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Coherent configurations I

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COHERENT CONFIGURATIONS I

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In the theory of G-spaces (G a finite group) there arise certain combinatorial configurations which we call *coherent* [6]. It is the purpose of the present paper to lay the ground work for a systematic study of this class of configurations in their own right. The first two sections, containing generalities about graphs, incidence structures and matrices, are included to establish our notation and to make the paper self-contained. Coherent (and semi coherent) configurations are defined in § 3 and the way which they arise in the theory of G-spaces is indicated. Our main line of attack in studying these configurations is through their semigroups of relations and centralizer rings, discussed in §§ 4-6 together with the decomposition into orbits. The methods of intersection matrices (cf. [5]) and coherent partitions are available for coherent configurations as is shown in §§ 7 and 8. In § 9 we consider the question of coarsening and refining given configurations and in § 10 we introduce a concept of primitivity and consider its relation to connectedness. Further work concerning extensions, automorphism groups, and some classification theorems will be discussed in a second paper.

Coherent configurations generalize the notion of *association scheme* (cf. [2]) and so encompass, for example, the strongly regular graphs of Bose [1]. Additional types of configurations, for example the symmetric block designs, are included as indicated at the end of § 6.

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1. Relations, graphs and incidence structures.

The sets X, Y, ... considered in this paper will all be assumed to be finite. A *relation* from a set X to a set Y is a subset of $X \times Y$; we denote the totality of these by Rel (X, Y), simplifying the notation to Rel X in case $X=Y$. The *converse* of $f \in$ Rel (X, Y) is the relation $f^{\cup}=\{y, x) \mid (x, y) \in f\}$ in Rel (Y, X). For $x \in X$ we put

$$f(x)=\{y \in Y \mid (x, y) \in f\}$$

and

$$\operatorname{dom} f=\{x \in X \mid f(x) \neq \emptyset\}, \qquad \operatorname{range} f=\{y \in Y \mid f^{\cup}(y) \neq \emptyset\}.$$

Relations $f \in$ Rel (X, Y) and $g \in$ Rel (Y, Z) are composed according to $fg=\{(x, z) \mid f(x) \cap g^{\cup}(z) \neq \emptyset\}$ to give $fg \in$ Rel (X, Z). The identity relation

$$I_X=\{(x, x) \mid x \in X\}$$

is an identity for this composition so we have a category, the category of relations (between finite sets), which we denote by Rel.

We denote by Map (X, Y) the set of maps from X to Y, writing Map X for Map (X, X), and regard the category Map of maps (between finite sets) as a subcategory of Rel.

With $f \in$ Rel (X, Y) there is associated a graph $\mathfrak{H}_f$ having the disjoint union $X+Y$ of X and Y as vertex set and f as edge set.

The *left equivalence kernel* $L_f=\bigcup_{n \geq 0} (ff^{\cup})^n$ and *right equivalence kernel* $R_f=\bigcup_{n \geq 0} (f^{\cup}f)^n$ of $f \in$ Rel (X, Y), $f \neq \emptyset$, are equivalence relations in Rel X and Rel Y respectively, and $L_f f=fR_f$. We see that any two of the conditions (i) $L_f=X \times X$, (ii) $L_f f=X \times Y$, and (iii) $R_f=Y \times Y$, imply the third; $\mathfrak{H}_f$ is said to be *connected* if these three conditions hold.

An incidence structure $\mathfrak{J}_f$ having $\mathfrak{H}_f$ as its Levi graph (cf. [3]) is obtained by taking X, Y and f respectively as the points, blocks and flags.

