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# DUAL-DEDEKIND SUBGROUPS IN FINITE GROUPS

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If $G$ is a group and $H$ is a subgroup of $G$, $H$ is dual-Dedekind in $G$, or a $\breve{\mathfrak{D}}$-subgroup of $G$ (written $H \breve{\mathfrak{D}} G$) if the following conditions are fulfilled:

i) $X \supseteq Y \Rightarrow (Y \cup H) \cap X = Y \cup (H \cap X)$

ii) $H \supseteq Y \Rightarrow (Y \cup X) \cap H = Y \cup (X \cap H)$

for every pair $X$, $Y$ of subgroups of $G$ (for the dual notion, namely that of Dedekind subgroups, there called « modular subgroups », see [4]). In this paper we are particularly concerned with the properties of « minimum » $\breve{\mathfrak{D}}$-subgroups (i.e. minimal in the set of non identity dual-Dedekind subgroups of a given group $G$); we establish some necessary conditions in order that a finite group $G$ has non-trivial (i.e. different from 1, $G$) $\breve{\mathfrak{D}}$-subgroups. From these it will follow that a finite group having non-trivial $\breve{\mathfrak{D}}$-subgroups cannot be simple (Theorem 3.3) — a similar result for Dedekind subgroups is proved in [2]; it is perhaps worth noting that the converse is false: $G$ non-simple is not a sufficient condition for $G$ to have a non-trivial $\breve{\mathfrak{D}}$-subgroup. The proposition « if $N \trianglelefteq G$, then $N \breve{\mathfrak{D}} G$ » for arbitrary $G$ is false; in the second half of the paper we determine all finite soluble groups where such a condition holds. The main result in this section is (Theorem 4.6): $G$ is soluble and every

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normal subgroup of $G$ is dual-Dedekind in $G$ iff $G=H_1\times H_2\times\ldots\times H_t$ with each $H_i$ a Hall subgroup of $G$ and either

i) $H_i$ is a modular $p$-group; or

ii) $H_i=(P_{i1}\times\ldots\times P_{is_i})Q_i$ with $P_{ij}$, $Q_i$ Sylow subgroups of $G$ for different primes, $P_{ij}$ abelian of odd order ($j=1,\ldots,s_i$), $Q_i=\langle b_i\rangle$ with $b_i$ inducing a non-identity power automorphism on each $P_{ij}$.

**1.** Let $L$ be a lattice. An element $a\in L$ is a dual-Dedekind element of $L$ (a $\breve{\mathfrak{D}}$-element of $L$, $a\breve{\mathfrak{D}}L$) if

i) $x\geq y\Rightarrow(y\cup a)\cap x=y\cup(a\cap x)$;

ii) $a\geq y\Rightarrow(y\cup x)\cap a=y\cup(x\cap a)$

hold for every pair $(x,y)$ of elements in $L$. Notice that $a\breve{\mathfrak{D}}L$ if and only if $a$ is a Dedekind element in $\breve{L}$ (the dual lattice of $L$); hence properties of $\breve{\mathfrak{D}}$-elements of $L$ are properties of Dedekind elements of $\breve{L}$. We shall use mostly (see [4]):

I) $a\breve{\mathfrak{D}}L$ iff for every $b\in L$ the maps

$$\varphi^b : x\to x\cup b \qquad \varphi^b : [a/a\cap b]\to[a\cup b/b]$$

$$\varphi_a : y\to y\cap a \qquad \varphi_a : [a\cup b/b]\to[a/a\cap b]$$

are inverse lattice-isomorphisms.

II) if $a\breve{\mathfrak{D}}L$ and $b\in L$, then $a\cap b\breve{\mathfrak{D}}(b)$.

III) if $a_1\breve{\mathfrak{D}}L$ and $a_2\breve{\mathfrak{D}}L$, then $a_1\cap a_2\breve{\mathfrak{D}}L$.

IV) if $a_1\breve{\mathfrak{D}}L$ and $a_2\breve{\mathfrak{D}}(a_1)$, then $a_2\breve{\mathfrak{D}}L$.

V) whenever $\varphi$ is a surjective lattice isomorphism of $L$ onto $L'$, we have $a\varphi\breve{\mathfrak{D}}L'$ iff $a\breve{\mathfrak{D}}L$.

A subgroup $H$ of a group $G$ is dual-Dedekind in $G$ ($H$ is a $\breve{\mathfrak{D}}$-subgroup of $G$, $H\breve{\mathfrak{D}}G$) if $H$ is a $\breve{\mathfrak{D}}$-element of the lattice $\mathfrak{L}(G)$ of all subgroups of $G$. Normal subgroups are usually not $\breve{\mathfrak{D}}$-subgroups; the following are dual-Dedekind subgroups in any group $G$:

*a*) $G$ and the identity subgroup 1 of $G$ (they will be referred to as the « trivial » $\breve{\mathfrak{D}}$-subgroups);

*b*) the subgroups of the centre $Z(G)$;

*c*) all normal cyclic subgroups;

*d*) the subgroups of the so-called « kernel »

$$K(G) = \cap \{\mathfrak{N}_G(X) \mid X \in \mathfrak{L}(G)\}.$$

All definitions and notations will be standard; throughout this paper « group » means « finite group ».

**2.** The existence of non-trivial $\breve{\mathfrak{D}}$-elements in $\mathfrak{L}(G)$ rather severely restricts the structure of $G$. The following two lemmas provide examples where, in very simple cases, the structure of $G$ is completely determined.

