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KRIPKE MODELS AND MANIVALUED LOGICS, II

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0. Introduction.

In November 1969, the author had the opportunity of giving a talk on the semantics of manivalued logics, at the International Symposium on Model Theory in Rome, Istituto Nazionale di Alta Matematica. The material underlying the first part of this talk (The semantics of the propositional calculus) was later published in [1]; the present paper covers the material underlying the second part of the same talk, namely the semantics of the predicate calculus of the first order, and of course makes free use of [1]'s results.

By and large, arguments in this paper will be patterned (as far as possible) after their counterparts in [1], with minor changes in notation. In particular, the symbol N will always denote *the set of all positive integers*, the symbol $\mathfrak{B}$ will denote *the elementary boolean lattice* $\{0, 1\}$ with $0<1$, and the symbols $\sim$, $\cap$, $\cup$, $>$ will denote, respectively, *complementation*, *meet operation*, *join operation* and *implication* in $\mathfrak{B}$, according to the tables:

	$\sim$
0	1
1	0

,

$\cap$	0	1
0	0	0
1	0	1

,

$\cup$	0	1
0	0	1
1	1	1

,

$>$	0	1
0	1	1
1	0	1

.

Let us recall that, given a partially ordered set $\mathfrak{X}=\langle X, \leqq_0\rangle$,

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the symbol $\mathfrak{B}^{\mathfrak{X}}$ *denotes the set of all isotonic mappings of* $\mathfrak{X}$ *into* $\mathfrak{B}$, *endowed with a « pointwise » partial ordering;*

the partially ordered set $\mathfrak{B}^{\mathfrak{X}}$ *turns out to be a complete implicative lattice, hence a completely distributive and pseudocomplemented one;*

the least element, the greatest element, the meet operation, the join operation of lattice $\mathfrak{B}^{\mathfrak{X}}$ *have, all of them, a strictly « pointwise » character;*

implication and pseudocomplementation of lattice $\mathfrak{B}^{\mathfrak{X}}$ *fail, in general, to have a « pointwise » character, and are to be described, respectively, as follows*

$$(f > g)(t) = \inf \{ f(x) > g(x) : t \leqq_0 x \in X \}, \tag{1}$$

for $t \in X$, *for* f, $g \in \mathfrak{B}^{\mathfrak{X}}$, *and*

$$(\sim f)(t) = \inf \{ \sim f(x) : t \leqq_0 x \in X \}, \tag{2}$$

for $t \in X$, *for* $f \in \mathfrak{B}^{\mathfrak{X}}$;

they do have a « pointwise » character, however, if $\mathfrak{X}$ *is partially ordered by equality: in this case we write* X *rather than* $\mathfrak{X}$, *and* $\mathfrak{B}^{X}$ *rather than* $\mathfrak{B}^{\mathfrak{X}}$.

Let us recall, too, that: *if* $\mathfrak{Y}$ *is a distributive lattice of finite length, then there is some partially ordered set* $\mathfrak{X}$ *such that* $\mathfrak{Y} = \mathfrak{B}^{\mathfrak{X}}$.

For details, see [1], § 1, Th. 6, Th. 7, and [2], p. 59, Th. 3.

In this paper, we shall have occasion to consider certain sequences of sequences of terms of different nature; in general, given any such « sequence of sequences »

$$s = ((s_{-1,1}, s_{-1,2}, s_{-1,3}, \ldots), (s_{01}, s_{02}, s_{03}, \ldots), (s_{11}, s_{12}, s_{13}, \ldots), (s_{21}, s_{22}, s_{23}, \ldots), \ldots),$$

we shall use the symbol $\pi_{ij}(s)$ to denote the element s_{ij} [$i = -1, 0, 1, 2, \ldots$; $j = 1, 2, 3, \ldots$].

1. The predicate calculus of the first order: syntax.

We are going to consider the predicate calculus of the first order, defined by the following stipulations.

THE BASIC ALPHABET.

A1. *Propositional variables*: a_k $[k=1, 2, 3, \ldots]$.

A2. *Free object variables*: b_k $[k=1, 2, 3, \ldots]$.

A3. *Bound object variables*: x_k $[k=1, 2, 3, \ldots]$.

A4. *Place marker*: $*$.

A5. *Predicate variables*: p_{rk} $[r, k=1, 2, 3, \ldots]$.

A6. *Quantifiers*: $\bigwedge$, $\bigvee$.

A7. *Propositional connectives*: $\neg$, $\wedge$, $\vee$, $\ulcorner$.

A8. *Punctuation signs*: (,).

A9. *Predicator*: $\leqq$.

We shall use the symbol y_k to denote either b_k or x_k, indifferently; r will be called *the degree* of the predicate variable p_{rk} $[r, k=1, 2, 3, \ldots]$.

Any nonempty word in the basic alphabet will be called *a form.* A simple inductive argument shows that, if c is any letter of the basic alphabet, other than $*$, and A is a form, the normal algorithm with the scheme $\begin{cases} * \rightarrow c \\ \quad \rightarrow \blacksquare \end{cases}$ concludingly transforms the form A into a form where letter $*$ does not occur: the latter form will be called $A[c]$.

Let us define, now, the set $\mathscr{F}$ of all *formulae* of the predicate calculus of the first order.

FORMULAE: AN INDUCTIVE DEFINITION.

F1. *Every propositional variable is a formula.*

F2. $p_{rk}b_{k_1}b_{k_2} \ldots b_{k_r}$ *is a formula* $[r, k, k_1, k_2, \ldots, k_r=1, 2, 3, \ldots]$.

F3. *If A and B are fomulae, so are* $\neg(A)$, $(A) \wedge (B)$, $(A) \vee (B)$, $(A) \ulcorner (B)$.

F4. *If A is a form where neither b_h nor x_k occur, and $A[b_h]$ is a formula, then both $(\wedge x_k)(A[x_k])$ and $(\vee x_k)(A[x_k])$ are formulae* $[h, k=1, 2, 3, ...]$.

ENTAILMENTS.

E1. *If A and B are formulae, then $A \leqq B$ is an entailment.*

In the sequel, and if no ambiguity arises, some [occurences of] punctuation signs may be omitted in the displaying or the mentioning of a formula, for short.

2. The predicate calculus of the first order: truth-functional reading.

In order to introduce the notion of *truth-functional reading* for formulae of the predicate calculus, we use *a truth-lattice, a universe, an acquaintance function.*

The *truth-lattice* is a given complete implicative [hence completely distributive and pseudocomplemented] lattice $\mathfrak{V}$, whose elements are called *truth-values*, whose least and greatest element, whose order relation, pseudocomplementation, meet operation, join operation and implication are denoted by the symbols 0_1, 1_1, $\leqq_1$, $\neg_1$, $\wedge_1$, $\vee_1$, $\sqsupset_1$, respectively; the set of all truth-values is denoted by V, so that $\mathfrak{V}=\langle V, \leqq_1\rangle$.

The *universe* is a given nonempty set U; its elements are called *objects*. The *acquaintance function* is a given mapping $\gamma_1 : U \to V$, of the universe into the truth-lattice $\mathfrak{V}$.

Let us consider, now, the following sets, defined in terms of truth-lattice, universe and acquaintance function:

$$P_r=\{p \in V^{U^r} : \text{ for all } u_1, u_2, ..., u_r \in U, \quad p(u_1, u_2, ..., u_r) \leqq_1 \gamma_1(u_1) \wedge_1 \gamma_1(u_2) \wedge_1 ... \wedge_1 \gamma_1(u_r)\} \tag{3}$$

for $r=1, 2, 3, ...$;

$$T=U^N \times V^N \times \mathop{\mathsf{X}}_{r=1}^{+\infty} P_r^N \, ; \; W=V^T \, ; \; \mathfrak{W}=\mathfrak{V}^T \, . \tag{4}$$

The typical member of T, denoted by t, is a sequence $(t_{-1}, t_0, t_1, t_2, \ldots)$ with $t_{-1} \in U^N$, $t_0 \in V^N$ and $t_r \in P_r^N$ $[r=1, 2, 3, \ldots]$, i.e. a sequence such that, in turn

(5_{-1}) *t_{-1} is a sequence $(t_{-1,1}, t_{-1,2}, t_{-1,3}, \ldots)$ of objects,*

(5_0) *t_0 is a sequence $(t_{01}, t_{02}, t_{03}, \ldots)$ of truth-values,*

(5_r) *t_r is a sequence $(t_{r1}, t_{r2}, t_{r3}, \ldots)$ of members of P_r, hence a sequence of mappings of U^r into V $[r=1, 2, 3, \ldots]$.*

Let us notice here [see § 0] that the partially ordered set $\mathfrak{W}$ is a complete implicative lattice, hence a completely distributive and pseudo-complemented one; that its least element, its greatest element, order relation, pseudocomplementation, meet operation, join operation, implication, all have a strctly « pointwise » character: they will be denoted respectively by the symbols 0_2, 1_2, $\leqq_2$, $\neg_2$, $\wedge_2$, $\vee_2$, $\sqsupset_2$; of course, $\mathfrak{W} = \langle W, \leqq_2 \rangle$.