Assume that $X=Y$. With $f \in$ Rel X there is associated a second graph $\mathfrak{G}_f$ having vertex set X and edge set f. The graph $\mathfrak{G}_f$ is *connected* if the *kernel* $E_f = \bigcup_{m\geq 0} f^n$ of f is equal to $X\times X$. It is, of course, quite possible that either one of the graphs $\mathcal{H}_f$ or $\mathcal{G}_f$ is connected while the other is not, but we note that

(1.1) *If* E_f *is an equivalence relation for some* f *in* Rel X, *then* $\mathcal{G}_f$ *is connected if* $\mathcal{H}_f$ *is.*

2. Matrices.

The category Mat_K of matrices in K, where K is a coefficient ring which we will choose to be commutative with identity element, has as objects the finite sets $X, Y, \ldots$ and as morphism the maps $X\times Y\rightarrow K$. We write $\mathrm{Mat}_K(X, Y)$ for the totality of these and compose $\varphi\in\mathrm{Mat}_K(X, Y)$ and $\psi\in\mathrm{Mat}_K(Y, Z)$ according to matrix multiplication

$$\varphi\psi(x, z)=\sum\varphi(x, y)\psi(y, z) \qquad (x, z)\in X\times Z$$

to obtain $\varphi\psi\in\mathrm{Mat}_K(X, Z)$. We write $\mathrm{Mat}_K X$ for $\mathrm{Mat}_K(X, X)$.

The maps $\mathrm{Rel}(X, Y) \underset{\mathrm{spt}}{\overset{\mathrm{char}}{\rightleftarrows}} \mathrm{Mat}_K(X, Y)$, where char f, $f\in\mathrm{Rel}(X, Y)$, is the characteristic function of f, and spt φ, $\varphi\in\mathrm{Mat}_K(X, Y)$, is the support of φ, are such that $\mathrm{spt}\circ\mathrm{char}=1$. Of course these maps are not functorial, but, e.g., if $\sigma\in\mathrm{Map}(X, Y)$, $f\in\mathrm{Rel}(X, Z)$ then

$$\mathrm{char}\,\sigma f=\mathrm{char}\,\sigma\cdot\mathrm{char}\,f.$$

Given orderings of X and Y we represent $\varphi\in\mathrm{Mat}_K(X, Y)$ by the matrix

$$B_\varphi=(\varphi(x, y))_{x,y}\,.$$

We write B_f for B_φ if $\varphi=\mathrm{char}\,f$, $f\in\mathrm{Rel}(X, Y)$, so that $\mathcal{H}_f$ has adjacency matrix $\begin{pmatrix}0 & B_f\\ B_f^t & 0\end{pmatrix}$. and $\mathfrak{J}_f$ has incidence matrix B_f. In case $X=Y$, B_f is the adjacency matrix of $\mathcal{G}_f$.

We see that for $f\in\mathrm{Rel}(X, Y)$, blocking B_f according to L_f and

R_f corresponds to full decomposition of B_f under permutation equivalence. In particular

(2.1) *For $f \in \operatorname{Rel}(X, Y)$, $\mathcal{H}_f$ is connected if and if B_f is indecomposable under permutation equivalence.*

If $a \in X$ and $f \in \operatorname{Rel}(X, Y)$, we put

$$\Gamma_f^{(i)}(a) = \{x \in X \mid (a, x) \in (ff^{\cup})^i \text{ and } (a, x) \notin (ff^{\cup})^j \text{ for } j < i\},$$

and

$$\Delta_f^{(i)}(a) = \{y \in Y \mid (a, y) \in (ff^{\cup})^i f \text{ and } (a, y) \notin (ff^{\cup})^j f \text{ for } j < i\}.$$

(2.2) *Assume that $\mathcal{H}_f$ is connected. Blocking the rows of B_f according to the $\Gamma_f^{(i)}(a)$ and the columns according to the $\Delta_{(i)}^f(a)$, $i = 0, 1, \ldots,$ corresponds to transformation under permutation equivalence of B_f to the form*

$$\begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & 0 \\ & b_{32} & b_{33} & \\ & 0 & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \quad \cdot \end{bmatrix}$$

where b_{11} is a 1-rowed matrix, and for all i, each row of the block b_{ii} and each column of the block b_{i+1i} contains an entry $\neq 0$.

We now consider $f \in \operatorname{Rel} X$ such that E_f is an equivalence relation. We see that a reduction of B_f under permutation similarity is automatically a decomposition and that blocking B_f according to E_f corresponds to full reduction of B_f under permutation similarity. In particular

(2.3) *If $f \in \operatorname{Rel} X$ is such that E_f is an equivalence relation, then the following statements are equivalent.*

a) $\mathcal{G}_f$ is connected.

b) B_f is irreducible under permutation similarity, and

c) B_f is indecomposable under permutation similarity.