LEMMA 2.1. *Assume $G = \langle a, b \rangle$, $|a| = |b| = p$, $p$ a prime, $G$ non cyclic. Then $\langle a \rangle \breve{\mathfrak{D}} G$ iff either $|G| = p^2$ or $G$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$).*

Since in both cases $\mathfrak{L}(G)$ is a modular lattice, the sufficiency of the condition is obvious. The condition is also necessary: the intervals $[\langle a, b \rangle / \langle b \rangle]$ and $[\langle a \rangle / \langle a \rangle \cap \langle b \rangle]$ are isomorphic, hence $\langle b \rangle$ is a maximal subgroup of $G$ and, if $G$ is a $p$-group, then $|G| = p^2$. Assume that $G$ is not a $p$-group; then $\langle b \rangle$ is a cyclic $p$-Sylow subgroup, $\langle b \rangle$ is its own normalizer in $G$ and has a normal complement $N$. $\langle a \rangle$, as a conjugate of $\langle b \rangle$, is maximal in $G$ and $\langle a \rangle \cap N = 1$; for a $c \in N$ of prime order $q$ one has $N = (\langle a \rangle \cup \langle c \rangle) \cap N = \langle c \rangle \cup (\langle a \rangle \cap N) = \langle c \rangle$, and the conclusion follows.

LEMMA 2.2. *Assume $G = \langle a, b \rangle$, $|a| = p$, $|b| = q$, $p$, $q$ different prime numbers. Then $\langle a \rangle \breve{\mathfrak{D}} G$ iff either $|G| = pq$, or $G$ contains an elementary abelian $p$-subgroup $N \triangleleft G$ such that $G = N\langle b \rangle$, and $\langle b \rangle$ operates irreducibly on $N$.*

If $\langle a \rangle \breve{\mathfrak{D}} G$, then $\langle b \rangle$ is a maximal subgroup of $G$ and from $\langle b \rangle \triangleleft G$ follows $|G| = pq$. If $\langle b \rangle$ is not normal, then $\langle b \rangle$ is a $q$-Sylow subgroup which is its own normalizer in $G$, hence $\langle b \rangle$ has a normal complement $N$. Since any conjugate of $\langle b \rangle$ is maximal in $G$, no proper non-trivial subgroup of $N$ is normalized by $b$ nor by a conjugate of $b$. Let $S$ be a $p$-Sylow subgroup of $N$ containing $\langle a \rangle$; by the Frattini argument $G = N\mathfrak{N}_G(S)$, hence $\mathfrak{N}_G(S)$ contains a conjugate of $\langle b \rangle$, and $S = N$. The Frattini subgroup $\Phi(N) \neq N$, $\Phi(N) \lhd G$, so that $\Phi(N) = 1$ and the « only if » part is proved. Conversely, neglecting the case $|G| = pq$ where everything is obvious, we have to prove that if $1 \neq a \in N$ and $X \in \mathfrak{L}(G)$ is arbitrary, $a \notin X$ implies that $X$ is maximal in $\langle a, X \rangle$. But $X \subseteq N$ implies that $\langle a, X \rangle$ is abelian, whereas if $X \not\subseteq N$ a conjugate of $\langle b \rangle$, say $\langle c \rangle$, lies in $X$, hence $X = \langle c \rangle$ is a maximal subgroup of $G = \langle a, c \rangle$.

3. Definition. Let $H$ be a subgroup of the group $G$. We shall say that $H$ is a minimum $\breve{\mathfrak{S}}$-subgroup of $G$ if $H$ is minimal in the set of all non-identity $\breve{\mathfrak{S}}$-subgroups of $G$.

Theorem 3.1. *Let $H$ be a minimum $\breve{\mathfrak{S}}$-subgroup of $G$. If $|H|$ is not a prime number, then*

i) *$H$ is normal in $G$;*

ii) *for every prime number $p$ dividing $|H|$, all the elements of $G$ of order $p$ are in $H$; and*

iii) $\mathfrak{C}_G(H) = \{x \in G \mid (|x|, |H|) = 1\}$.

