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## Some applications of Saalschütz's theorem


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# SOME APPLICATIONS OF SAALSCHÜTZ'S THEOREM

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**1.** Put

$$(1)\qquad {}_rF_s\left[\begin{matrix} a_1\,,\ a_2\,,\ \ldots,\ a_r \\ b_1\,,\ b_2\,,\ \ldots,\ b_s \end{matrix}\right] = \sum_{\infty}^{k=0} \frac{(a_1)_k(a_2)_k\ \ldots\ (a_r)_k}{k!(b_1)_k\ \ldots\ (b_s)_k},$$

where

$$(a)_k = a(a+1)\ \ldots\ (a+k-1), \qquad (a)_0 = 1.$$

The formula

$$(2)\qquad {}_3F_2\left|\begin{matrix} & c, & d \\ -n, & a, & b \end{matrix}\right| = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

where $n$ is a nonnegative integer and

$$(3)\qquad c+d = -n+a+b+1,$$

is Saalschütz's theorem [1, p. 9].

Changing the notation slightly, (2) becomes

$$(4)\qquad \sum_{j=0}^{k} \frac{(-k)_j(a+k)_j(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j} = \frac{(a-b)_k(a-c)_k}{(b+1)_k(c+1)_k}.$$

Then

$$\sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!(b+1)_k(c+1)_k} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \frac{(-k)_j(a)_{j+k}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j}$$

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$$=\sum_{j=0}^{\infty}(-1)^j\frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j}x^j\sum_{k=0}^{\infty}\frac{(a+2j)_k}{k!}x^k.$$

Since

$$\sum_{k=0}^{\infty}\frac{(a+2j)_k}{k!}x^k=(1-x)^{-a-2j},$$

we get the identity

$$(5)\quad \sum_{k=0}^{\infty}\frac{(a)_k(a-b)_k(a-c)_k}{k!(b+1)_k(c+1)_k}x^k=\sum_{j=0}^{\infty}(-1)^j\frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j}x^j(1-x)^{-a-2j}.$$

In particular if we take $a=-2n$, $x=1$, (5) becomes

$$(6)\quad \sum_{k=0}^{2n}\frac{(-2n)_k(-2n-b)_k(-2n-c)_k}{k!(b+1)_k(c+1)_k}=(-1)^n\frac{(2n)!(b+c+2n+1)_n}{n!(b+1)_n(c+1)_n}$$

In the hypergeometric notation (1), we have

$$(7)\quad {}_3F_2\left[\begin{matrix}-2n, & -2n-b, & -2n-c\\ & b+1, & c+1\end{matrix}\right]=(-1)^n\frac{n!(b+1)_n(c+1)_n}{(2n)!(b+c+2n+1)_n},$$

which is the finite version of Dixon's theorem [1, p. 13].

We remark that, for $x=-1$, (5) reduces to

$$(8)\quad \sum_{k=0}^{\infty}(-1)^k\frac{(a)_k(a-b)_k(a-c)_k}{j!(b+1)_j(c+1)_j}=\sum_{j=0}^{\infty}\frac{(a)_{2j}(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j}2^{-a-2j}.$$

Also, for $a=-n$, $b=c=0$, (5) becomes

$$(9)\quad \sum_{k=0}^{\infty}\binom{n}{k}^3x^k=\sum_{2j\le n}\frac{(j!)^3(n-2j)!}{(n+j)!}x^j(1+x)^{n-2j}.$$

This formula, attributed to Foata, is proved in an entirely different way in [3, p. 41]. The special case

$$(10)\quad \sum_{k=0}^{\infty}\binom{n}{k}^3=\sum_{2j\le n}\frac{(n+j)!}{(j!)^3(n-2j)!}2^{n-2j}$$

is due to MacMahon.

**2.** In the next place we consider the sum

(11) $$S=\sum_{k=0}^{\infty}\frac{k!(b+1)_k(c+1)_k(d+1)_k(e+1)_k}{(a)_k(a-b)_k(a-c)_k(a-d)_k(a-e)_k}=$$

$$={}_5F_4\left[\begin{matrix}a,\ a-b,\ a-c,\ a-d,\ a-e\\ b+1,\ c+1,\ d+1,\ e+1\end{matrix}\right].$$