Given $f \in \operatorname{Rel} X$ and $x, y \in X$ we define: $\rho_f(x, y) =$ the least integer $i \geq 0$ such that $(x, y) \in f^i$, or ∞ if no such i exists.

For $i \geq 0$, the *f-circle* with center x and radius i is

$$\Lambda_f^{(i)}(x) = \{y \in X \mid \rho_f(x, y) = i\}.$$

The diameter of the graph $\mathcal{G}_f$ is defined to be

$$\max\{\rho_f(x, y) \mid x, y \in X\}.$$

(2.4) *Assume that $\mathcal{G}_f$ is connected. Then blocking B_f according to the circles $\Lambda_f^{(i)}(x)$, $i = 1, 2, \ldots$, for given $x \in X$ corresponds to transformations under permutation similarity of B_f to the form*

$$\begin{bmatrix} 0 & c_{12} & & & & \\ & c_{22} & c_{23} & & 0 & \\ & & c_{33} & \cdot & & \\ & & & \cdot & \cdot & \\ & & * & & \cdot & \cdot \\ & & & & \cdot & c_{m-1m} \\ & & & & & c_{mm} \end{bmatrix}$$

in which c_{12} is a 1-rowed matrix, each block c_{ii} is square, and each column of the block c_{ii+1} contains an entry $\neq 0$. The number m of blocks on the main disgonal is one greater than the diameter of $\mathcal{G}_f$.

3. Coherent and semi coherent configurations.

We think of configurations consisting of finite collections of disjoint sets together with finite collections of relations between them, and visualize the relations in terms of the associated graphs and incidence structures as described in § 2. By an obvious device we can reduce the number of sets to one, so a sufficiently general definition of *combinatorial configuration* is a pair $(X, \mathcal{O})$ consisting of a set X and a set $\mathcal{O}$ of relations on X. A configuration $\mathfrak{g}_0 = (X_0, \mathcal{O}_0)$ *will* be called a *subconfiguration* of $\mathfrak{g} = (X, \mathcal{O})$, written $\mathfrak{g}_0 \leq \mathfrak{g}$, if $X_0 \subseteq X$ and $\mathcal{O}_0 \subseteq \mathcal{O} \mid X_0$, where $\mathcal{O} \mid X_0 = \{f \cap (X_0 \times X_0) \mid f \in \mathcal{O}\}$. The *full* subconfiguration on $X_0 \subseteq X$ is $(X_0, \mathcal{O} \mid X_0)$.

We now single out the class of configurations of particlar interest to us here. Given a configuration $(X, \mathcal{O})$ and $f, g, h \in \mathcal{O}$, we define an (f, g, h)-*triangle* to be an ordered triple (x, y, z) such that $(x, y) \in f$, $(y, z) \in g$ and $(x, z) \in h$. We call $(X, \mathcal{O})$ *coherent* if

1) $\mathcal{O}$ is a partition of $X \times X$,

2) the identity relation I_X is a union of members of $\mathcal{O}$,

3) $f \in \mathcal{O}$ implies $f^{\cup} \in \mathcal{O}$, and

4) for given $f, g, h \in \mathcal{O}$ and $(x, z) \in h$, the number of (f, g, h)-triangles $(x\ y, z)$ is independent of the choice of (x, z); we denote this number by a_{fgh}.

Our motivation for this definition comes from almost obvious fact that

(3.1) *If X is a G-space for some group G, and if $\mathcal{O}_G(X)$ is the set of G-orbits in $X \times X$ (under componentwise action), then $(X, \mathcal{O}_G(X))$ is coherent.*

On the one hand this provides us with a rich source of coherent configurations, and on the other with a bridge between group theory and combinatorics. A coherent configuration obtained from the action of a group G on a set X as in (2.1) will be said to be *realized* by G, or by the action of G on X, or by the G-space X.