First of all, notice that the minimality of $H$ and III, V of section 1 imply that for every $g \in G$ either $g^{-1}Hg \cap H = 1$ or $g^{-1}Hg = H$; furthermore, if $1 \neq A \subseteq H$, then $\mathfrak{N}_G(A) \subseteq \mathfrak{N}_G(H)$: thus, if $g^{-1}Ag = A$, then $1 \neq A \subseteq H \cap g^{-1}Hg$, whence $g^{-1}Hg = H$. Choose now an element $a \in H$ of prime order $p$. For any $b \notin H$ such that $|b| = p$, $\langle a \rangle = \langle a, b \rangle \cap H \breve{\mathfrak{S}} \langle a, b \rangle$; by lemma 2.1 either $|\langle a, b \rangle| = p^2$, or $\langle a, b \rangle$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$). In the first case $[a, b] = 1$, hence $b \in \mathfrak{N}_G(H)$; moreover, for every $x \in H$, $\langle x \rangle = \langle x, b \rangle \cap H \lhd \langle x, b \rangle$, i.e. $b$ is in the normalizer of every subgroup of $H$. Since the same conclusion holds for $ab$, it would follow that $a$ is in the kernel of $H$, so that $\langle a \rangle \breve{\mathfrak{S}} H$; by IV of section 1 this would imply $\langle a \rangle \breve{\mathfrak{S}} G$ and $H = \langle a \rangle$, which contradicts our assumption on the order of $H$. In the second case $\langle b\rangle$ and $\langle a\rangle$ are conjugate in $\langle b\rangle \cup H$; $b$ lies in a conjugate $H_1$ of $H$ $(H_1 \neq H)$ and $\langle b\rangle = (\langle b\rangle \cup H) \cap H_1 \breve{\mathfrak{D}} \langle b\rangle \cup H$, whence again $\langle a\rangle \breve{\mathfrak{D}} \langle b\rangle \cup H$, $\langle a\rangle \breve{\mathfrak{D}} H$ and $H = \langle a\rangle$, thus contradicting our hypothesis on $|H|$. We can now prove that $H \lhd G$: let $g \in G$ have order $q^n$ with $q$ a prime number; if $q \mid |H|$, by a previous remark $\langle g^{q^{n-1}}\rangle \subseteq H$ and $g \in \mathfrak{N}_G(H)$; if $q \nmid |H|$, for $a \in H$ of prime order we have $\langle g^{-1}ag\rangle \subseteq \subseteq \langle a, g\rangle \cap H = \langle a\rangle$, and by the same remark $g \in \mathfrak{N}_G(H)$. Moreover, in the latter case, for every $x \in H$ we get $\langle x\rangle = \langle x, g\rangle \cap H \lhd \langle x, g\rangle$ and, assuming $|x|$ to be a prime number, from $[x, g] \neq 1$ it would follow that $x$ too normalizes every subgroup of $H$, which clearly cannot happen; i.e. $g$ centralizes $H$. On the other hand in our hypothesis $Z(H) = 1 = = H \cap \mathfrak{C}_G(H)$, and if $(|g|, |H|) \neq 1$ then $\langle g\rangle \cap H \neq 1$; all this implies that $\mathfrak{C}_G(H)$ is exactly the set of all the elements of $G$ whose order is prime to $|H|$.

The above theorem does not cover the minimum $\breve{\mathfrak{D}}$-subgroups of prime order; they will be dealt with in the following

THEOREM 3.2. *Let $a \in G$ have prime order $p$. If $\langle a\rangle \breve{\mathfrak{D}} G$, then either* i) *$\langle a\rangle^G$ is an elementary abelian $p$-group, or* ii) *$G = S(N \times K)$, where $K = \mathfrak{C}_G(\langle a\rangle^G)$ is a Hall subgroup of $G$, $N$ is an elementary abelian $q$-group with $q$ a prime greater than $p$, $S$ is a $p$-Sylow subgroup of $G$ which is cyclic or generalized quaternion, and $\langle a\rangle^G = \langle a\rangle N$ is a P-group.*