Last, let us introduce the mapping $\gamma_2 : U \to W$, defined as follows. Since γ_1 is a mapping of U into V, for each $u \in U$ we have $\gamma_1(u) \in V$; well, for each $u \in U$ let us call $\gamma_2(u)$ that mapping of T into V, which maps all members of T into the single truth-value $\gamma_1(u)$: of course $\gamma_2(u) \in W$; this way, γ_2 actually maps the universe U into the lattice $\mathfrak{W}$; in other words,

(6) $$[\gamma_2(u)](t) = \gamma_1(u),$$

for all $u \in U$, $t \in T$.

2.1. To each formula A of the predicate calculus, we are now going to associate a member $|A|$ of the lattice $\mathfrak{W}$, *the truth-functional reading of the formula A*, relative to the truth-lattice $\mathfrak{V}$, the universe U, the acquaintance function γ_1. To this effect, we consider a sequence $(u_1, u_2, u_3, \ldots)$ of parameters, and adopt the following stipulations.

TRUTH-FUNCTIONAL READING: AN INDUCTIVE DEFINITION.

V1. $|a_k|(t) = \pi_{0k}(t)$ *for* $t \in T$ $[k=1, 2, 3, \ldots]$.

V2. $|b_k|(t) = \pi_{-1,k}(t)$ *for* $t \in T$ $[k=1, 2, 3, \ldots]$.

V3. $|x_k|(t)=u_k$ *for* $t\in T$ $[k=1, 2, 3, ...]$.

V4. $|p_{rk}|(t)=\pi_{rk}(t)$ *for* $t\in T$ $[r, k=1, 2, 3, ...]$.

V5. $|p_{rk}y_{k_1}y_{k_2} \ldots y_k|(t)=[|p_{rk}|(t)][|y_{k_1}|(t), |y_{k_2}|(t), \ldots, |y_{k_r}|(t)]$ *for* $t\in T$ $[r, k, k_1, k_2, \ldots, k_r=1, 2, 3, \ldots]$.

V6. $|\neg A|=\neg_2|A|$, $|A\wedge B|=|A|\wedge_2|B|$,

$|A\vee B|=|A|\vee_2|B|$, $|A\vdash B|=|A|\vdash_2|B|$.

V7. $|(\bigwedge x_k)(A)|=\inf\{\gamma_2(u_k)\vdash_2|A| : u_k\in U\}$ *in* $\mathfrak{W}$,

$|(\bigvee x_k)(A)|=\sup\{|A| : u_k\in U\}$ *in* $\mathfrak{W}$.

We want to prove that

TH. 1. *Clauses* V1 *through* V7 *define inductively a truth-functional reading* $|A|$ *for each formula A of the predicate calculus; such truth-functional readings are members of the complete implicative lattice* $\mathfrak{W}$.

2.2. In order to prove Th. 1, we need the auxiliary notion of *open formula*, as well as a definition of the property « *in the open formula C there is an unquantified occurrence of the bound object variable* x_h »; we stipulate the following.

OPEN FORMULAE: AN INDUCTIVE DEFINITION.

OF1. *Every propositional variable is an open formula; in it there is no unquantified occurrence of any bound object variable whatsoever.*

OF2. $p_{rk}y_{k_1}y_{k_2} \ldots y_{k_r}$ *is an open formula; in it there is an unquantified occurrence of the bound object variable* x_h, *iff some of the symbols* $y_{k_1}, y_{k_2}, \ldots, y_{k_r}$ *stand actually for* x_h $[r, k, k_1, k_2, \ldots, k_r, h=1, 2, 3, \ldots]$.

OF3. *If A and B are open formulae, so are* $\neg(A)$, $(A)\wedge(B)$, $(A)\vee(B)$, $(A)\vdash(B)$; *in* $\neg(A)$ *there is an unquantified occurrence of the bound object variable* x_h, *iff in A itself there is an unquantified occurrence of the bound object variable* x_h; *in* $(A)+(B)$ — *where the symbol* $+$ *stands for any one of the symbols* $\wedge$, $\vee$, $\vdash$ — *there is an* *unquantified occurence of the bound object variable* x_h *iff there is an unquantified occurrence of the bound object variable* x_h *in* A *and/or in* B $[h=1, 2, 3, \ldots]$.

OF4. *If* A *is an open formula, then both* $(\bigwedge x_k)(A)$ *and* $(\bigvee x_k)(A)$ *are open formulae; in* $(\bigwedge x_k)(A)$, *as well as in* $(\bigvee x_k)(A)$, *there is no unquantified occurrence of the bound object variable* x_k; *for* $h \neq k$, *there is in* $(\bigwedge x_k)(A)$, *as well as in* $(\bigvee x_k)(A)$, *an unquantified occurrence of the bound object variable* x_h *iff in* A *itself there is an unquantified occurrence of the bound object variable* x_h $[h, k=1, 2, 3, \ldots]$.

Since an open formula is anyway a (finite) word in the basic alphabet, a straightforward induction argument shows that

LEMMA 1. *Given an open formula* C, *and the bound obiect variables* $x_h, x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, *the following two properties are decidable: « There is in* C *some unquantified occurrence of* x_h *», and « There is in* C *no unquantified occurrence of any bound object variable, other than* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$ *»* $[h, h_1, h_2, \ldots, h_s, s=1, 2, 3, \ldots]$.

Another straightforward induction argument shows that

LEMMA 2. *If* C *is a formula of the predicate calculus, then* C *is an open formula, and in* C *there is no unquantified occurrence of any bound object variables.*

Let us prove, now, the following

LEMMA 3. *Given an open formula* C, *and* s *bound object variables* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, *if there is in* C *no unquantified occurrence of any bound object variable other than* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, *then clauses* V1 *through* V7 *define for* C *a truth-functional reading* $|C|$, *which turns out to be a member of the complete implicative lattice* $\mathfrak{W}$, *possibily dependent on* s *parameters* $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$, *but independent of any other parameter.*

We shall give the proof of this lemma in the next section 2.3, but notice right away that the conjunction of Lemmas 2 and 3 yields just Th. 1.

2.3. (*a*) Under V1 and (5_0), whatever $t \in T$, $|a_k|(t)$ is a member of V, hence the truth-functional reading $|a_k|$, of the propositional variable a_k, is, under V1, a mapping of T into V, i.e. is a member of W [$k=1, 2, 3, ...$]. In other words, *if the open formula C is given by clause* OF1, *then clause* V1 *defines its truth-functional reading* $|C|$ *as a member of the complete implicative lattice* $\mathfrak{W}$, *independent of any parameter whatsoever.*

(*b*) Whatever $t \in T$, under V4 and (5_r), $|p_{rk}|(t)$ is a certain mapping of U^r into V; moreover, whatever $t \in T$, under V2 and (5_{-1}), $|b_{k_1}|(t)$, $|b_{k_2}|(t)$, ..., $|b_{k_r}|(t)$ are certain members of U, while, whatever $u_{k_1}, u_{k_2}, ..., u_{k_r} \in U$ and whatever $t \in T$, under V3, $|x_{k_1}|(t)$, $|x_{k_2}|(t)$, ..., $|x_{k_r}|(t)$ are certain r members of U as well; consequently, whatever $u_{k_1}, u_{k_2}, ..., u_{k_r} \in U$ and whatever $t \in T$, $[|p_{rk}|(t)][|y_{k_1}|(t), |y_{k_2}|(t), ..., |y_{k_r}|(t)]$ is a certain member of V, which will depend on the parameters $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$ and $t \in T$ — but on no other ones — if and only if the bound object variables $x_{h_1}, x_{h_2}, ..., x_{h_s}$ do occur in the open formula $p_{rk}y_{k_1}y_{k_2} ... y_k$, but no other ones do; of course, if no bound object variable occurs in $p_{rk}y_{k_1}y_{k_2} ... y_{k_r}$, then, whatever $t \in T$, $[|p_{rk}|(t)][|y_{k_1}|(t), ..., |y_{k_r}|(t)] = [|p_{rk}|(t)][|b_{k_1}|(t), ..., |b_{k_r}|(t)]$, which is a member of V, dependent on the only parameter $t \in T$. Anyway, under clauses V2 through V5, $|p_{rk}y_{k_1}y_{k_2} ... y_{k_r}|$ is a certain mapping of T into V — i.e. a certain member of W — which will be independent of any parameter in case no bound object variable occurs in $p_{rk}y_{k_1}y_{k_2} ... y_{k_r}$, while it will depend on no other parameter than $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$ in case no bound object variable occurs in $p_{rk}y_{k_1}y_{k_2} ... y_{k_r}$ other than $x_{h_1}, x_{h_2}, ..., x_{h_s}$. In other words, *if the open formula C is given by clause* OF2 *and in C there is no unquantified occurrence of bound object variables other than* $x_{h_1}, x_{h_2}, ..., x_{h_s}$, *then clauses* V2 *through* V5 *define the truth-functional reading* $|C|$ *as a member of the complete implicative lattice* $\mathfrak{W}$, *dependent on no parameter other than* $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$; *in case the open formula C is still given by* OF2 *but in C there is no unquantified occurrence of any bound object variable whatsoever, then clauses* V2 *through* V5 *define the truth-functional reading* $|C|$ *as a member of the complete implicative lattice* $\mathfrak{W}$, *independent of any parameter whatsoever.*

(*c*) Suppose C to be given by clause OF3, and consider first the case that $C = \neg(A)$ where A is some open formula. If in C there is no unquantified occurrence of any bound object variable, then, by OF3, in A there is no unquantified occurrence of any bound object variable either and, by the inductive assumption, clauses V1 through V7 define the truth-functional reading $|A|$ as a member of the lattice $\mathfrak{W}$, independent of any parameter whatsoever; but then clause V6 will define $|C| = \neg_2 |A|$ also as a member of lattice $\mathfrak{W}$, independent of any parameter whatsoever. If in C there is no unquantified occurrence of any bound object variable other than $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, then, by OF3, in A there is no unquantified occurrence of any bound object variable other than $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, and, by the inductive assumption, clauses V1 through V7 define the truth-functional reading $|A|$ as a member of the lattice $\mathfrak{W}$, independent of all parameters other than $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$; but then clause V6 will define $|C| = \neg_2 |A|$ also as a member of lattice $\mathfrak{W}$, independent of all parameters other than $u_{h_1}, u_{h_2}, \ldots, u_{h_s}$.