We carry over much of the terminology of the theory of G-spaces (cf. [6]) to general combinatorial configurations $\mathfrak{g} = (X, \mathcal{O})$ The number $n = |X|$ will be called the *degree* of $\mathfrak{g}$. The relations in $\mathcal{O}$ will be referred to as the *orbital relations* of $\mathfrak{g}$ and their number $r = |\mathcal{O}|$ as the *rank* of $\mathfrak{g}$. The mapping $f \mapsto f^{\cup}$, $f \in \mathcal{O}$, is called the *pairing* on $\mathfrak{g}$. The numbers a_{fgh} of 4) will be called the *intersection numbers of* $\mathfrak{g}$. Association schemes (cf. [2]) are just those coherent configurations with $I_X \in \mathcal{O}$ and trivial pairing.

Assume that $\mathfrak{g}$ is coherent. For $x \in X$, the set $\{f(x) \mid f \in \mathcal{O}, f(x) \neq \emptyset\}$ is a partition of X. Moreover

(3.2) *For $f, g, h \in \mathcal{O}$ and $(x, z) \in h$,*

$$a_{fgh} = |f(x) \cap g^{\cup}(z)|.$$

Hence

(3.3) *For $f, g, h \in \mathcal{O}$, $a_{fgh} = a_{g^{\cup} f^{\cup} h^{\cup}}$.*

For some purposes it is convenient to replace condition 4) of the definition of coherent configuration by

4′) Given f, g, $h \in \mathcal{O}$, and $(x_i, z_i) \in h$, $i=1, 2$, there exists an (f, g, h)-triangle (x_1, y_1, z_1) if and only if there exists an (f, g, h)-triangle (x_2, y_2, z_2); we write $\widehat{a}_{fgh}=1$ or 0 according as such triangles exist or not.

Of course 4) implies 4′), with $\widehat{a}_{fgh}=1$ or $\widehat{a}_{fgh}=0$ according as $a_{fgh}\neq 0$ or $a_{fgh}=0$. We call $\mathfrak{g}$ *semi coherent* if 1), 2), 3), and 4′) are satisfied.

4. The semigroup of relations and the centralizer ring.

Let $\mathfrak{g}=(X, \mathcal{O})$ be a semicoherent configuration. Then the set $R=R(\mathfrak{g})$ consisting of $\emptyset$ and all unions of members of $\mathcal{O}$ is a subsemigroup of Rel X under composition. In fact, given 1), a necessary and sufficient condition for 4′) is that R is closed under composition. We refer to R as the *semigroup of relations* of $\mathfrak{g}$. The elements of R we refer as $\mathfrak{g}$-relations; in case $\mathfrak{g}$ is realized by a group, these are the G-relations [7].

Regarding Rel X as a vector space over $\boldsymbol{F}_2$ with respect to symmetric difference, so that R is the subspace spanned by $\mathcal{O}$, the product of f and g in $\mathcal{O}$ is

$$fg= \sum \widehat{a}_{fgh}h.$$

If the pairing $f \mapsto f^{\cup}$ on $\mathfrak{g}$ is trivial, then R is commutative.

From now on in this section we assume that $\mathfrak{g}$ is coherent, and choose a cofficient ring K, commutative with 1. The set $C=C_K(\mathfrak{g})$ of all $\varphi \in \mathrm{Mat}_K X$ which are constant on f for every $f \in \mathcal{O}$ is a free K-submodule of $\mathrm{Mat}_K X$ with the set $\mathfrak{B}=\{\mathrm{char}\, f \mid f \in \mathcal{O}\}$ as basis. We have for f, $g \in \mathcal{O}$, that

$$\mathrm{char}\, f \cdot \mathrm{char}\, g= \sum a_{fgh}\, \mathrm{char}\, h$$

so that C is a K-subalgebra of $\mathrm{Mat}_K X$. We refer to C as the *centralizer ring of* $\mathfrak{g}$ *over* K; if $\mathfrak{g}$ is realized by the action of a group G on the set X, then C is the centralizer ring of the G-space X. If the pairing on $\mathfrak{g}$ is trivial, then every $\varphi \in C$ is symmetric, hence

(4.1) *If the pairing* $\mathcal{G}$ *is trivial, then* C *is commutative.* Since C is closed under the transpose map.