Let us first show that if we can find in $G$ an element $b$ of order $p$ such that $\langle a\rangle \cap \langle b\rangle = 1$ but $[a, b] = 1$, then $a$ permutes with every element of order $p$ in $G$; hence it will follow that, if this is the case, $\langle a\rangle^G$ is elementary abelian. Thus, choose if possible $c \in G$ such that $|c| = p$, $[a, c] \neq 1$; by lemma 2.1 $\langle a, c\rangle = \langle a, d\rangle$ where $\langle d\rangle \lhd \langle a, c\rangle$ and $|d| = q$ (notice that $a$ and $c$ are conjugate). If $[b, c] = 1$, then $\langle a, b, c\rangle = \langle a, c\rangle \times \langle b\rangle$, $|db| = pq$, whereas $|adb| = p$ and lemma 2.1 imply that no elements of composite order are in $\langle a, adb\rangle$, so that we can assume $[b, c] \neq 1$. $\langle b\rangle$ is then conjugate to $\langle c\rangle$, whence $\langle b\rangle \breve{\mathfrak{D}} G$. If $b \in \mathfrak{N}_G(\langle d\rangle)$ we get $\langle a, c\rangle \lhd \langle a, b, c\rangle = \langle a, c\rangle \times \langle b'\rangle$, where $b'$ is a suitable element of $\langle a\rangle \times \langle b\rangle$, and the above technique leads to a contradiction. Lemma 2.2 now implies $\langle b, d\rangle = N\langle d\rangle$ with $N$ an elementary abelian normal $p$-subgroup of $\langle b, d\rangle$ which in turn is normal in $\langle a, b, c\rangle$; $a \notin N$, and for every $x \in N$ we have $\langle x\rangle = \langle x, a\rangle \cap N \lhd \langle x, a\rangle$, i.e. $a \in \mathfrak{C}_G(N)$. Hence $\langle a\rangle = \langle a, d\rangle \cap \mathfrak{C}_{\langle a,b,c\rangle}(N) \lhd \langle a, d\rangle$, thus contradicting an earlier statement. So far we proved that, if $\langle a\rangle^G$ is not an elementary abelian $p$-group, then $[a, b] \neq 1$ for every $b \in G$ such that $|b| = p$, $\langle a\rangle \cap \langle b\rangle = 1$; as a consequence, all $p$-Sylow subgroups of $G$ are either cyclic or generalized quaternion. We now proceed to show that for any pair $x$, $y$ of elements of $G$ such that $|x| = |y| = p$, $\langle x\rangle \cap \langle y\rangle = 1$, the subgroup $\langle x, y\rangle$ is non abelian and $|\langle x, y\rangle| = pq$, $q$ being independent from the choice of $x$, $y$; since there is in $G$ just one class of conjugate subgroups of order $p$, it is enough if we prove that $|b| = |c| = p$, $\langle a\rangle \cap \langle b\rangle = \langle a\rangle \cap \langle c\rangle = 1$ implies $|\langle a, b\rangle| = |\langle a, c\rangle|$. Let $u \in G$ be such that $\langle a, b\rangle = \langle a, u\rangle$, $|u| = q$, $\langle u\rangle \quad \langle a, b\rangle$; $c \in \mathfrak{N}_G(\langle u\rangle)$ (were this not the case, by lemma 2.2 two independent conjugates of $a$ would permute), hence $\langle u\rangle \lhd \langle a, b, c\rangle = \langle u\rangle\langle a, c\rangle$. Looking at $\langle a, c\rangle$, which by lemma 2.1 is also non abelian of order, say, $pr$, we see that $\langle a, c\rangle = \langle a, v\rangle$ where $|v| = r$, $\langle v\rangle \lhd \langle a, c\rangle$ and $\langle v\rangle = \langle a, c\rangle \cap \mathfrak{C}_{\langle a,b,c\rangle}(u)$, so that $\langle a, b, c\rangle = (\langle u\rangle \times \langle v\rangle)\langle a\rangle$. The subgroups $\langle au\rangle$, $\langle av\rangle$, being conjugate to $\langle a\rangle$, are dual-Dedekind in $G$; by lemma 2.1 no element of composite order lies in $\langle au, av\rangle$, hence $v^{-1}u \in \langle au, av\rangle$ has prime order: but then $q = r$ (notice that we have also proved that every element of order $p$ normalizes every subgroup of order $q$). By an easy induction argument one can now prove that any set of elements of order $p$ generates a $P$-group of order $pq^n$ for a suitable $n$, so that $\langle a\rangle^G$, which is generated by all such elements of $G$, is a $P$-group: $\langle a\rangle^G = \langle a\rangle N$, with $N$ an elementary abelian $q$-subgroup on which $a$ induces a non identity power automorphism. Our next step is to prove that for every pair $x$, $y$ of elements of $G$ such that $|x| = q^m$, $|y| = p$, one has $\langle x\rangle \lhd \langle x, y\rangle$; by an earlier remark we can assume $m > 1$ and use induction. $\langle x^q\rangle$ is then normal in $\langle y, x^q\rangle$; $\langle y, x^q\rangle/\langle x^q\rangle \breve{\mathfrak{D}} \langle y, x\rangle/\langle x^q\rangle$: if $|\langle y, x\rangle/\langle x^q\rangle| = pq$ we are through. If this is not the case, then $\langle y, x\rangle/\langle x^q\rangle = (\langle x\rangle/\langle x^q\rangle)(N/\langle x^q\rangle)$ with $N/\langle x^q\rangle$ an elementary abelian normal $p$-subgroup of $\langle y, x\rangle/\langle x^q\rangle$ (lemma 2.2); $\langle x^q\rangle$ is a $q$-Sylow subgroup of $N$, whence $N = M\langle x^q\rangle$ for a suitable elementary abelian $p$-subgroup $M$ containing $\langle y\rangle$. But then $M = \langle y\rangle$ and again $|\langle y, x\rangle/\langle x^q\rangle| = pq$. For a conjugate $b$ of $a$ such that $\langle b\rangle \cap \langle a\rangle = 1$ one has either $\langle x\rangle \cap \langle a, b\rangle \neq 1$, which implies $[a, x]\neq 1$; or $\langle x\rangle \cap \langle a, b\rangle = 1$, and $\langle x, a, b\rangle = (\langle x\rangle \times \langle u\rangle)\langle a\rangle$, where $\langle u\rangle$ is the $q$-Sylow subgroup of $\langle a, b\rangle$, and $a$ induces a non identity power automorphism on $\langle x\rangle \times \langle u\rangle$, whence again $[a, x]\neq 1$; but then $\langle x\rangle = \langle [a, x]\rangle \subseteq \langle a\rangle^G$, i.e. $N$ is the (unique) $q$-Sylow subgroup of $G$. Now put $K = \mathfrak{C}_G(\langle a\rangle^G)$; $K\cap \langle a\rangle^G = 1$ and, since a $p$-Sylow subgroup is either cyclic or generalized quaternion and its subgroup of order $p$ lies in $\langle a\rangle^G$, $K \subseteq \{g\in G \mid (|g|, pq) = 1\}$. On the other hand, if $(|g|, pq) = 1$, for every $y\in G$ with $|y| = p$ one has $\langle y\rangle = \langle y, g\rangle \cap \langle a\rangle^G \lhd \langle y, g\rangle$; therefore $g$ is in the normalizer of every subgroup of order $pq$ in $\langle a\rangle^G$: but this implies $[g, \langle a\rangle^G] = 1$, which concludes the proof of the theorem.

The following result is a trivial corollary to theorems 3.1, 3.2:

THEOREM 3.3. *Let $G$ be a finite group. If $G$ has non-trivial dual-Dedekind subgroups, then $G$ is not simple.*

REMARK. Finite non simple groups with no non-trivial $\breve{\mathfrak{D}}$-subgroups do exist: e.g. the symmetric group $S_n$ is such whenever $n>3$ (it is a simple matter to verify that no normal subgroup of $S_n$ satisfies the theorems 3.1, 3.2); the case $n=4$ provides an example of a soluble group which has no non-trivial $\breve{\mathfrak{D}}$-subgroups.

