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ON A PROBLEM OF J. CZIPSZER

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L. Moser and M. G. Murdeshwar [1] attribute the following problem to J. Czipszer: Let $A_0=\{a_1, a_2, ..., a_n\}$ where $a_1<a_2< ... <a_n$ are arbitrary integers and let A_k denote the translated set

$$A_k=\{a_1+k,\ a_2+k,\ ...,\ a_n+k\},$$

k any arbitrary non-zero integer. Denote by M_k the number of new elements generated by the shift of A_0 by k, i.e., M_k is the number of elements in $A_k \cap A_0^c$, A_0^c being the complement of A_0 with respect to the integers and, finally, let $M=\min\limits_{A_0}\max\limits_{0<|k|\leqq n} M_k$. The problem is to find or estimate M. Czipszer proved $n/2\leqq M\leqq 2n/3$ and conjectured that $M=2n/3$.

Our concern here is with the lower bound of M. We present two elementary methods. The first is based on a probabilistic interpretation of the problem. Although this derivation does not yield an essential improvement of the lower bound of Czipszer it is presented because the method in itself seems to be of some interest. The second consists of a simple counting process.

The first method. Let $f_k(x)=1$ if $x\in A_k$ and $f_k(x)=0$ if $x\notin A_k$, $k=0, \pm 1, \pm 2, ..., \pm n$. Then clearly the number d of elements in $A_0 \cap A_k$ equals

$$\sum_j f_0(j)f_k(j)=\sum_j f_0(j-k)f_0(j)=\sum_j f_0(j+k)f_0(j),$$

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the last step following from the symmetry of shifts to the right or to the left. Then

$$M_k = n - \sum_j f_0(j+k) f_0(j)$$

and the problem consists of finding

$$\max_{A_0} \min_{0<|k|\leq n} \sum_j f_0(j+k) f_0(j). \tag{1}$$

Consider two independent random variables X, Y each taking on the values in A_0 with equal probability:

$$\text{Prob}\{X=j\} = \text{Prob}\{Y=j\} = \frac{f_0(j)}{n}.$$

Let $Z=X-Y$. Then we have

$$\text{Prob}\{Z=k\} = \frac{1}{n^2} \sum_j f_0(j+k) f_0(j).$$

We shall now obtain an upper bound for $\min_{0<|k|\leq n} \text{Prob}\{Z=k\} \equiv m$, say. Z is a symmetric random variable with range

$$D^* = \{a_j - a_i,\ i,\ j=1,\ 2,\ \ldots,\ n\}.$$

Further,

$$\text{Prob}\{Z=0\} = \frac{1}{n}$$

and

$$\text{Prob}\{Z=\pm d\} = \begin{cases} \dfrac{v(d)}{d^2} & \text{if } d \in D^* \\ 0 & \text{if } d \notin D^* \end{cases}$$

where $v(d)$, $d \in D^*$, is the multiplicity of d, i.e., $v(d)$ is the number of distinct pairs (i, j) such that $a_j - a_i = d$. By virtue of the symmetry of Z, and the obvious fact that the minimum multiplicity cannot occur for

$Z=0$, we may restrict our attention to $D=\{d_{ij}=a_j-a_i\,,\ j>i\}$; D is a set with possibly repeated members and clearly $D\subset D^*$. The problem is to find that $k\in D$, $k\leqq n$, so that k has least probability:

$$m\leqq \frac{\text{Prob}\,\{0<Z\leqq n\}}{n}. \tag{2}$$

A generalized form of Markov's inequality ([2], p. 157) states: If g is a non-negative function, even and non-decreasing on $[0,\ \infty)$, then for all $a\geqq 0$ and any random variable Z the following inequality holds

$$\frac{E\{g(Z)\}-g(a)}{\sup g(Z)}\leqq \text{Prob}\,\{|\,Z\,|\geqq a\}\leqq \frac{E\{g(Z)\}}{g(a)} \tag{3}$$

where E is the expectation. Choose $g(z)=|\,z\,|$ and $a=n+1$ in (3) to obtain

$$\frac{E\{|\,Z\,|\}-(n+1)}{\max\,|\,Z\,|}\leqq \text{Prob}\,\{|\,Z\,|\geqq n+1\}\leqq \frac{E\{|\,Z\,|\}}{n+1}. \tag{4}$$

Since

$$\text{Prob}\,\{|\,Z\,|\geqq n+1\}=1-\text{Prob}\,\{Z=0\}-2\,\text{Prob}\,\{0<Z\leqq n\}$$

from (4) follows

$$\frac{n-1}{2n}-\frac{E\{|\,Z\,|\}}{2(n+1)}\leqq \text{Prob}\,\{0<Z\leqq n\}\leqq \frac{n-1}{2n}-\frac{E\{|\,Z\,|-(n+1)\}}{2(a_n-a_1)}.$$

Without loss of generality, we may take $a_1=0$. Thus

$$m\leqq \frac{\text{Prob}\,\{0<Z\leqq n\}}{n}\leqq \frac{n-1}{2n^2}-\frac{E\{|\,Z\,|\}-(n+1)}{2na_n}. \tag{5}$$

The random variable Z takes on the values ± 1, ± 2, ..., $\pm n$, each at least m times. $|\,Z\,|$ takes on the values 1, 2, ..., m, each at least $2m$ times. The total number of elements of D^* including duplicates is n^2.