(4.2) *If* K *is a field of characteristic* 0, *then* $C_K(\mathcal{G})$ *is semisimple.*

Taking K to be the complex number field $\boldsymbol{C}$, we have that

(4.3) $r=\sum_{i=1}^{m} e_i^2$ *where* $e_1, \ldots, e_m$ *are the structure constants of* $C_{\boldsymbol{C}}(\mathcal{G})$, i.e., *the degrees of the inequivalent irreducible representations of* $C_{\boldsymbol{C}}(\mathcal{G})$.

We refer to the e_i as the *structure constants* of $\mathcal{G}$.

Now suppose that X is a set $\neq\emptyset$ and $\mathfrak{B}$ is a subset of $\mathrm{Mat}_{\boldsymbol{Z}} X$ such that

a) $\varphi(x, y)\in\{0, 1\}$ for all $\varphi\in\mathfrak{B}$ and all $x, y\in X$,

b) $\sum \varphi(x, y)=1$ for all $x, y\in X$,

c) $\mathrm{char}\, I_X$ is a sum members of $\mathfrak{B}$,

d) $\varphi\in\mathfrak{B}$ implies $\varphi^t\in\mathfrak{B}$, and

e) the $\boldsymbol{Z}$-submodule B of $\mathrm{Mat}_{\boldsymbol{Z}} X$ spanned by $\mathfrak{B}$ is a subalgebra.

Then it is not difficult to see that $\mathcal{G}=(X, \mathcal{O})$, where $\mathcal{O}=\{\mathrm{spt}\,\varphi \mid \varphi\in\mathfrak{B}\}$, is a coherent configuration with $C_{\boldsymbol{Z}}(\mathcal{G})=B$.

5. Orbits.

Let $\mathcal{G}=(X, \mathcal{O})$ be a semicoherent configuration. Then

$$I_X=E_1+E_2+\ldots+E_t,$$

where the E_i are uniquely determined elements of $\mathcal{O}$. We have $\mathrm{dom}\, E_i=\mathrm{range}\, E_i$, and denoting this set by X_i, E_i induces the identity relation on X_i. The X_i, which we refer to as *orbits* of $\mathcal{G}$, or the $\mathcal{G}$-*orbits*, constitute a partition of X; in case $\mathcal{G}$ is realized by a group, the X_i are the G-orbits in X. We call $\mathcal{G}$ *transitive* if $t=1$, i.e., if $I_X\in\mathcal{O}$. The *degree* of X_i is defined to be $n_i=|X_i|$, and

(5.1) $$n=n_1+n_2+\ldots+n_t.$$

We now observe that

(5.2) *For* $f \in \mathcal{O}$, dom $f = X_i$ *and* range $f = X_j$ *for some* i, j.

PROOF. We have dom $f \cap X_i \neq \emptyset$ for some i, and then $\widetilde{a}_{E_i ff} = 1$. If $x \in$ dom f, there exists an element $z \in X$ such that $(x, z) \in f$. There must exist an (E_i, f, f)-triangle (x, y, z) and hence $x \in$ dom $E_i = X_i$, That is dom $f \subseteq X_i$.

Again, if $x \in$ dom f and $(x, z) \in f$, then (x, z, x) is an $(f, f^{\cup}, E_i)$-triangle, so $\widetilde{a}_{ff^{\cup}E_i} = 1$. Hence if $y \in X_i$, there must exist an $(f, f^{\cup}, E_i)$-triangle (y, w, y), and hence $y \in$ dom f. Thus dom $f = X_i$, and range $f =$ $=$ dom $f^{\cup} = X_j$ for some j.

We put $\mathcal{O}^{ij} = \{f \in \mathcal{O} \mid \text{dom } f = X_i \text{ and range } f = X_j\}$, so that $\{\mathcal{O}^{ij}\}$ is a partition of $\mathcal{O}$ by (5.2). Note that if $f, g, h \in \mathcal{O}$, then $\widetilde{a}_{fgh} = 1$ implies that $f \in \mathcal{O}^{ij}$, $g \in \mathcal{O}^{jk}$ and $h \in \mathcal{O}^{ik}$ for some i, j, k. The number $r_{ij} = |\mathcal{O}^{ij}|$ will be called the *rank* of the pair X_i, X_j; in case $\mathfrak{g}$ is realized by a group G, $r_{ij} = \text{rank}_G(X_i, X_j)$ (cf. [6]). We have

$$r_{ij} = r_{ji},$$

(5.3)
$$1 \leq r_{ij} \leq n_i n_j, \text{ and}$$

$$r = \sum_{1 \leq i, j \leq t} r_{ij}.$$

It is clear that full subconfiguration on the union of any nonempty set of orbits of a (semi) coherent configuration $\mathfrak{g}$ is (semi) coherent. In particular, the full subconfiguration $\mathfrak{g}^i$ on X_i is (semi) coherent of degree n_i and rank r_{ii}, and is transitive.