A similar argument takes care of the case that C equals either $(A) \wedge (B)$, or $(A) \vee (B)$ or $(A) \vdash (B)$.

(*d*) Suppose C to be given by clause OF4, and consider first the case that $C = (\bigwedge x_k)(A)$, where A is some open formula. If in C there is no unquantified occurrence of any bound object variable, then, by OF4, in A there is no unquantified occurrence of any bound object variable except possibly x_k, and, by the inductive assumption, clauses V1 through V7 define the truth-functional reading $|A|$ as a member of the complete implicative lattice $\mathfrak{W}$, possibly dependent upon the parameter $u_k \in U$; but, whatever $u_k \in U$, $\gamma_2(u_k)$ is also a member of the implicative lattice $\mathfrak{W}$ (see end of sec. 2), and so is $\gamma_2(u_k) \vdash_2 |A|$, and it does make sense to consider the set $\{\gamma_2(u_k) \vdash_2 |A| : u_k \in U\}$ of elements of the complete lattice $\mathfrak{W}$, and the infimum of this set in the lattice itself; but then clause V7 defines $|C| = |(\bigwedge x_k)(A)|$ as a member of the lattice $\mathfrak{W}$, independent of all parameters. If, instead, there is in C no unquantified occurrence of any bound variable except possibly $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$, then, by OF 4, there is in A no unquantified occurrence of any bound object variable except possibly $x_k, x_{h_1}, x_{h_2}, \ldots, x_{h_s}$: again, by the inductive assumption and by clause V7, also the truth-functional reading $|C| = |(\bigwedge x_k)(A)|$ will be defined as a member of the complete implicative lattice $\mathfrak{W}$, dependent on no parameter except possibly $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$.

A similar argument takes care of the case that $C = (\bigvee x_k)(A)$.

This concludes the proof of lemma 3.

2.4. After the proof of Theorem 1, and because of clause V6.

TH. 2. *Clauses* V1 *through* V7 *define a [homomorphic] mapping* $A \rightarrow |A|$ *of the set* $\mathscr{F}$ *of all formulae of the predicate calculus [under « operations »* $\neg, \wedge, \vee, \vdash$*] into the set* W *[under operations* $\neg_2, \wedge_2, \vee_2, \vdash_2$].

With a reasonable extension of the customary terminology, the image of $\mathscr{F}$ under the mapping $A \rightarrow |A|$ can be called *the Lindenbaum algebra of the predicate calculus,* and is itself an implicative sublattice $|\mathscr{F}|$ of $\mathfrak{W}$.

Two formulae A and B will be said *truth-functionally equivalent,* when they have the same truth-functional reading, i.e. when $|A|=|B|$; if $|A| \leqq_2 |B|$ only, we will say that A *truth-functionally entails* B, or that *the entailment* $A \leqq B$ *is truth-functionally valid.*

Only at this stage it would be sensible to describe an inferential structure of the predicate calculus, with this target in mind, e.g., that an entailment be inferentially valid if and only if it is truth-functionally so; but this is not relevant to my subject.

I wish to stress again, however, that the notions introduced above are relative to the truth-lattice $\mathfrak{V}$, to the universe U, to the acquaintance function γ_1.

3. A manivalued theory.

The classical two-valued semantics for the predicate calculus of the first order is obtained from the one described above, by specializing the truth-lattice to be the elementary boolean lattice $\mathfrak{B}$, and the acquaintance function to be the constant 1. Intuitively, this means to acknowledge a *univocal* [absolute] notion of truth and existence; concretely, this means to restrict the applicability of the calculus to those concrete theories only, where exactly *one* criterion of truth and *one* criterion of existence are admitted, respectively able, at least in principle, to tell of each proposition of the theory whether it is true, to tell of each object of the theory whether it exists.

But suppose we adopt a *pluralistic* [relative] notion of truth and of existence. Specifically, suppose we have a theory $\mathcal{T}$ for which *several* criteria of truth and *several* criteria of existence are recognized, perhaps embodied in some formal metatheory; let us call M' the set of all recognized truth-criteria, M'' the set of all recognized existence-criteria, and for the sake of peace let's admit each one of these criteria to be two-valued: a very familiar instance of this state of affairs can be found in any theory where exactly two truth-criteria are recognized, one of them merely necessary and the other one merely sufficient, and/or exactly two existence-criteria are recognized, one of them nonconstructive and the other one constructive.

Anyway, beside the sets M' and M'', we will consider also their cartesian product, the set $M = M' \times M''$ of all pairs consisting of one recognized truth-criterion and one recognized existence-criterion; such pairs will be called *witnesses*, because, so to speak, each one of them is asked to testify about the truth of each proposition and the existence of each object pertaining to the theory $\mathcal{T}$.

Now, if we ask all the witnesses about the truth of a given proposition, or about the existence of a given object, we shall get a [possibly different] answer out of each one of them, viz. a *yes* or a *no* as the case may be, in symbols respectively a 1 or a 0; it is only natural to consider the aggregate result of this questioning as *the truth-value of the proposition*, or as *the existence-value of the object*, under scrutiny, in the opinion of the given panel of witnesses: such aggregate result is just a member of the boolean lattice $\mathfrak{B}^M$.

To avoid possible misunderstandings,let us stipulate once and for all, that a witness's *yes* is to be construed as *an assertion* [= « Yes, I know that this proposition is true » or « that this object exists »], while a witness's *no* is to be construed as *a refusal to assert* [= « No, I do not know whether this proposition is true » or « whether this object exists »], not as a denial. For short, we will say that a given witness *asserts* or *fails to assert* [alternatively *knows as true* or *does not know as true*] a given proposition, that a given witness *knows* or *does not know* a given object, as the case may be.

In general, given any two witnesses m' and m'', we will say that *m' is less knowledgeable than m''* [in symbols, $m' \leqq_0 m''$], or that *m'' is more knowledgeable than m'* [in symbols, $m'' \geqq_0 m'$], when, on metatheoretical grounds, all objects known to witness m' are also known to witness m'', and all propositions known as true to witness m' are known as true to witness m'' as well; we will also identify any two witnesses to whom, on metatheoretical grounds, exactly the same objects are known, and exactly the same proposition are known as true, i.e. any two *equally knowledgeable* witnesses.

This way, the panel M of witnesses is turned into *a partially ordered set* $\mathfrak{M}=\langle M, \leqq_0\rangle$, and only those members of the boolean lattice $\mathfrak{B}^M$ can possibly be truth-values of propositions or existence-values of objects, which map $\mathfrak{M}$ *isotonically* into $\mathfrak{B}$, i.e. which are actually members of the complete implicative lattice $\mathfrak{B}^{\mathfrak{M}}$. Of course, the case that all witnesses are independent [i.e. that no witness is more, or less, knowledgeable than any other one] is subsumed under the general case, the panel of witnesses being partially ordered by the relation of equality.

Under these stipulations,

the least element of the lattice $\mathfrak{B}^{\mathfrak{M}}$ *is the truth-value of a proposition, whose truth is not known to any witness, and the existence-value of an object, whose existence is unknown to all witnesses*;

the greatest element of the lattice $\mathfrak{B}^{\mathfrak{M}}$ *is the truth-value of a proposition, whose truth is known to all witnesses, and the existence-value of an object, whose existence is also known to all witnesses*;

all other elements of the lattice $\mathfrak{B}^{\mathfrak{M}}$ *are possible truth-values of propositions whose truth is known to some, but not to all, of the witnesses, and/or possible existence-values of objects, which are known to some, but not to all, of the witnesses.*

3.1. After these preliminaries, let us see more in detail what our theory $\mathcal{T}$ is supposed to be. Theory $\mathcal{T}$ must comprise a set $\mathfrak{V}$ of ($\mathcal{T}$-theoretical) *propositions*, a nonempty set U of ($\mathcal{T}$-theoretical) *objects*, and, for each $r=1, 2, 3, ...,$ a set $\mathfrak{P}_r$ of ($\mathcal{T}$-theoretical) *r-adic predicates*; the set U is called the ($\mathcal{T}$-theoretical) *universe*. Metatheoretically, *a panel* $\mathfrak{M}=\langle M, \leqq_0\rangle$ *of witnesses* must be available, partially ordered by knowledgeability; the complete implicative lattice $\mathfrak{Y}=\mathfrak{B}^{\mathfrak{M}}$ is taken as *truth-lattice*, with the same notations as in sec. 2.