By (4) we have

$$S=\sum_{k=0}^{\infty}\frac{(a)_k(a-d)_k(a-e)_k}{k!(d+1)_k(e+1)_k}\sum_{j=0}^{k}\frac{(-k)_j(a+k)_j(-a+b+c+1)_j}{j!(b+1)_j(c+1)_j}=$$

$$S=\sum_{j=0}^{\infty}(-1)^j\frac{(a)_{2j}(-a+b+c+1)_j(a-d)_j(a-e)_j}{j!(b+1)_j(c+1)_j(d+1)_j(e+1)_j}\cdot$$

$$\cdot\sum_{k=0}^{\infty}\frac{(a+2j)_k(a-d+j)_k(a-e+j)_k}{k!(d+j+1)_k(e+j+1)_k}.$$

If we take $a=-2n$ the inner sum is

$${}_3F_2\left[\begin{matrix}-2n+2j,\ -2n-d+j,\ -2n-e+j\\ d+j+1,\quad e+j+1\end{matrix}\right]$$

which, by (7), is equal to

$$(-1)^{n-j}\frac{(2n-2j)!(d+e+2n+1)}{(n-j)!(d+j+1)_{n-j}(e+j+1)_{n-j}}.$$

Thus

$$S=(-1)\sum_{j=0}^{n}\frac{(-2n)_{2j}(b+c+2n+1)_j(-2n-d)_j(-2n-e)_j}{j!(b+1)_j(c+1)_j(d+1)_j(e+1)_j}\cdot$$

$$\cdot\frac{(2n-2j)!(d+e+2n+1)_{n-j}}{(nj)!(d+j+1)_{n-j}(e+j+1)_{n-j}}=$$

$$=(-1)^n\sum_{j=0}^{n}\frac{(2n)!(b+c+2n+1)_j(-2n-d)_j(-2n-e)_j(d+e+2n+1)_{n-j}}{j!(n-j)!(b+1)_j(c+1)_j(d+1)_n(e+1)_n}=$$

$$=(-1)^n\frac{(2n)!(d+e+2n+1)_n}{n!(d+1)_n(e+1)_n}.$$

$$\cdot \sum_{0=!}^{n} \frac{(-n)_j(b+c+2n+1)_j(-2n-d)_j(-2n-e)_j}{j!(b+1)_j(c+1)_j(-d-e3-n)_j}.$$

If we assume that

$$b+c+d+e+5n+1=0, \tag{12}$$

the last sum reduces to

$$\sum_{j=0}^{n} \frac{(-n)_j(-2n-d)_j(-2n-e)_j}{j!(b+1)_j(c+1)_j}. \tag{13}$$

In view of (12), we can sum (13) by means of (2) to get

$$\frac{(b+d+2n+1)_n(c+d+2n+1)_n}{(b+1)_n(c+1)_n}.$$

We have therefore

$$S=(-1)^n \frac{(2n)!(b+d+2n+1)_n(c+d+2n+1)_n(d+e+2n+1)_n}{n!(b+1)_n(c+1)_n(d+1)_n(e+1)_n}, \tag{14}$$

provided (12) is satisfied.

We may restate this result in the following form.

(15
$$ {}_5F_4\left[\begin{matrix} a, \ a-b, \ a-c, \ a-d, \ a-e \\ b+1, \ c+1, \ d+1, \ e+1 \end{matrix}\right] =$$

$$=(-1)^n\left[\frac{(2n)!(b+d+2n+1)_n(c+d+2n+1)_n(d+e+2n+1)_n}{n!(b+1)_n(c+1)_n(d+1)_n(e+1)_n}\right]=$$

$$=(-1)^n \frac{(2n)!(b+e+2n+1)_n(c+e+2n+1)_n(d+e+2n+1)_n}{n!(b+1)_n(c+1)_n(d+1)_n(e+1)_n},$$

where $a=-2n$ and

$$b+c+d+e+5n+1=0.$$

In particular, for $b=c=d=0$, $e=-5n-1$, (15) reduces to

$$(16) \quad {}_5F_4\left[\begin{matrix} -2n, & -2n, & -2n, & -2n, & 3n+1 \\ & 1, & 1, & 1, & -5n \end{matrix}\right] = (-1)^n \frac{((3n)!)^3(4n)!}{(n!)^4((2n)!)^2(5n)!}.$$

If $p=5n+1$ is prime, then (16) implies the following curious result.

$$(17) \qquad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^5 \equiv -\frac{1}{[n!(2n)!]^5} \pmod{p}.$$

For other congruences of this kind see [2].

## REFERENCES

[1] Bailey, V. N.: *Generalized hypergeometric series*, Cambridge, 1935.

[2] Carlitz, L.: *Some hypergeometric congruences*, Portugaliae Mathematica, vol. 12 (1953), pp. 119-128.

[3] Riordan, J.: *Combinatorial Identities*, New York, 1968.

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