**4.** We have already pointed out that, generally speaking, normal subgroups need not be $\breve{\mathfrak{D}}$-subgroups; in order to evaluate, in a sense, the gap between these two classes we proceed to study the groups where every normal subgroup is also a $\breve{\mathfrak{D}}$-subgroup (in the main result of this section we restrict ourselves to soluble groups).

PROPOSITION 4.1. *Assume that every normal subgroup of the group $G$ is a $\breve{\mathfrak{D}}$-subgroup of $G$. If $N \trianglelefteq G$, then every normal subgroup of $G/N$ is a $\breve{\mathfrak{D}}$-subgroup of $G/N$.*

Thus, $K/N \trianglelefteq G/N$ implies $K \trianglelefteq G$, $K\breve{\mathfrak{D}}G$, hence $K\breve{\mathfrak{D}}[G/N]$ and obviously $K/N\breve{\mathfrak{D}}G/N$.

PROPOSITION 4.2. *Let $N$ be a minimum normal subgroup of $G$. If every normal subgroup of $G$ is also a $\breve{\mathfrak{D}}$-subgroup, then $N$ is simple.*

Assume first that $N$ is abelian; then $|N| = p^\alpha$ with $p$ a prime and $\alpha \geq 1$; the number $k$ of its subgroups of order $p$ is congruent to 1 (mod $p$). The normal subgroup $P=\cap\{\mathfrak{N}_G(H)\mid H\subseteq N,\ |H|=p\}$ contains every element of $G$ whose order is prime to $p$: thus, if $(|x|, p)=1$, then $\langle x\rangle\cap N=1$ and, for any such an $H$, $H=(H\cup\langle x\rangle)\cap N\lhd H\cup\langle x\rangle$. So $G$ acts as a $p$-group of permutations on the set of the $k$ subgroups of order $p$ in $N$, hence it has at least a fixed point, i.e. $|N|=p$. Assume now that $N$ is abelian; let $N_1$ be a simple direct factor of $N$. If $N_1\neq N$ and $x\in G$ is such that $x^{-1}N_1x\neq N_1$, then $N_1\times x^{-1}N_1x\subseteq$ $\subseteq(N_1\cup\langle x\rangle)\cap N=N_1(\langle x\rangle\cap N)$ and $x^{-1}N_1x$ would be isomorphic to a subgroup of $\langle x\rangle$, which is clearly not the case.

COROLLARY 4.3. *Let $G$ be a soluble group. If every normal subgroup of $G$ is a $\breve{\mathfrak{D}}$-subgroup of $G$, then $G$ is supersoluble.*

PROPOSITION 4.4. *Let $G$ be a nilpotent group. If $H\breve{\mathfrak{D}}G$, then $H$ is quasi-normal in $G$.*

This is a trivial consequence of a result of Napolitani, [1].

PROPOSITION 4.5. *Let $G$ be a $p$-group ($p$ a prime). If every normal subgroup of $G$ is a $\breve{\mathfrak{D}}$-subgroup, then $G$ is modular.*

For $u\in Z(G)$, with $|u|=p$, $G/\langle u\rangle$ is by induction a modular $p$-group. Assume that $G/\langle u\rangle$ is either abelian or Hamiltonian: for arbitrary $x\in G$, $\langle x, u\rangle$ is abelian, hence $\langle x\rangle\breve{\mathfrak{D}}\langle x, u\rangle$; moreover $\langle x, u\rangle\trianglelefteq G$ implies $\langle x, u\rangle\breve{\mathfrak{D}}G$ and $\langle x\rangle\breve{\mathfrak{D}}G$; by proposition 4.4 $\langle x\rangle$ is a quasi-normal subgroup of $G$, i.e. $G$ is modular. We may then assume that $G/\langle u\rangle$ is neither abelian nor Hamiltonian, so that $G=\langle t, A\rangle$ with $u\in A$, $A/\langle u\rangle$ abelian, $t^{-1}at=a^{1+p^s}u^{\alpha(a)}$ for every $a\in A$ and suitable $\alpha(a)$, $s\geq 2$ if $p=2$ ([3], p. 13). Just as before one sees that every subgroup of $A$ is dual-Dedekind, whence quasi-normal, in $G$; it follows that $A$ is a modular group. Moreover $A^p=\{a^p\mid a\in A\}$ is a subgroup of $A$, any of whose subgroups is normalized by $t$; $t^p$ normalizes every subgroup of $A$, inducing on every cyclic subgroup a power automorphism which is congruent to 1 (mod. $p$), and congruent to 1 (mod. 4) if $p=2$. $A$ cannot be a Hamiltonian group: thus, if $A=Q\times B$ with $Q$ a quaternion group of order 8 and $B^2=1$, from $u\in Q$ it would follow that $G/\langle u\rangle$ is abelian, whereas, if $u\notin Q$, $G/\langle u\rangle$ would be a modular 2-group containing a quaternion group, and $G/\langle u\rangle$ would be a Hamiltonian group. There are two cases left:

i) $A$ is abelian. By a previous remark, $\langle t^p, A\rangle$ is modular and all its subgroups are quasi-normal in $G$. Let $y\in G$ be such that $y\notin\langle t^p, A\rangle$, so that $G=A\langle y\rangle$. If $\langle y\rangle\cap A\neq 1$, since $\langle y\rangle\cap A\trianglelefteq G$, then by induction $G/\langle y\rangle\cap A$ is modular, hence $\langle y\rangle$ is quasi-normal in $G$. Assume now $\langle y\rangle\cap A=1$: for every $a\in A$ we get $\langle a\rangle=\langle a, y\rangle\cap A\triangleleft\langle a, y\rangle$, i.e. $y$ induces a power automorphism on the abelian group $A$, which is congruent to 1 (mod. $p$). If $p\neq 2$ there is nothing more to prove; if $p=2$ we remark that, if we had $A^4=1$, $G/\langle u\rangle$ would be abelian; hence $A^4\neq 1$, $G/A^4$ is by induction a modular group, and the power induced by $y$ is congruent to 1 (mod. 4), which implies that $G$ is modular.

ii) $A$ is neither abelian nor Hamiltonian. We have $A=\langle v, B\rangle$, $B$ abelian, $v^{-1}xv=x^n$ with $n\equiv 1$ (mod. $p$) for every $x\in B$ and $n$ independent from the choice of $x$ ($n\equiv 1$ (mod. 4) if $p=2$; we remark here that $B^4\neq 1$, otherwise $A$ would be abelian). $A^p\subseteq Z(A)$, hence every subgroup of $A^p$ is normal in $G$; both of $A/A^p$ and $A/B$ are abelian, so that $\langle u\rangle=A'\subseteq A^p\cap B$; moreover, we can write $B$ as $B=\langle b\rangle\times B_1$ where $u\in\langle b\rangle$, $\exp B_1<|b|$ and $|b|\geq 8$ if $p=2$. We will show that $\langle g_1, g_2\rangle=$ $=\langle g_1\rangle\langle g_2\rangle$ for every pair $g_1$, $g_2$ of elements of $G$ (without loss of generality, we can assume $g_i\notin A$, since every subgroup of $A$ is quasi-normal in $G$). Write $\langle g_1\rangle=\langle a_1t^{p^h}\rangle$, $\langle g_2\rangle=\langle a_2t^{p^k}\rangle$; assuming $0\leq h\leq k$ we get $g_2\in A\langle g_1\rangle$, $\langle g_1, g_2\rangle=\langle g_1, a\rangle$ for suitable $a_1$, $a_2$, $a\in A$. Should $\langle g_1\rangle$ contain a non-identity normal subgroup $K$ of $G$, since $G/K$ would be a modular group by the induction hypothesis, then $\langle g_1\rangle$ would be quasi-normal in $G$; hence we can assume $\langle g_1\rangle\cap A^p=1$, which implies $u\notin\langle g_1\rangle$. Suppose $\langle g_1\rangle\cap A=1$; then $\langle a\rangle=\langle a, g_1\rangle\cap A\triangleleft\langle a, g_1\rangle$, and, if $p\neq 2$, $\langle a, g_1\rangle$ is modular, whence $\langle g_1, g_2\rangle=\langle g_1\rangle\langle g_2\rangle$. Under the same assumptions, but with $p=2$, $g_1$ induces a power automorphism on the abelian group $B$; $G/B^4$ being modular, this power is congruent to 1 (mod. 4), so that if $a\in B$ then $\langle a, g_1\rangle$ is modular. Let now $a\notin B$; $u\in\langle a, g_1\rangle$ if and only if $u\in\langle a\rangle$, hence if either $u\notin\langle a\rangle$ or $u\in\langle a^4\rangle$ we again conclude that $\langle a, g_1\rangle$ is modular; we are left with one more possibility: $u=a^2=b^{2^l}$; but $[g_1, b^{2^{l-1}}]=1$ (for $\langle g_1, b\rangle$ is modular), $[g_1, ab^{2^{l-1}}]=1$ since $|ab^{2^{l-1}}|=2$, so that $\langle g_1, a\rangle$ is abelian. Assume now $1=\langle g_1\rangle\cap A^p\subset$ $\subset\langle g_1\rangle\cap A=\langle c\rangle$ with $|c|=p$, $u\in\langle a, g_1\rangle\cap A=\langle a, c\rangle$; if $|a|=p$ then $\langle g_1\rangle\triangleleft\langle a, g_1\rangle=\langle g_1, g_2\rangle=\langle g_1\rangle\langle g_2\rangle$; if $|a|>p$ but $u\notin\langle a\rangle$ we should have $c\in\langle a\rangle\times\langle u\rangle$, whence $c\in\langle a^p\rangle\times\langle u\rangle\subseteq A^p$, contradicting an earlier hypothesis. We have then $|a|>p$, $u\in\langle a\rangle$: so $\langle a\rangle\trianglelefteq G$ and, if either $p\neq 2$ or $p=2$, $u\in\langle a^4\rangle$, $\langle g_1, a\rangle$ is modular. It follows that we are left with one last case: $p=2$, $u=a^2=b^{2^l}$. Since $\langle b\rangle\lhd G$, $u\in\langle b^4\rangle$ and $G/\langle u\rangle$ is modular, we see that $\langle g_1, b\rangle$ is also modular, whence $[g_1, b^{2^{l-1}}]=1$; if $a\in\langle b\rangle$, $\langle g_1, a\rangle$ is abelian, whereas, if $a\notin\langle b\rangle$, $|a^{-1}b^{2^{l-1}}|=2$, $a^{-1}b^{2^{l-1}}\in\mathfrak{N}_G(\langle g_1\rangle)$ and finally $\langle g_1\rangle\lhd\langle g_1, g_2\rangle\subseteq\langle g_1, b^{2^{l-1}}, a^{-1}b^{2^{l-1}}\rangle$, which disposes of the case and ends the proof.