Given any proposition $\mathfrak{v}\in\mathfrak{V}$, the symbol $\widehat{\mathfrak{v}}$ will denote the truth-value of $\mathfrak{v}$, so that, for each $m\in M$, $\widehat{\mathfrak{v}}(m)=1$ iff proposition $\mathfrak{v}$ is known as true by witness m, $\widehat{\mathfrak{v}}(m)=0$ otherwise; of course, $\widehat{\mathfrak{v}}\in\mathfrak{Y}$. Similarly,given any object $u\in U$, the symbol $\widehat{u}$ will denote the existence-value of u, so that, for each $m \in M$, $\widehat{u}(m)=1$ iff object u is known to witness m, $\widehat{u}(m)=0$ otherwise; of course, $\widehat{u} \in \mathfrak{V}$. The mapping $\gamma_1 : u \to \widehat{u}$, of the universe U into the truth-lattice $\mathfrak{V}$, is taken as the ($\mathcal{T}$-theoretical) *acquaintance function.*

About the set $\mathfrak{V}$ of all propositions, we stipulate the following.

$\mathcal{T}1$. *For each $v \in V$, there is some proposition $\mathfrak{v} \in \mathfrak{V}$ such that $\widehat{\mathfrak{v}}=v$.*

$\mathcal{T}2$. *Given $\mathfrak{p} \in \mathfrak{P}_r$, whatever $u_1, u_2, \dots, u_r \in U$, $\mathfrak{p}u_1u_2 \dots u_r$ is a proposition fulfilling the following condition*

$$(7) \qquad (\mathfrak{p}u_1u_2 \dots u_r)^{\wedge} \leqq_1 \widehat{u_1} \wedge_1 \widehat{u_2} \wedge_1 \dots \wedge_1 \widehat{u_r} ;$$

this condition simply means that the typical witness will refrain from asserting proposition $\mathfrak{p}u_1u_2 \dots u_r$ unless he knows all objects $u_1, u_2, \dots, u_r$.

$\mathcal{T}3$. *Given two propositions $\mathfrak{v}, \mathfrak{v}' \in \mathfrak{V}$, there is in $\mathfrak{V}$:*

- *a proposition $\sim_1\mathfrak{v}$, which is asserted by the typical witness m, iff each witness more knowledgeable than m (including m) fails to know whether proposition $\mathfrak{v}$ is true;*
- *a proposition $\mathfrak{v} \cap_1 \mathfrak{v}'$, which is asserted by the typical witness m, iff m himself knows both $\mathfrak{v}$ and $\mathfrak{v}'$ to be true;*
- *a proposition $\mathfrak{v} \cup_1 \mathfrak{v}'$, which is asserted by the typical witness m, iff m himself knows $\mathfrak{v}$ and/or $\mathfrak{v}'$ to be true;*
- *a proposition $\mathfrak{v} \succ_1 \mathfrak{v}'$, which is asserted by the typical witness m, iff each witness more knowledgeable than m (including m) knows $\mathfrak{v}'$ to be true and/or fails to know whether $\mathfrak{v}$ is true.*

$\mathcal{T}4$. *Given a mapping $\varphi : U \to \mathfrak{V}$, there is in $\mathfrak{V}$:*

- *a proposition $(\forall_1 u)(\varphi(u))$, which is asserted by the typical witness m, iff each witness m′ more knowledgeable than m (including m), for each object u′ known to himself, m′, knows proposition $\varphi(u')$ to be true;*
- *a proposition $(\exists_1 u)(\varphi(u))$, which is asserted by the typical witness m, iff there is some object u′ such that proposition $\varphi(u')$ is known as true by m.*

We introduce also the following notion:

$\mathcal{T}5$. *Given two proposition* $\mathfrak{v}, \mathfrak{v}' \in \mathfrak{V}$, $\mathfrak{v}$ *is said to entail* $\mathfrak{v}'$ [*in symbols,* $\mathfrak{v} \supset_1 \mathfrak{v}'$], *when* $\mathfrak{v}'$ *is asserted by every witness who knows* $\mathfrak{v}$ *as true.*

Condition $\mathcal{T}3$ can be restated, more succinctly, as follows:

(8_1) $[\sim_1 \mathfrak{v}]\hat{}(m) = \inf\{\sim[\widehat{\mathfrak{v}}(m')] : m \leqq_0 m' \in M\}$, *for* $m \in M$,

(8_2) $[\mathfrak{v} \cap_1 \mathfrak{v}']\hat{}(m) = [\widehat{\mathfrak{v}}(m)] \cap [\widehat{\mathfrak{v}}'(m)]$, *for* $m \in M$,

(8_3) $[\mathfrak{v} \cup_1 \mathfrak{v}']\hat{}(m) = [\widehat{\mathfrak{v}}(m)] \cup [\widehat{\mathfrak{v}}'(m)]$, *for* $m \in M$,

(8_4) $[\mathfrak{v} >_1 \mathfrak{v}']\hat{}(m) = \inf\{[\widehat{\mathfrak{v}}(m')] > [\widehat{\mathfrak{v}}'(m')] : m \leqq_0 m' \in M\}$, *for* $m \in M$,

using notations introduced in sec. 0. Under (1), (2), and because of the « pointwise » character of the meet and the join operations in the lattice $\mathcal{V} = \mathfrak{B}^{\mathfrak{M}}$, and of clause $\mathcal{T}5$, we have the following

TH. 2. *Given any two propositions* $\mathfrak{v}, \mathfrak{v}' \in \mathfrak{V}$,

$(8'_1)$ $[\sim_1 \mathfrak{v}]\hat{} = \neg_1 \widehat{\mathfrak{v}}$,

$(8'_2)$ $[\mathfrak{v} \cap_1 \mathfrak{v}']\hat{} = \widehat{\mathfrak{v}} \wedge_1 \widehat{\mathfrak{v}}'$,

$(8'_3)$ $[\mathfrak{v} \cup_1 \mathfrak{v}']\hat{} = \widehat{\mathfrak{v}} \vee_1 \widehat{\mathfrak{v}}'$,

$(8'_4)$ $[\mathfrak{v} >_1 \mathfrak{v}']\hat{} = \widehat{\mathfrak{v}} \vdash_1 \widehat{\mathfrak{v}}'$,

(9) $\mathfrak{v} \supset_1 \mathfrak{v}'$ *if and only if* $\mathfrak{v} \leqq_1 \widehat{\mathfrak{v}}'$.

This theorem is just a replica of [1], Th. 8; because of it, the mapping $\mathfrak{v} \to \widehat{\mathfrak{v}}$ is a « homomorphism » of the set $\mathfrak{V}$ of all propositions (under « operations » $\sim_1$, $\cap_1$, $\cup_1$, $>_1$, and relation $\supset_1$) into the set $\mathfrak{V}$ of all truth-values (under operations $\neg_1$, $\wedge_1$, $\vee_1$, $\vdash_1$, and relation $\leqq_1$).

A result without counterpart in [1] is the following.

TH. 3. *Given a mapping* $\varphi : U \to \mathfrak{V}$, *we have*

(10_1) $[(\forall_1 u)(\varphi(u))]\hat{} = \inf\{\widehat{u} \vdash_1 [\varphi(u)]\hat{} : u \in U\}$ *in* $\mathcal{V}$,

(10_2) $[(\exists_1 u)(\varphi(u))]^\wedge = \sup\{[\varphi(u)]^\wedge : u \in U\}$ *in* $\mathfrak{V}$.

PROOF. (*a*) Given an object u' and a witness m, consider the following inequality between members of the lattice $\mathfrak{B}$:

(11) $\widehat{u'}(m) \cap [(\forall_1 u)(\varphi(u))]^\wedge(m) \leqq [\varphi(u')]^\wedge(m)$.

In case $\widehat{u'}(m)=0$, as well as in case $[\varphi(u')]^\wedge(m)=1$, inequality (11) is trivial. Consider then the case that, concurrently, $\widehat{u'}(m)=1$ and $[\varphi(u')]^\wedge(m)=0$; in this case, witness m knows object u', but he fails to know whether proposition $\varphi(u')$ is true, hence he cannot assert proposition $(\forall_1 u)(\varphi(u))$, and consequently $[(\forall_1 u)(\varphi(u))]^\wedge(m)=0$, which again proves inequality (11).

As (11) holds for all $m \in M$, whatever $u' \in U$, also inequality

$$\widehat{u'} \wedge_1 [(\forall_1 u)(\varphi(u))]^\wedge \leqq_1 [\varphi(u')]^\wedge$$

holds whatever $u' \in U$, hence inequality

$$[(\forall_1 u)(\varphi(u))]^\wedge \leqq_1 \widehat{u'} \rightarrow_1 [\varphi(u')]^\wedge$$

holds whatever $u' \in U$, therefore

(12) $[(\forall_1 u)(\varphi(u))]^\wedge \leqq_1 \inf\{\widehat{u} \rightarrow_1 [\varphi(u)]^\wedge : u \in U\}$ *in* $\mathfrak{V}$.

(*b*) Given $v \in V$, such that

(13) $v \leqq_1 \widehat{u'} \rightarrow_1 [\varphi(u')]^\wedge$ *for all* $u' \in U$,

given any witness m, consider the following inequality between members of $\mathfrak{B}$:

(14) $v(m) \leqq [(\forall_1 u)(\varphi(u))]^\wedge(m)$.

