L(G)$ of the graph G* if $P(L(G))$ can be put in one-to-one correspondence with $E(G)$ in such a way that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent,

— *the subgraph G' of the graph G* if $P(G') \subset P(G)$ and each line in $E(G)$ joining points in $P(G')$ also belongs to $E(G')$,

— *of type T_1 in G* if its point-set $\overline{P}$ and its edge-set $\overline{E}$ respectively are subsets of $P(G)$ and $E(G)$, and it has at least three common lines with every complete subgraph on 4 points of the subgraph G' of G with $P(G')=\overline{P}$.

— *of type T_2 in G* if it is of type T_1 in G and no point not from its point-set is adjacent to more than one point in its point-set,

— *hamiltonian* if it possesses a hamiltonian circuit,

— *hamiltonian-connected* if every pair of distinct points is connected by a hamiltonian path,

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— *randomly hamiltonian* if every path is contained in a hamiltonian circuit,

— *k-path hamiltonian* ($0 \leq k \leq p-2$, where p is the number of vertices) if every path of length not exceeding k is contained in a hamilonian circuit (for $k=0$ it is meant a hamiltonian graph, and for $k=p-2$ one obtains a randomly hamiltonian graph),

— *weakly k-path hamiltonian* if every path of length not exceeding k and of type T_2 is contained in a hamiltonian circuit.

If M is a subset of the point-set of a graph then the number of all points not in M each of which is adjacent to some point of M is called *the degree of M*. If a is a point of a graph, then $\rho(a)$ denotes the degree of $\{a\}$.

2. Let G be a graph on p points.

G. Chartrand and H. Kronk [3] gave necessary and sufficient conditions for G to be $(p-2)$-path hamiltonian (randomly hamiltonian).

Results of O. Ore ([5], [6], [7]) giving sufficient conditions for the graph G to be hamiltonian have been extended in the following ways to provide sufficient conditions for G to be k-path hamiltonian ($0 \leq k \leq p-3$).

Proposition 1 (H. Kronk [4]). *G is k-path hamiltonian if for any pair of non-adjacent vertices a and b,*

$$\rho(a)+\rho(b) \geq p+k.$$

Proposition 2 (H. Kronk [4]). *G is k-path hamiltonian if it has at least* $\frac{1}{2}(p-1)(p-2)+k+2$ *edges.*

Theorem 1 will give another sufficient condition for a graph to be k-path hamiltonian.

The next two Propositions contain sufficient conditions for a graph such that its (iterated) line-graph is hamiltonian.

Proposition 3 (G. Chartrand [1], [2]). *G is sequential if and only if L(G) is hamiltonian.*

PROPOSITION 4 (G. Chartrand [1], [2]). *If G is not a path, then $L^{p+k-3}(G)$ is hamiltonian for all $k \geq 0$.*

Theorems 2 and 2a will give necessary conditions for a graph to be k-path hamiltonian, and Theorem 3 together with its Corollaries will complete some results in [2].

3. THEOREM 1. *If each subgraph of G on at least $p-k+1$ vertices is hamiltonian-connected, then G is k-path hamiltonian* *).

PROOF. Let K be a k-path (a path of length at most k) in G, of endpoints a, b. Since the subgraph G' of G with $P(G')=(P(G))-$ $-P(K)) \cup \{a, b\}$ is hamiltonian-connected, a and b are joined by a hamiltonian path Π in G'. Then $K \cup \Pi$ is a hamiltonian circuit of G.

That Theorem 1 may be used in cases in which Propositions 1 and 2 fail to apply, it can be seen from the following example:

Let G be the graph obtained by joining each point of the edgeless graph E_4 on 4 points with each point of the complete graph K_7 on 7 points and also joining another point v with 5 vertices of K_7. G does not satisfy the sufficient conditions of Proposition 1 for being 1-path hamiltonian, because for some vertex w of E_4

$$\rho(v)+\rho(w)=12,$$

while $p+k=13$. Also, G fails to satisfy the sufficient conditions of Proposition 2 because its number of edges is 54, while $\frac{1}{2}(p-1)(p-2)+$ $+k+2=58$. By applying Theorem 1, G is even 3-path hamiltonian. (We note that for $k=1$ Theorem 1, though not false, is uninteresting since hamiltonian-connectedness directly implies the property of being 1-path hamiltonian).

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*) It can be proved that this theorem is stronger than Theorem 8 of C. Berge in « Graphes et hypergraphes », Dunod 1970, p. 197 (regarded as a sufficient condition for a graph to be k-path hamiltonian), and that both Propositions 1 and 2 are weaker than the mentioned result of C. Berge.

4. THEOREM 2. *If G is k-path hamiltonian, then $L(G)$ is weakly $(k+1)$-path hamiltonian.*

PROOF. Let Λ be a $(k+1)$-path of type T_2 in $L(G)$. The edges in $E(G)$ corresponding to the vertices of Λ form a set

$$V=\{v_0, ..., v_{k+1}\}$$

such that v_i and v_{i+1} are adjacent ($i=0, ..., k$). Let

$$\{v_0, v_{n_1}, ..., v_{n_l}, v_{k+1}\} \quad (n_1< ... <n_l)$$

be a subset of V forming a path of maximal length. Evidently, each edge v_i ($i=0, ..., k+1$) is adjacent to some edge of the path K generated by

$$\{v_{n_1}, ..., v_{n_l}\}.$$

Since $l \leq k$, K may be extended to a hamiltonian circuit C in G. Each edge of G not in V is adjacent to some edge of $E(C)-V$. Now, all the edges in $E(G)-V$ may be arranged in an obvious manner to form a sequence

$$\{a_1, ..., a_m\}$$

such that v_{n_1} and a_1 are adjacent, a_i and a_{i+1} are adjacent ($i=1, ..., m-1$), and a_m and v_{n_1} are adjacent. The points in $L(G)$ corresponding to the cycle of edges

$$\{v_0, ..., v_{k+1}, a_1, ..., a_m, v_0\}$$

are consecutively adjacent, thus providing a hamiltonian circuit which includes Λ.

The proof of Theorem 2 suggests the following improvement of its statement.

THEOREM 2 a. *If G is k-path hamiltonian, then each $(k+1)$-path of type T_1 in $L(G)$ whose $(k-1)$-subpath obtained by removing its endpoints (and adjacent edges) is of type T_2 in $L(G)$, is extendable to a hamiltonian circuit of $L(G)$.*

Using Theorem 2 a it can be seen that for $k=0$, 1, Theorem 2 may be stated in the following stronger form:

THEOREM 3. *If G is k-path hamiltonian, then L(G) is* $(k+1)$*-path hamiltonian* $(k=0$ *or* $1)$.

COROLLARY 1. *If G is hamiltonian, then L(G) is 1-path hamiltonian and* $L^n(G)$ *is 2-path hamiltonian for every* $n\geq 2$.

The above corollary improves Corollary 1 B in [2].

Proposition 3 together with Corollary 1 imply:

COROLLARY 2. *If G is sequential, then* $L^2(G)$ *is 1-path hamiltonian and* $L^n(G)$ *is 2-path hamiltonian for every* $n\geq 3$.

Proposition 4 together with Corollary 1 yield the following improvement of Proposition 4.

COROLLARY 3. *If G is not a path, then* $L^{p+k-3}(G)$ *is* $\min\{2, k\}$*-path hamiltonian* $(k\geq 0)$.

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