THEOREM 4.6. *The group $G$ is soluble and every normal subgroup of $G$ is dual-Dedekind in $G$ if and only $G=H_1\times H_2\times\ldots\times H_t$ with $H_i$ a Hall subgroup of $G$ ($i=1, \ldots, t$) and either*

1) *$H_i$ is a modular p-group; or*

2) *$H_i=(P_{i1}\times\ldots\times P_{is_i})Q_i$ with $P_{ij}$, $Q_i$ Sylow subgroups of $G$ for different primes, $P_{ij}$ abelian of odd order ($j=1, \ldots, s_i$), $Q_i=\langle b_i\rangle$, and $b_i$ inducing a non identity power automorphism on each $P_{ij}$.*

PROOF OF NECESSITY. Assume $S$, a $p$-Sylow subgroup of $G$ for some prime $p$, is normal in $G$; then, unless $S$ is a direct factor of $G$, $S\subseteq\Gamma_\infty(G)$ where $\Gamma_\infty(G)$ denotes the intersection of all normal subgroups of $G$ whose factor group is nilpotent. Thus $S\bar{\mathfrak{D}}G$ and for $a\in S$, $x\in G$ such that $(|x|, p)=1$ we have $\langle a\rangle=\langle a\rangle\cup(\langle x\rangle\cap S)=\langle a, x\rangle\cap S\lhd\langle a, x\rangle$; if $S$ is not a direct factor of $G$, we can choose $a$, $x$ such that $[a, x]\neq 1$, but then $\langle[a, x]\rangle=\langle a\rangle$ and $a$ also induces a power automorphism on $S$. Let now $b$ be arbitrary in $S$; if $[b, x]\neq 1$ the above argument shows that $b$ operates on $S$ as a power automorphism, whereas if $[b, x]=1$ we have $[ab, x]\neq 1$ and the same conclusion holds for $ab$, hence for $b$. It follows that $S$ is abelian of odd order, $x^{-1}yx=y^r$ with $r\not\equiv 1$ (mod. $p$), $r$ independent from the choice of $y\in S$, $[G, S]=S$ and $S\subseteq\Gamma_\infty(G)$. Choosing for $p$ the maximum prime divisor of $|G|$, by the supersolubility of $G$ the $p$-Sylow subgroup is certainly normal, so that an easy induction proves that $\Gamma_\infty(G)$ is a Hall subgroup of $G$. Moreover $G$ has a normal 2-complement whose quotient group is clearly nilpotent, so that $|\Gamma_\infty(G)|$ is odd; again, by the supersolubility of $G$, $\Gamma_\infty(G)$ is nilpotent, hence it is a direct product of normal Sylow subgroups of $G$ which are all abelian by the preceding remark, and every element of $G$ operates by conjugation on $\Gamma_\infty(G)$ as a power automorphism. $G/\Gamma_\infty(G)$ is a direct product of modular $p$-groups for different primes; notice that every Sylow subgroup of $G$ which is a direct factor has trivial intersection with $\Gamma_\infty(G)$, and is modular; therefore, we can factor out all such subgroups, and write $G=T\times G_1$ with $T$ a modular, nilpotent, Hall subgroup of $G$ and $G_1$ also satisfying all our assumptions; from now on we shall assume $G=G_1$. Let $P$ be a normal Sylow subgroup of $G$; we have already seen that $P\subseteq\Gamma_\infty(G)$ and that every element of $G$ operates on $P$ as a power automorphism; we claim that $G/\mathcal{C}_G(P)$ is a (cyclic) group of prime power order. Deny: then there are a $q$-Sylow subgroup $Q$ and an $r$-Sylow subgroup $R$ of $G$ such that $[Q, R]=1$, $Q\cap\Gamma_\infty(G)=$ $=R\cap\Gamma_\infty(G)=1$, $[Q, P]=[R, P]=P$; choose $a\in Q$, $b\in R$, $u\in P$ such that $[a, P]\neq 1$, $[b, P]\neq 1$, $|u|=p$ $(p\mid |P|)$. The Hall subgroup $Q\Gamma_\infty(G)$ is normal, hence dual-Dedekind, in $G$, which is a contradiction to $\langle au\rangle=$ $=\langle au\rangle\cup(\langle b\rangle\cap Q\Gamma_\infty(G))\neq(\langle au\rangle\cup\langle b\rangle)\cap Q\Gamma_\infty(G)$ (this owing to the fact that the former group has $q$-power order, whereas the latter contains $\langle u\rangle=\langle[au, b]\rangle$ which has order $p$). Therefore we get $G=Q\mathcal{C}_G(P)$ for a suitable $q$-Sylow subgroup $Q$ of $G$; we shall now prove, by induction on $q^\beta=|Q|$, that $Q$ is cyclic. Without loss of generality we can assume $P=\Gamma_\infty(G)$ (were this not the case, we would work on $G/C$ with $C$ the complement of $P$ in $\Gamma_\infty(G)$). If $Q\cap\mathcal{C}_G(P)=1$, since $G/\mathcal{C}_G(P)$ is cyclic, then $Q$ is also cyclic. Assume then $Q\cap\mathcal{C}_G(P)\neq 1$; $\mathcal{C}_G(P)\cap Z(Q)$ is a non-trivial normal subgroup of $G$ and by the inductive hypothesis $Q/\mathcal{C}_G(P)\cap Z(Q)$ is cyclic; therefore $Q$ is abelian and all subgroups of $QP$ containing $P$ are normal, hence dual-Dedekind subgroups of $G$. If now $Q$ were not cyclic we could pick $a$ and $b$ in $Q$ in such a way that $a\notin\mathcal{C}_G(P)$, $a^q\in\mathcal{C}_G(P)$, $b\in Q$, $|b|=q$, $[b, P]=1$, $\langle a\rangle\cap\langle b\rangle=1$; for $u\in P$ with $|u|=p$ we would have $\langle au\rangle=\langle au\rangle\cup(\langle ab\rangle\cap\langle a\rangle P)=$ $=(\langle au\rangle\cup\langle ab\rangle)\cap\langle a\rangle P\supseteq\langle[au, ab]\rangle=\langle[u, a]\rangle=\langle u\rangle$ i.e. $[u, a]=1$ contrary to our choice of $a$. Now let $Q_1$ be a non normal Sylow subgroup of $G$, and let $P_{11}, P_{12}, \ldots, P_{1s_1}$ be those Sylow subgroups of $\Gamma_\infty(G)$ which are not centralized by $Q_1$; $H_1=(P_{11}\times\ldots\times P_{1s_1})Q_1$ is a direct factor of $G$, and if $G=H_1$ the theorem is proved. Assume $G\neq H_1$; let $Q_2$ be a normal Sylow subgroup of $G$, not contained in $H_1$, and let $P_{21}, \ldots, P_{2s_2}$ be those Sylow subgroups of $\Gamma_\infty(G)$ which are not centralized by $Q_2$: $H_2=(P_{21}\times\ldots\times P_{2s_2})Q_2$ is also a direct factor of $G$, and $H_1\cap H_2=1$; in this way we clearly get a decomposition of $G$ as a direct product of factors of the prescribed type.