In case $v(m)=0$, inequality (14) is trivial. Consider the case $v(m)=1$, and let m' be any witness more knowledgeable than m; since v is an isotonic mapping of $\mathfrak{M}$ into $\mathfrak{B}$, we have

(15) $v(m')=1$;

from (13) we get $\widehat{u'} \wedge_1 v \leqq_1 [\varphi(u)]^\wedge$ whathever $u' \in U$, and in particular

(16) $\widehat{u'}(m') \cap v(m') \leqq [\varphi(u')]^\wedge(m')$, *whatever* $u' \in U$;

from (15) and (16) we get

$$\widehat{u'}(m') = \widehat{u'}(m') \cap 1 = \widehat{u'}(m') \cap v(m') \leqq [\varphi(u')]^\wedge(m')$$

whatever $u \in U$, hence

(17) $\widehat{u'}(m') \leqq [\varphi(u')]^\wedge(m')$ *for all* $u' \in U$.

Inequality (17) means that witness m' asserts proposition $\varphi(u')$ for every object u' known to him; since this holds for every witness m' more knowledgeable than m, witness m must assert proposition $(\forall_1 u)(\varphi(u))$, therefore we have $[(\forall_1 u)(\varphi(u))]^\wedge(m) = 1$, which again proves inequality (14).

As inequality (14) holds for all $m \in M$, we have $v \leqq_1 [(\forall_1 u)(\varphi(u))]^\wedge$, and in particular, for $v = \inf\{\widehat{u} \rightharpoonup_1 [\varphi(u)]^\wedge \cdot u \in U\}$ *in* $\mathfrak{V}$, we have

(18) $\inf\{\widehat{u} \rightharpoonup_1 [\varphi(u)]^\wedge : u \in U\}$ *in* $\mathfrak{V} \leqq_1 [(\forall_1 u)(\varphi(u))]^\wedge$.

(*c*) From (12) and (18), we get (10_1).

(*d*) Given an object $u' \in U$ and a witness $m \in M$, consider the following inequality between members of the lattice $\mathfrak{B}$:

(19) $[\varphi(u')]^\wedge(m) \leqq [(\exists_1 u)(\varphi(u))]^\wedge(m)$.

In case $[\varphi(u')]^\wedge(m) = 0$, inequality (19) is trivial. In case $[\varphi(u')]^\wedge(m) = 1$, witness m knows proposition $\varphi(u')$ as true, therefore he must assert proposition $(\exists_1 u)(\varphi(u))$; we have $[(\exists_1 u)(\varphi(u))]^\wedge(m) = 1$, then, and this proves again inequality (18). As inequality (19) holds identically, we have also $[\varphi(u')]^\wedge \leqq_1 [(\exists_1 u)(\varphi(u))]^\wedge$ for all $u' \in U$, hence

(20) $\sup\{[\varphi(u')]^\wedge : u \in U\}$ *in* $\mathfrak{V} \leqq_1 [(\exists_1 u)(\varphi(u))]^\wedge$.

(*e*) Given $v \in V$, such that

(21) $[\varphi(u')]^\wedge \leqq_1 v$ *for all* $u' \in U$,

given any $m \in M$, consider the following inequality between members of $\mathfrak{B}$:

(22) $[(\exists_1 u)(\varphi(u))]^{\wedge}(m) \leqq v(m)$.

In case $v(m)=1$, inequality (22) is trivial. In case $v(m)=0$, on account of (21) we have also $[\varphi(u')]^{\wedge}(m)=0$ for all $u' \in U$: then witness m does not assert proposition $\varphi(u')$ for any object $u' \in U$, therefore he cannot assert proposition $(\exists_1 u)(\varphi(u))$; we have $[(\exists_1 u)(\varphi(u))]^{\wedge}(m)=0$, then, which again proves (22). Thus, inequality (22) holds whatever $m \in M$, consequently $[(\exists_1 u)(\varphi(u))]^{\wedge} \leqq_1 v$; in particular, for

$$v = \sup \{[\varphi(u)]^{\wedge} : u \in U\} \text{ in } \mathfrak{Y},$$

we get

(23) $[(\exists_1 u)(\varphi(u))]^{\wedge} \leqq_1 \sup \{[\varphi(u)]^{\wedge} : u \in U\}$ *in* $\mathfrak{Y}$.

(*f*) From (20) and (23), we get (10_2).

The proof is complete.

4. The predicate calculus of the first order: $\mathcal{T}$-theoretical reading.

We will now make use of theory $\mathcal{T}$, introduced in sec.s 3 and 3.1, in order to define the notion of *$\mathcal{T}$-theoretical reading* for formulae of the predicate calculus of the first order. To this effect, consider first the following sets, defined in terms of the ($\mathcal{T}$-theoretical) universe U, the set $\mathfrak{P}$ of all ($\mathcal{T}$-theoretical) propositions, and the sets $\mathfrak{P}_r$ of all ($\mathcal{T}$-theoretical) r-adic predicates, with $r=1, 2, 3, ...$:

(24) $\mathfrak{T} = U^N \times \mathfrak{P}^N \times \mathop{\mathsf{X}}\limits_{r=1}^{\infty} \mathfrak{P}_r^N, \quad \mathfrak{W} = \mathfrak{P}^{\mathfrak{T}}$.

The typical member of $\mathfrak{T}$, denoted by $\mathfrak{t}$, is a sequence $(\mathfrak{t}_{-1}, \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, ...)$ with $\mathfrak{t}_{-1} \in U^N$, $\mathfrak{t}_0 \in \mathfrak{P}^N$ and $\mathfrak{t}_r \in \mathfrak{P}_r^N$ $[r=1, 2, 3, ...]$, i.e. a sequence such that in turn

(25_{-1}) $\mathfrak{t}_{-1}$ *is a sequence* $(\mathfrak{t}_{-1,1}, \mathfrak{t}_{-1,2}, \mathfrak{t}_{-1,3}, ...)$ *of objects,*

(25_0) $\mathfrak{t}_0$ *is a sequence* $(\mathfrak{t}_{01}, \mathfrak{t}_{02}, \mathfrak{t}_{03}, \ldots)$ *of propositions,*

(25_r) $\mathfrak{t}_r$ *is a sequence* $(\mathfrak{t}_{r1}, \mathfrak{t}_{r2}, \mathfrak{t}_{r3}, \ldots)$ *of r-adic predicates* $[r=1, 2, 3, \ldots]$.

In the set $\mathfrak{W}$, we define now a unary operation $\sim_2$, three binary operations $\cap_2$, $\cup_2$, $>_2$, and a binary relation $\supset_2$, through the following stipulations:

(26_1) $(\sim_2 \mathfrak{w})(\mathfrak{t}) = \sim_1 [\mathfrak{w}(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$,

(26_2) $(\mathfrak{w} \cap_2 \mathfrak{w}')(\mathfrak{t}) = [\mathfrak{w}(\mathfrak{t})] \cap_1 [\mathfrak{w}'(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$,

(26_3) $(\mathfrak{w} \cup_2 \mathfrak{w}')(\mathfrak{t}) = [\mathfrak{w}(\mathfrak{t})] \cup_1 [\mathfrak{w}'(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$,

(26_4) $(\mathfrak{w} >_2 \mathfrak{w}')(\mathfrak{t}) = [\mathfrak{w}(\mathfrak{t})] >_1 [\mathfrak{w}(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$,

(26_5) $\mathfrak{w} \supset_2 \mathfrak{w}'$ *if and only if, for all* $\mathfrak{t} \in \mathfrak{T}$, $\mathfrak{w}(\mathfrak{t}) \supset_1 \mathfrak{w}'(\mathfrak{t})$;

this way, given any two members of the set $\mathfrak{W}$, say $\mathfrak{w}$ and $\mathfrak{w}'$, they are mappings of $\mathfrak{T}$ into $\mathfrak{V}$, so that, whatever $\mathfrak{t} \in \mathfrak{T}$, $\mathfrak{w}(\mathfrak{t})$ and $\mathfrak{w}'(\mathfrak{t})$ are propositions, and with them also

$$[\mathfrak{w}(\mathfrak{t})] \cap_1 [\mathfrak{w}'(\mathfrak{t})], \ [\mathfrak{w}(\mathfrak{t})] \cup_1 [\mathfrak{w}'(\mathfrak{t})], \ [\mathfrak{w}(\mathfrak{t})] >_1 [\mathfrak{w}'(\mathfrak{t})] \text{ and } \sim_1 [\mathfrak{w}(\mathfrak{t})]$$

are propositions; consequentely, clauses (26_1) through (26_4) define

$$\sim_2 \mathfrak{w}, \quad \mathfrak{w} \cap_2 \mathfrak{w}', \quad \mathfrak{w} \cup_2 \mathfrak{w}', \quad \mathfrak{w} >_2 \mathfrak{w}',$$

as mappings of $\mathfrak{T}$ into $\mathfrak{V}$, i.e. as members of the set $\mathfrak{W}$; trivially, clause (26_5) defines $\supset_2$ as a binary relation in the set $\mathfrak{W}$.