For the rest of this section we assume that $\mathfrak{g}$ is coherent.

For $f \in \mathcal{O}^{ij}$,

(5.4)
$$a_{E_k ff} = \delta_{ik}$$

and if we put $n_f = |f(x)|$, $x \in X_i$, then

(5.5)
$$a_{ff^{\cup}E_k} = \delta_{ik} n_f.$$

In particular, n_f is independent of the choice of $x \in X_i$, so

(5.6)
$$|f| = n_i n_f = n_j n_{f^{\cup}}, \text{ and } n_f = n_{f^{\cup}} \text{ if } i = j.$$

Moreover

(5.7) $$n_j = \sum_{f \in \mathcal{O}^{ij}} n_f .$$

The numbers n_f, $f \in \mathcal{O}$ are called the *subdegrees* of $\mathfrak{g}$.

If f, g, $h \in \mathcal{O}$ and there exists $x \in \operatorname{dom} f \cap \operatorname{dom} h$, then

$$\begin{aligned} a_{fgh} n_h &= \text{the number of } (f, g, h)\text{-triangles } (x, y, z) \\ &= \text{the number of } (h, g^{\cup}, f)\text{-triangles } (x, z, y) \\ &= a_{hg^{\cup}f} n_f . \end{aligned}$$

If $\operatorname{dom} f \cap \operatorname{dom} h = \emptyset$, then $a_{fgh} = a_{hg^{\cup}f} = 0$, so, in any case,

(5.8) *For* f, g, $h \in \mathcal{O}$, $a_{fgh} n_h = a_{hg^{\cup}f} n_f$.

In particular

(5.9) $a_{fgf^{\cup}} = a_{f^{\cup}g^{\cup}f}$ *for* f, $g \in \mathcal{O}^{ii}$.

Two further relations for the intersection numbers are

(5.10) *If* $g \in \mathcal{O}^{ik}$ *and* $h \in \mathcal{O}^{ik}$, *then*

$$\sum_{f \in \mathcal{O}} a_{fgh} = \sum_{f \in \mathcal{O}^{ij}} a_{fgh} = n_g ,$$

and

(5.11) *If* $f \in \mathcal{O}^{ij}$ *and* $h \in \mathcal{O}^{ik}$, *then*

$$\sum_{g \in \mathcal{O}} a_{fgh} = \sum_{g \in \mathcal{O}^{jk}} a_{fgh} = n_f .$$

For $\varphi = \mathit{char}\, f$, $f \in \mathcal{O}^{ij}$,

(5.12) $$\sum_{y \in X} \varphi(x, y) = \sum_{y \in X_j} \varphi(x, y) = \begin{cases} n_f & \text{if } x \in X_i \\ 0 & \text{if } x \notin X_i \end{cases}$$

and

(5.13) $$\sum_{x \in X} \varphi(x, y) = \sum_{x \in X} \varphi(x, y) = \begin{cases} n_{f^{\cup}} & \text{if } y \in X_j \\ 0 & \text{if } y \notin X_j . \end{cases}$$

6. Decomposition of the relation semigroup and centralizer ring.

Assume that $\mathfrak{G}$ is semi coherent. The semigroup R of relations decomposes into a disjoint union

$$R=\sum_{i,j} R^{ij}$$

where R^{ij} is the $\boldsymbol{F}_2$-subspace of R spanned by $\mathcal{O}^{ij}$. We have

$$R^{ij}R^{j_1k}=0 \text{ if } j\neq j_1, \qquad R^{ij}R^{jk}\subseteq R^{ik}.$$

Furthemore R^{ii} is isomorphic with the semigroup of relations of $\mathfrak{G}^i$.