Therefore there remain $n^2-n-2mn$ values of $|Z|$ each of which is at least 1. Thus we obtain

$$(6) \qquad E\{|Z|\} \geqq \frac{2m}{n^2} \sum_{j=1}^{n} j + \frac{n^2-n-2mn}{n^2} = \frac{n-1}{n}(m+1).$$

Combining (5) and (6) gives

$$m \leqq \frac{n-1}{2n^2} - \frac{1}{2na_n} \cdot \frac{n-1}{n}(m+1) + \frac{n+1}{2na_n}.$$

From this we obtain

$$(7) \qquad m \leqq \inf_{a_n} \left(\frac{n-1}{2n^2} + \frac{1}{2a_n} + \frac{1}{2n^2 a_n} \right) = \frac{n-1}{2n^2},$$

which gives us the bound $M \geqq \dfrac{n+1}{2}$.

The second method. Let $v = \min_{d \in N} v(d)$ where N denotes the set $\{1, 2, \ldots, n\}$. Then v is the smallest number of times that D covers N. To sharpen the result of our first method we need the following

LEMMA. If $d_{ij} \in D$, then $v(d_{ij}) \leqq n-(j-i)$.

PROOF. Display D as follows

$$(8) \qquad \begin{array}{llll} d_{12} & & & \\ d_{13} & d_{23} & & \\ d_{1i} & d_{2i} & \ldots\ldots & d_{i-1,i} \\ d_{1n} & d_{2n} & \ldots\ldots\ldots\ldots & d_{n-1,n}\,. \end{array}$$

Now each element is strictly greater than its neighbor above or to its right. Therefore, d_{ij} is different from any element in the same row and in the same column. And d_{ij} can't be equal to an element in the quadrants in the upper right or lower left obtained from (8) by removing the j-th row and the i-th column. In the top left quadrant d_{ij} can be equal to at most one entry per column, otherwise d_{ij} would be equal to two elements in the same row, which is not possible. Similarly, in the bottom right quadrant d_{ij} equals at most one element per row. From this we obtain

$$v(d_{ij}) \leqq (i-1)+(n-j)+1.$$

Now let $n=2p$, p a fixed integer, and suppose that $v=p-2$, $s \leqq p$. D must contain at least $2p(p-s)$ elements which are not necessarily distinct. The total number of elements of D is $\frac{(2p-1)2p}{2}$. If $j-i>p+s$, then by our Lemma $v(d_{ij})<p-s$. Such entries appear in a lower left triangle of entries in (8), and there are $\frac{(p-s)(p-s-1)}{2}$ of them. Now we wish to obtain an upper bound for s. From the above follows that there can't be s satisfying the inequality

$$2p(p-s)>\frac{(2p-1)2p}{2}-\frac{(p-s-1)(p-s)}{2}$$

or, equivalently, satisfying

$$p^2+p+s^2+s-6ps>0. \tag{9}$$

The roots of the equation

$$p^2+p+s^2+s-6ps=0$$

are

$$s=\frac{(6p-1)\pm\sqrt{32p^2-16p+1}}{2}.$$

The larger root is clearly greater than p, and can therefore be disregarded. Hence we must have

$$s<\frac{6p-1-\sqrt{32p^2-16p+1}}{2},$$

from which we get, in particular, that $p>13$ implies $s \leqq 0.2p$.

For $n=2p-1$, the condition corresponding to (9) is

$$s^2+3s+p^2+3p-6ps-2>0,$$

from which we are getting the same bound for s just as in the case of even n. Therefore, in either case we have the following

THEOREM. $v \leqq p-0.2p=0.8p=0.4n$

$$M \geqq 0.6n, \text{ for } n \geqq 26.$$

Comments.

1. No improvement of the bound is obtained if in our first method $g(z)$ is chosen to be $|z|^r$ or $\dfrac{|z|^r}{1+|z|^r}$, $r>0$, or simple truncations of these functions.

2. In [1], Moser and Murdeshwar generalize Çzipszer's problem to density functions. To obtain their result they use the theory of characteristic functions.

REFERENCES

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