We have to introduce, now, two more operations in the set $\mathfrak{W}$, of a more general nature. Let ψ be a mapping of U into $\mathfrak{W}$, so that $\psi(u)$ is a member of the set $\mathfrak{W}$, dependent on the parameter $u \in U$; in other words, for each $u \in U$, $\psi(u)$ is a mapping of $\mathfrak{T}$ into $\mathfrak{V}$, so that, in turn, for each given $\mathfrak{t} \in \mathfrak{T}$, $(\psi(u))(\mathfrak{t})$ is a member of $\mathfrak{V}$. For each given $\mathfrak{t} \in \mathfrak{T}$, now, consider the mapping $u \rightarrow (\psi(u))(\mathfrak{t})$ of the universe U into the set $\mathfrak{V}$: because of stipulation $\mathcal{T}4$, there are in $\mathfrak{V}$ two propositions, dependent of course on the given $\mathfrak{t} \in \mathfrak{T}$, to be denoted respectively by $(\forall_1 u)[(\psi(u))(\mathfrak{t})]$ and by $(\exists_1 u)[(\psi(u))(\mathfrak{t})]$, fulfilling certain conditions, specified under the same $\mathcal{T}4$. It makes sense, then given this mapping $\psi : U \rightarrow \mathfrak{W}$, to define

(27_1) $[(\forall_2 u)(\psi(u))](\mathfrak{t}) = (\forall_1 u)[(\psi(u))(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$,

(27_2) $[(\exists_2 u)(\psi(u))](\mathfrak{t}) = (\exists_1 u)[(\psi(u))(\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$.

For each given $\mathfrak{t} \in \mathfrak{T}$, $[(\forall_2 u)(\psi(u))](\mathfrak{t})$ and $[(\exists_2 u)(\psi(u))](\mathfrak{t})$ are defined by (27_1) and (27_2) as certain members of the set $\mathfrak{V}$, hence (27_1) and (27_2) define $[(\forall_2 u)(\psi(u))]$ and $[(\exists_2 u)(\psi(u))]$ as certain mappings of $\mathfrak{T}$ into $\mathfrak{V}$, i.e. as certain members of the set $\mathfrak{W}$.

4.1. To each formula A of the predicate calculus, we are now going to associate a member $\| A \|$ of the set $\mathfrak{W}$, to be called *the $\mathcal{T}$-theoretical reading* of the formula A; such $\mathcal{T}$-theoretical reading will depend on the chosen theory $\mathcal{T}$, of course. To this effect, let us consider a sequence u_1, u_2, u_3, ..., of parameters and stipulate the following.

$\mathcal{T}$-THEORETICAL READING: AN INDUCTIVE DEFINITION.

$\mathfrak{V}1$. $\| a_k \| (\mathfrak{t}) = \pi_{0k}(\mathfrak{t})$ *for* $\mathfrak{t} \in \mathfrak{T}$ $[k = 1, 2, 3, ...]$.

$\mathfrak{V}2$. $\| b_k \| (\mathfrak{t}) = \pi_{-1,k}(\mathfrak{t})$ *for* $\mathfrak{t} \in \mathfrak{T}$ $[k = 1, 2, 3, ...]$.

$\mathfrak{V}3$. $\| x_k \| (\mathfrak{t}) = u_k$ *for* $\mathfrak{t} \in \mathfrak{T}$ $[k = 1, 2, 3, ...]$.

$\mathfrak{V}4$. $\| p_{rk} \| (\mathfrak{t}) = \pi_{rk}(\mathfrak{t})$ *for* $\mathfrak{t} \in \mathfrak{T}$ $[r, k = 1, 2, 3, ...]$.

$\mathfrak{V}5$. $\| p_{rk} y_{k_1} y_{k_2} \dots y_{k_r} \| (\mathfrak{t}) = [\| p_{rk_r} \| (\mathfrak{t})][\| y_{k_1} \| (\mathfrak{t}), \dots, \| y_{k_r} \| (\mathfrak{t})]$ *for* $\mathfrak{t} \in \mathfrak{T}$ $[r, k, k_1, k_2, \dots, k_r = 1, 2, 3, ...]$.

$\mathfrak{V}6$. $\| \neg A \| = \sim_2 \| A \|$, $\| A \wedge B \| = \| A \| \cap_2 \| B \|$,
$\| A \vee B \| = \| A \| \cup_2 \| B \|$, $\| A \rightarrow B \| = \| A \| \supset_2 \| B \|$.

$\mathfrak{V}7$. $\| (\bigwedge x_k)(A) \| = (\forall_2 u_k)(\| A \|)$, $\| (\bigvee x_k)(A) \| = (\exists_2 u_k)(\| A \|)$.

In close analogy to Th. 1, we have the following

TH. 4. *Clauses* $\mathfrak{V}1$ *through* $\mathfrak{V}7$ *define inductively a* $\mathcal{T}$*-theoretical reading* $\| A \|$ *for each formula A of the predicate calculus; such* $\mathcal{T}$*-theoretical readings are members of the set* $\mathfrak{W}$.

This Th. 4 is a trivial consequence of the conjunction of lemma 2 with the following

LEMMA 4. *Given an open formula C* [*and s bound object variables* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$], *if in C there is no unquantified occurrence of any bound object variable* [*other than* $x_{h_1}, x_{h_2}, \ldots, x_h$], *then clauses* $\mathfrak{V}1$ *through* $\mathfrak{V}7$ *define for C a* $\mathcal{T}$*-theoretical reading* $\| C \|$, *which turns out to be a member of the set* $\mathfrak{W}$ [*independent of any parameter other than* $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$].

The proof of this lemma is analogous to the proof of lemma 3. Here it is.

4.2. (*a*) Under $\mathfrak{V}1$ and (25_0), $\| a_k \| (\mathfrak{f})$ is a member of $\mathfrak{V}$ for all $\mathfrak{f} \in \mathfrak{T}$, hence the $\mathcal{T}$-theoretical reading $\| a_k \|$, of the propositional variable a_k, is a mapping of $\mathfrak{T}$ into $\mathfrak{V}$, i.e. a member of $\mathfrak{W}$. In other words, *if the open formula C is given by clause* OF1, *then clause* $\mathfrak{V}1$ *defines its* $\mathcal{T}$*-theoretical reading* $\| C \|$ *as a member of the set* $\mathfrak{W}$ [independent of any parameter whatsoever].

(*b*) Given $\mathfrak{f} \in \mathfrak{T}$, and $u_{k_1}, u_{k_2}, \ldots, u_{k_r} \in U$, $\| p_{rk} \| (\mathfrak{f})$ is a certain r-adic predicate under $\mathfrak{V}4$ and (25), while $\| b_{k_1} \| (\mathfrak{f})$, $\| b_{k_2} \| (\mathfrak{f})$, ..., $\| b_{k_r} \| (\mathfrak{f})$ are certain r objects under $\mathfrak{V}2$ and (25_{-1}), and $\| x_{k_1} \| (\mathfrak{f})$, $\| x_{k_2} \| (\mathfrak{f})$, ..., $\| x_{k_r} \| (\mathfrak{f})$ are certain r objects under $\mathfrak{V}3$; consequently, if in the open formula $p_{rk} y_{k_1} y_{k_2} \ldots y_{k_r}$ there is no occurrence of any bound object variable [other than $x_{h_1}, x_{h_2}, x_{h_s}$], then under $\mathfrak{V}5$,

$$[\| p_{rk} \| (\mathfrak{f})][\| y_{k_1} \| (\mathfrak{f}), \ldots, \| y_{k_r} \| (\mathfrak{f})]$$

is a certain member of $\mathfrak{V}$, which depends on no parameter other than $\mathfrak{f} \in \mathfrak{T}$ [and $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$]. Thus, under clauses $\mathfrak{V}2$ through $\mathfrak{V}5$, if no bound object variable [other than $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$] occurs in the open formula $p_{rk} y_{k_1} y_{k_2} \ldots y_{k_r}$, then the $\mathcal{T}$-theoretical reading $\| p_{rk} y_{k_1} y_{k_2} \ldots y_{k_r} \|$ is a certain mapping of $\mathfrak{T}$ into $\mathfrak{V}$ — i.e. a certain member of $\mathfrak{W}$ — which depends on no parameter [other than $u_{h_1}, u_{h_2}, \ldots, u_{h_s}$].

In other words, *if the open formula C is given by clause* OF2 *and in C there is no unquantified occurrence of any bound object variable* [*other than x_{h_1}, x_{h_2}, ..., x_{h_s}*], *then clauses* $\mathfrak{V}2$ *through* $\mathfrak{V}5$ *define the $\mathcal{T}$-theoretical reading $\| C \|$ as a member of the set* $\mathfrak{W}$, *dependent on no parameter* [*other than u_{h_1}, u_{h_2}, ..., u_{h_s}*].

(*c*) Suppose C is given by clause OF3, and consider first the case that $C = \neg(A)$, where A is some open formula. If in C there is no unquantified occurrence of any bound object variable [other than x_{h_1}, x_{h_2}, x_{h_s}], then, by OF3, in A there is no unquantified occurrence of any bound object variable [other than x_{h_1}, x_{h_2}, x_{h_s}], and, by the inductive assumption, clauses $\mathfrak{V}1$ through $\mathfrak{V}7$ define the $\mathcal{T}$-theoretical reading $\| A \|$ as a member of the set $\mathfrak{W}$, independent of any parameter [other than u_{h_1}, u_{h_2}, ..., $u_{h_s} \in U$]; then clause $\mathfrak{V}6$ defines $\| C \| = \sim_2 \| A \|$ also as a member of the set $\mathfrak{W}$, independent of any parameter [other than u_{h_1}, u_{h_2}, ..., $u_{h_s} \in U$].

A similar argument takes care of the case that C be equal to either $(A) \wedge (B)$ or $(A) \vee (B)$ or $(A) \vdash (B)$.