Now assume that $\mathfrak{G}$ is coherent. Taking a commutative coefficient ring K with identity element, we obtain a similar decomposition of the centralizer ring $C=C_K(\mathfrak{G})$ by putting $C^{ij}=\{\varphi\in C \mid \text{spt}\,\varphi\subseteq X_i\times X_j\}$. Then C^{ij} is a K-submodule of C with K-basis $\mathfrak{B}^{ij}=\{\text{char}\, f \mid f\in\mathcal{O}^{ij}\}$, and

$$C=\sum_{i,j} C^{ij} \qquad \text{(direct sum)}$$

as a K-module. We have $C^{ij}C^{j_1k}=0$ if $j\neq j_1$, $C^{ij}C^{jk}\subseteq C^{ik}$ and C^{ii} is isomorphic with the centralizer ring of $\mathfrak{G}^i$.

Taking an ordering of X consistent with the partition $\{X_i\}$ and associating with each $\varphi\in C$ the matrix $B_\varphi=(\varphi(x,y))_{x,y}$ we see that for $\varphi\in C^{ij}$, B_φ has the form

$$\begin{array}{c} \overbrace{}^{X_j} \\ X_i\left\{\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\right. \end{array}.$$

From (5.12) and (5.13) it follows that if $\varphi=$ char f, $f\in\mathcal{O}^{ij}$, then B_φ has row sum n_f and column sum $n_{f^\cup}$. In particular,

(6.1) *Each C^{ii} has a linear representation.*

From (4.3) and (6.1) we see that

(6.2) *If $r_{ii}\leq 4$, then C^{ii} is commutative.*

(If $\mathfrak{g}$ is realized by a group, then C^{ii} is commutative for $r_{ii}=5$ too.)

We now consider the case in which $\mathfrak{g}$ has two orbits, i.e., $t=2$, so that, in terms of matrices, the centralize ring $C=C_C(\mathfrak{g})$ is a semi-simple subalgebra of $\boldsymbol{C}_n$ and

$$C=C^{11}\oplus C^{12}\oplus C^{21}\oplus C^{22}$$

where $C^{ij}=I_iCI_j$, with $I_1=\begin{pmatrix}0 & I_{n_2}\\ 0 & 0\end{pmatrix}$ and $I_2=\begin{pmatrix}0 & 0\\ I_{n_1} & 0\end{pmatrix}$. The rings C^{11} and C^{22} are semisimple, and we regard C^{12} as a module over the semi-simple ring $C^{11}\otimes_C C^{22}$. Then we see without difficulty that the decomposition

$$C^{ii}=\sum_{\alpha=1}^{m_i} C^{ii}_\alpha$$

of C^{ii} into simple two-sided ideals, and the decomposition

$$C^{12}=\sum_{\alpha=1}^{s} C^{12}_\alpha$$

of C^{12} into simple submodules over $C^{11}\otimes_C C^{22}$, can be so arranged that C^{12}_α is faithful for $C^{11}_\alpha\otimes_C C^{22}_\alpha$, $1\leq\alpha\leq s$. Hence we see that the simple components of C are

$$C^{11}_\alpha\oplus C^{12}_\alpha\oplus C^{21}_\alpha\oplus C^{22}_\alpha,$$

$1\leq\alpha\leq s$, where $C^{21}_\alpha=[C^{12}_\alpha]^t$, together with C^{ii}_α, $s+1\leq\alpha\leq m_i$, $i=1, 2$. This means that

(6.3) *If $t=2$, then the structure constants $e_{i\alpha}$, $1\leq\alpha\leq m_i$, for $\mathfrak{g}^i$, $i=1, 2$, can be so numbered that the structure constants for $\mathfrak{g}$ are $e_{1\alpha}+e_{2\alpha}$, $1\leq\alpha\leq s$, together with $e_{i\alpha}$, $s+1\leq\alpha\leq m_i$, $i=1, 2$, and then*

$$r_{12}=\sum_{\alpha=1}^{s} e_{1\alpha}e_{2\alpha}\,.$$

Thus, for example, if $r_{11}=2$ and $r_{22}=3$, then by (6.2) and (6.3), $r_{12}\leq 2$.