PROOF OF SUFFICIENCY. Since such a decomposition as is described in the theorem is both group- and lattice-theoretical, it will be enough if we prove the theorem for each one of the factors (nothing is to be proved for the modular ones). Without loss of generality, we can assume $G=(P_1\times\ldots\times P_s)Q$ where the $P_i$'s and $Q$ are Sylow subgroups of $G$, $Q$ is cyclic, $P_i$ is abelian of odd order ($i=1, \ldots, s$) and $Q$ operates on $P_1\times\ldots\times P_s$ as a group of power automorphisms, with $\mathfrak{C}_Q(P_i)\neq Q$. Let $H\trianglelefteq G$; we renumber the $P_i$'s so hat $[H, P_i]=P_i$ for $i=1, \ldots, r$ and $[H, P_i]=1$ for $i=r+1, \ldots, s$. We shall prove that $\varphi^K: X\to X\cup K$ ($\varphi^K: [H/H\cap K]\to[HK/K]$) and $\varphi_H: Y\to Y\cap H$ ($\varphi_H: [HK/K]\to\to[H/H\cap K]$) are inverse lattice isomorphisms, whenever $K$ is a subgroup of $G$; since $H\trianglelefteq G$, we have only to prove that $X\varphi^K\varphi_H=X$ for every $X\in[H/H\cap K]$. Assume first that

$$K\subseteq(P_1\times\ldots\times P_s)H=(H\cap Q)(P_1\times\ldots\times P_r)\times(P_{r+1}\times\ldots\times P_s);$$

we have

$$K=(K\cap(H\cap Q)(P_1\times\ldots\times P_r))\times(K\cap(P_{r+1}\times\ldots\times P_s))=(H\cap K)L$$

with $L=K\cap(P_{r+1}\times\ldots\times P_s)\trianglelefteq G$. We have $H\cup K=H\cup(H\cap K)\cup L==H\cup L$ and, for every $X\in[H/H\cap K]$,

$$X\varphi^K\varphi_H=(X\cup K)\cap H=(X\cup(H\cap K)\cup L)\cap H=$$
$$=(X\cup L)\cap H=X\cup(L\cap H)=X.$$

Assume now that $K\nsubseteq(P_1\times\ldots\times P_s)H$; there exists a $q$-Sylow subgroup $T$ of $G$ with $T\cap K$ $q$-Sylow in $K$; we have $T\cap H\subseteq T\cap K$. If we call $M=H\cap(P_1\times\ldots\times P_s)$, then $H=M(T\cap H)=M(H\cap K)$; notice that, since every subgroup of $M$ is normal in $G$, $M\breve{\mathfrak{D}}G$. Now for every $X\in[H/H\cap K]$ we get $X=(X\cap M)\cup(H\cap K)$ and

$$(X\cup K)\cap H=X\varphi^K\varphi_H=((X\cap M)\cup(H\cap K)\cup K)\cap H=$$
$$=((X\cap M)\cup K)\cap H=(X\cap M)\cup(H\cap K)=X,$$

thus ending our proof.

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