(*d*) Suppose C is given by clause OF4, and consider first the case that $C = (\bigwedge x_k)(A)$, where A is some open formula. If in C there is no unquantified occurrence of any bound object variable [other than x_{h_1}, x_{h_2}, ..., x_{h_s}], then, by OF4, there is in A no unquantified occurrence of any bound object variable other than x_k[, x_{h_1}, x_{h_2}, ..., x_{h_s}], and by the inductive assumption clauses $\mathfrak{V}1$ through $\mathfrak{V}7$ define the $\mathcal{T}$-theoretical reading $\| A \|$ as a member of the set $\mathfrak{W}$, dependent on no parameter other than u_k[, u_{h_1}, u_{h_2}, ..., u_{h_s}] $\in U$. Then clause $\mathfrak{V}7$ defines $\| C \| = (\forall_2 u_k)(\| A \|)$ as a member of the set $\mathfrak{W}$, dependent on no parameter [other than u_{h_1}, u_{h_2}, ..., $u_{h_s} \in U$].

A similar argument takes care of the case that $C = (\bigvee x_k)(A)$.
This concludes the proof of lemma 4.

4.3. After the proof of Th. 4, and because of clause $\mathfrak{V}6$,

TH. 5. *Clauses* $\mathfrak{V}1$ *through* $\mathfrak{V}7$ *define a* [*homomorphic*] *mapping $A \to \| A \|$ of the set $\mathcal{F}$ of all formulae of the predicate calculus* [*under « operations » $\neg$, $\wedge$, $\vee$, $\vdash$*] *into the set* $\mathfrak{W}$ [*under operations* $\sim_2$, $\cap_2$,

$\cup_2, >_2]$. *The image of* $\mathscr{F}$ *under such mapping is itself a substructure* $\| \mathscr{F} \|$ *of the structure* $\langle \mathfrak{W}, \sim_2, \cap_2, \cup_2, >_2 \rangle$.

Two formulae A and B are said to be *$\mathcal{T}$-theoretically equivalent*, when they have the same $\mathcal{T}$-theoretical reading, i.e. when $\| A \| = \| B \|$; if simply $\| A \| \supset_2 \| B \|$, then one says that *the entailment* $A \leqq B$ *is $\mathcal{T}$-theoretically valid*, or that A *$\mathcal{T}$-theoretically entails* B.

The content of sec.s 2 and 4 up to this point can be summarized in the diagram shown in figure 1.

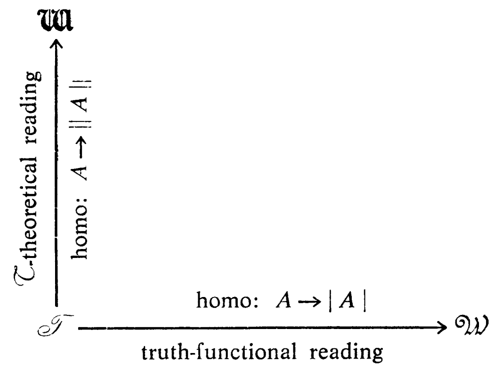

Fig. 1.

4.4. In order to state the main theorem of this section, we need a mapping $*$ of the set $U \cup \mathfrak{V} \cup \bigcup_{r=1}^{\infty} \mathfrak{P}_r$ into the set $U \cup V \cup \bigcup_{r=1}^{\infty} P_r$, and a mapping $\#$ of the set $\mathfrak{T}$ into the set T, which we are going to define, under the common sense assumption that sets U, V, P_r $\mathfrak{V}$, $\mathfrak{P}_r$ $[r = 1, 2, 3, ...]$ be mutually disjoint. We stipulate the following:

(28₁) $u^* = u$, *for all* $u \in U$,

(28₂) $\mathfrak{v}^* = \widehat{\mathfrak{v}}$, *for all* $\mathfrak{v} \in \mathfrak{V}$,

(28₃) $\mathfrak{p}^*(u^*_1, u^*_2, ..., u^*_r) = (\mathfrak{p} u_1 u_2 \ldots u_r)^*$, *for* $u_1, u_2, ..., u_r \in U$ *and for* $\mathfrak{p} \in \mathfrak{P}_r$;

moreover, *for all* $\mathfrak{f} \in \mathfrak{T}$,

(29) $\mathfrak{f}^\# = [(\mathfrak{f}^*_{-11}, \mathfrak{f}^*_{-12}, \mathfrak{f}^*_{-13}, ...), (\mathfrak{f}^*_{01}, \mathfrak{f}^*_{02}, \mathfrak{f}^*_{03}, ...), (\mathfrak{f}^*_{11}, \mathfrak{f}^*_{12}, \mathfrak{f}^*_{13}, ...),$
$(\mathfrak{f}^*_{21}, \mathfrak{f}^*_{22}, \mathfrak{f}^*_{23}, ...), ...]$.

Under (28_1), the restriction of mapping $*$ to the set U is simply *the identical mapping*; under (28_2), the restriction of mapping $*$ to the set $\mathfrak{V}$ of all proposition is simply the « *truth-value of* » mapping, which of course maps $\mathfrak{V}$ into V.

Under $\mathcal{T}2$, given $\mathfrak{p} \in \mathbb{P}_r$ and $u_1, u_2, ..., u_r \in U$, $\mathfrak{p} u_1 u_2 ... u_r$ is a member of $\mathfrak{V}$, and (28_3) will be equivalent, because of (28_1) and (28_2), to

$$(28'_3) \qquad \mathfrak{p}^*(u_1, u_2, ..., u_r) = (\mathfrak{p} u_1 u_2 ... u_r)^{\hat{}};$$

this ($28'_3$) shows that $\mathfrak{p}^*$ is a mapping of U^r into V, while condition (7) proves that $\mathfrak{p}^*$ actually belongs to the set P_r [see (3)]. Thus, $*$ actually maps the set $\mathbb{P}_r$ into the set P_r.

As for (29), for each given $\mathfrak{f} \in \mathfrak{T}$ it defines $\mathfrak{f}^\#$ to be a sequence of sequences, with

$$(29') \qquad \pi_{-1k}(\mathfrak{f}^\#) = [\pi_{-1k}(\mathfrak{f})]^*, \ \pi_{0k}(\mathfrak{f}^\#) = [\pi_{0k}(\mathfrak{f})]^*, \ \pi_{rk}(\mathfrak{f}^\#) = [\pi_{rk}(\mathfrak{f})]^* \quad [r, k = 1, 2, 3, ...].$$

But, under (28_1) and (25_{-1}), $[\pi_{-1k}(\mathfrak{f})]^*$ is an object; under (28_2) and (25_0), $[\pi_{0k}(\mathfrak{f})]^*$ is a truth-value; under (28_3) and (25_r), $[\pi_{rk}(\mathfrak{f})]^*$ is a member of P_r; then, for each $\mathfrak{f} \in \mathfrak{T}$, under (29) $\mathfrak{f}^\#$ must be a member of T.

Let us consider, now, an open formula C of the predicate calculus, a member $\mathfrak{f}$ of the set $\mathfrak{T}$ [and s bound object variables $x_{h_1}, x_{h_2}, ..., x_{h_s}$], under the assumption that in C there is no unquantified occurence of any bound object variable [other than $x_{h_1}, x_{h_2}, ..., x_{h_s}$]. By lemma 4, the $\mathcal{T}$-theoretical reading $\| C \|$ of C is a member of $\mathfrak{W}$ — i.e. a mapping of $\mathfrak{T}$ into $\mathfrak{V}$ — independent of any parameter [other than $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$], therefore $\| C \| (\mathfrak{f})$ is a member of $\mathfrak{V}$, independent of any parameter [except $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$], and $[\| C \| (\mathfrak{f})]^*$ is, under (28_2), a truth-value independent of any parameter [other than $u_{h_1}, u_{h_2}, ..., u_{h} \in U$]. Similarly, by lemma 3, the truth-functional reading $| C |$ of C is a member of W — i.e. a mapping of T into V — independent of any parameter [other than $u_{h_1}, u_{h_2}, ..., u_{h_s} \in U$], while $\mathfrak{f}^\#$ is a member of T, therefore $| C | (\mathfrak{f}^\#)$ is a member of V — i.e. a truth-value — independent of any parameter [other than $u_{h_1}, u_{h_2} ..., u_{h_s} \in U$]. Let us prove the following

LEMMA 5. *Given an open formula C of the predicate calculus, given a member* $\mathfrak{f}$ *of the set* $\mathfrak{T}$ *[and s bound object variables* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$*], if in C there is no unquantified occurrence of any bound object variables [other than* $x_{h_1}, x_{h_2}, \ldots, x_{h_s}$*], then we have*

$$(30) \quad [\| C \| (\mathfrak{f})]^* = | C | (\mathfrak{f}^\#),$$

[identically with respect to the s parameters $u_{h_1}, u_{h_2}, \ldots, u_{h_s} \in U$*].*

PROOF. (*a*) If C is given by OF1, we have $C = a_k$ for some $k = 1, 2, \ldots$, hence, under $\mathfrak{V}1$, (29′), V1, we have also

$$[\| C \| (\mathfrak{f})]^* = [\| a_k \| (\mathfrak{f})]^* = [\pi_{0k}(\mathfrak{f})]^* = \pi_{0k}(\mathfrak{f}^\#) = | a_k | (\mathfrak{f}^\#) = | C | (\mathfrak{f}^\#).$$

(*b*) Under $\mathfrak{V}2$, (29′), V2, we have $[\| b_k \| (\mathfrak{f})]^* = [\pi_{-1k}(\mathfrak{f})]^* = \pi_{-1k}(\mathfrak{f}^\#) = | b_k | (\mathfrak{f}^\#)$, for $k = 1, 2, 3, \ldots$; similarly, under $\mathfrak{V}3$, V3, we have $[\| x_k \| (\mathfrak{f})]^* = u_k = | x_k | (\mathfrak{f}^\#)$, for $k = 1, 2, 3, \ldots$; this can be summed up in the equality $[\| y_k \| (\mathfrak{f})]^* = | y_k | (\mathfrak{f}^\#)$, valid for all $k = 1, 2, 3, \ldots$.

Moreover, under $\mathfrak{V}4$, (29′), V4, we have

$$[\| p_{rk} \| (\mathfrak{f})]^* = [\pi_{rk}(\mathfrak{f})]^* = \pi_{rk}(\mathfrak{f}^\#) = | p_{rk} | (\mathfrak{f}^\#), \text{ for } r, k = 1, 2, 3, \ldots$$

Hence, under $\mathfrak{V}5$, (28_3), V5, we have

$$\begin{aligned} &[\| p_{rk} y_{k_1} y_{k_2} \ldots y_{k_r} \| (\mathfrak{f})]^* = \\ &= ([\| p_{rk} \| (\mathfrak{f})][\| y_{k_1} \| (\mathfrak{f}), \ldots, \| y_{k_r} \| (\mathfrak{f})])^* = \\ &= [\| p_{rk} \| (\mathfrak{f})]^*([\| y_{k_1} \| (\mathfrak{f})]^*, \ldots, [\| y_{k_r} \| (\mathfrak{f})]^*) = \\ &= [| p_{rk} | (\mathfrak{f}^\#)][| y_{k_1} | (\mathfrak{f}^\#), \ldots, | y_{k_r} | (\mathfrak{f}^\#)] = | p_{rk} y_{k_1} y_{k_2} \ldots y_{k_r} | (\mathfrak{f}^\#), \end{aligned}$$

whatever $r, k, k_1, k_2, \ldots, k_r = 1, 2, 3, \ldots$, and identically with respect to all parameters $u_1, u_2, \ldots$.

We reach the conclusion that, if C is given by OF2, the lemma's thesis holds.

(*c*) Suppose C is given by clause OF3, and consider first the case $C = \neg A$, where A is some open formula fulfilling the inductive assumption that $[\| A \| (\mathfrak{f})]^*$ be identically equal to $| A | (\mathfrak{f}^\#)$. Then, under $\mathfrak{V}6$, (26_1), (28_2), ($8'_1$), (28_2), the inductive assumption, the « pointwise » definition of $\neg_2$, V6, we have

$$[\| C \| (\mathfrak{f})]^* = \{[\sim_2 \| A \|](\mathfrak{f})\}^* = (\sim_1 [\| A \| (\mathfrak{f})])^* =$$

$$= (\sim_1 [\| A \| (\mathfrak{f})])^{\wedge} = \neg_1 [\| A \| (\mathfrak{f})]^{\wedge} = \neg_1 [\| A \| (\mathfrak{f})]^* =$$

$$= \neg_1 [| A | (\mathfrak{f}^\#)] = [\neg_2 | A |](\mathfrak{f}^\#) = | \neg A | (\mathfrak{f}^\#) = | C | (\mathfrak{f}^\#), \qquad \text{Q.E.D.}$$

A similar argument takes care of the case that C be equal to either $(A) \wedge (B)$ or $(A) \vee (B)$ or $(A) \vdash\!\!- (B)$.

(d) Suppose C is given by clause OF4, and consider first the case that $C = (\bigwedge x_k)(A)$, where A is some open formula fulfilling the inductive assumption that $[\| A \| (\mathfrak{f})]^*$ be identically equal to $| A | (\mathfrak{f}^\#)$. Then, under $\mathfrak{V}7$, (27_1), (28_2), (10_1), the inductive assumption, the « pointwise » character of operations in $\mathfrak{M}$, and V7, we have

$$[\| C \| (\mathfrak{f})]^* = [\| (\bigwedge x_k)(A) \| (\mathfrak{f})]^* = \{[(\forall_2 u_k)(\| A \|)](\mathfrak{f})\}^* =$$

$$= \{(\forall_1 u_k)[\| A \| (\mathfrak{f})]\}^* = \{(\forall_1 u_k)[\| A \| (\mathfrak{f})]\}^{\wedge} =$$

$$= \inf \{\widehat{u_k} \vdash_1 [\| A \| (\mathfrak{f})]^{\wedge} : u_k \in U\} = \inf \{\widehat{u_k} \vdash_1 [| A | (\mathfrak{f}^\#)] : u_k \in U\} =$$

$$= (\inf \{\gamma_2(u_k) \vdash_2 | A | : u_k \in U\})(\mathfrak{f}^\#) = | (\bigwedge x_k)(A) | (\mathfrak{f}^\#) = | C | (\mathfrak{f}^\#), \quad \text{Q.E.D.}$$

A similar argument takes care of the case that C be equal to $(\bigvee x_k)(A)$.

By structural induction, the lemma is proved.

By lemmas 2 and 5, by (28_2), we draw the corollary

LEMMA 6. *For each formula C of the predicate calculus, for all $\mathfrak{f} \in \mathfrak{T}$,*

(31) $[\| C \| (\mathfrak{f})]^{\wedge} = | C | (\mathfrak{f}^\#)$.

We are now in position to prove the main result of this section:

TH. 6. *For any two formulae A and B of the predicate calculus, if $| A | \leqq_2 | B |$, then $\| A \| \supset_2 \| B \|$.*

PROOF. Assuming $|A| \leqq_2 |B|$, by the « pointwise » character of the order relation in $\mathfrak{W}$ we have $|A|(t) \leqq_1 |B|(t)$ for all $t \in T$, hence $|A|(\mathfrak{f}^\#) \leqq_1 |B|(\mathfrak{f}^\#)$ for all $\mathfrak{f} \in \mathfrak{T}$; by lemma 6, then, $[\|A\|(\mathfrak{f})]^\wedge \leqq_1 \leqq_1 [\|B\|(\mathfrak{f})]^\wedge$ for all $\mathfrak{f} \in \mathfrak{T}$, hence, by (9), $\|A\|(\mathfrak{f}) \supset_1 \|B\|(\mathfrak{f})$ for all $\mathfrak{f} \in \mathfrak{T}$, i.e., by (26_5), $\|A\| \supset_2 \|B\|$, Q.E.D.

5. The predicate calculus of the first order: semantics.

We are now in position to state the following theorem.

TH. 7. *Given a nonempty set U, a partially ordered set* $\mathfrak{M}$, *and a mapping* $\gamma_1 : U \to \mathfrak{B}^{\mathfrak{M}}$,

if $A \leqq B$ *is a truth-functionally valid entailment, relative to the Universe U, to the truth-lattice* $\mathfrak{V} = \mathfrak{B}^{\mathfrak{M}}$, *to the acquaintance function* γ_1;

then $A \leqq B$ *is also a* $\mathcal{T}$*-theoretically valid entailment, for every theory* $\mathcal{T}$ *whose universe of objects equals (up to bijections) the set U, whose panel of witnesses equals (up to order isomorphisms) the partially ordered set* $\mathfrak{M}$, *and whose acquaintance function equals* γ_1.

This theorem is actually a restatement of theorem 6. But furthermore

TH. 8. *Given a nonempty set U, a partially ordered set* $\mathfrak{M}$, *and a mapping* $\gamma_1 : U \to \mathfrak{B}^{\mathfrak{M}}$, *there is a theory* $\mathcal{T}$ *fulfilling the following conditions*:

the universe of objects for theory $\mathcal{T}$ *equals U (up to bijections)*;

the panel of witnesses associated to theory $\mathcal{T}$ *equals the partially ordered set* $\mathfrak{M}$ *(up to order isomorphisms)*;

the acquaintance function for theory $\mathcal{T}$ *equals* γ_1;

given two formulae A and B, the entailment $A \leqq B$ *is truth-functionally valid, relative to the Universe U, the truth-lattice* $\mathfrak{V} = \mathfrak{B}^{\mathfrak{M}}$, *the acquaintance function* γ_1, *if and only if it is* $\mathcal{T}$*-theoretically valid.*

The proof of this theorem is just a variant of the proof of Th. 12 in [1]; loosely speaking, theory $\mathcal{T}$ mentioned in the statement of this theorem is just a formal version of the description of the « truth-functional reading » of a formula for the predicate calculus: there is no point in carrying it out explicitly.

These two theorems again substantiate my contention, that *a manivalued semantics can be interpreted in a quite reasonable and commonsensical way, and, whatever that semantics might be, it will have its own field of application, however restricted*; a recognition that the arguments used in this paper can be formalized in two-valued logics substantiates my contention that, *whatever can be accomplished by using manivalued logic, it can also be accomplished (albeit perhaps in a clumsier way) by using two-valued logic.*

It is true that this paper takes only those manivalued logics into consideration, whose truth-lattices are of type $\mathfrak{B}^{\mathfrak{M}}$, where $\mathfrak{M}$ is an arbitrary partially ordered set; it is also true, however, that (as recalled in sec. 0) among such manivalued logics, all those logics are to be found, whose truth-lattices are distributive lattices of finite length. Of course, it would be interesting a study of the semantics of manivalued logics of a more general type.

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