We have remarked in § 3 that the transitive configuration with trivial pairing are just the association schemes. In general, if the pairing on $\mathcal{G}$ is trivial, $\mathcal{G}$ coherent then $\mathcal{G}^i$ is an association scheme, and if we put $\bar{f}=f\cap(X_i\times X_j)$, $f\in\mathcal{O}^{ij}$, then $\{\mathcal{G}_{\bar{f}}\mid f\in\mathcal{O}^{ii}\}$ is a concordant family of graphs in the sense of Bose and Mesner [2], and $\mathfrak{I}_{\bar{f}}$, $f\in\mathcal{O}^{ij}$, is a partially balanced incomplete block design.

At this point we look briefly at the types of configurations arising for some special values of r.

Let $\mathcal{G}=(X, \mathcal{O})$ be a coherent configuration of degree n and rank r, with t orbits $X_1, X_2, \ldots, X_t$.

1) The extreme possibilities for r are $r=1$ and $r=n^2$. If $r=1$ then $n=1$ so this case is included in the case $r=n^2$. If $r=n^2$, then $\mathcal{O}$ is the totality of one-element subsets of $X\times X$, the orbits are the one-element subsets of X, so that $t=n$. R is the full semigroup Rel X of relations on X and C is the full matrix ring $\mathrm{Mat}_K X$. In this case we refer to $\mathcal{G}$ as a trivial configuration. It is realized by any group G acting trivially on X.

2) There is just one coherent configuration of rank 2 on a given set $|X|$, $|X|\geq 2$. It is realized by any group G acting doubly transitively on X. Here $t=1$ and $C\approx K\oplus K$.

3) If $r=3$, then $t=1$ so $I_X\in\mathcal{O}$, and $\mathcal{O}=\{I_X, f, g\}$. If the pairing is trivial, so that $f=f^{\cup}$ and $g=g^{\cup}$, then the graphs $\mathcal{G}_f$ and $\mathcal{G}_g$ are a complementary pair of strongly regular graphs. Conversely, given a strongly regular graph with vertex set X and edge set f, the configuration $(X, \mathcal{O})$,

$$\mathcal{O}=\{I_X, f, X\times X-(f+I_X)\},$$

is coherent of rank 3. Any rank 3 permutation group of even order realizes a configuration of this type, while the rank 3 groups of odd order realize rank 3 configurations with non trivial pairing.

4) If $r\leq 8$, then $t\leq 2$. If $r\leq 7$ and $t=2$ then $r_{12}=1$ so that the two orbits are joined together in a trivial way. The first interesting case in which $t=2$ is $r=8$ with $r_{11}=r_{22}=r_{12}=2$. If we write $\mathcal{O}^{12}=\{f, g\}$ and $f=f\cap(X_1\times X_2)$ and $g=g\cap(X_1\times X_2)$ then $\mathfrak{I}_{\bar{f}}$ and $\mathfrak{I}_f$ are a complementary pair of (possibly degenerate) blanced incomplete block designs which are symmetric if and only if $n_1=n_2$.

7. Intersection matrices.

We consider once more a coherent configuration $\mathcal{G}=(X, \mathcal{O})$, maintaining the notation of the preceding two sections. For purposes of computation we take an ordering of X consistent with the decompotision into orbits, and a numbering of the elements of each $\mathcal{O}^{ij}$,

$$\mathcal{O}^{ij}=\{f_\alpha^{ij} \mid \alpha=1, 2, \ldots, r_{ij}\},$$

such that

(7.1) $$f_1^{ii}=E_i \text{ and } (f_\alpha^{ij})^{\cup}=\begin{cases} f_\alpha^{ji} & \text{if } i\neq j \\ f_{\alpha'}^{ii} & \text{if } i=j \end{cases}$$

where, of course, the pairing $\tau(i): \alpha \to \alpha'$ depends on i. We write $n_i=|X_i|$ and $n_\alpha^{ij}=n_f$ for $f=f_\alpha^{ij}$, so that

(7.2) $$n=\sum_{i=1} n_i, \qquad n_j=\sum_{\alpha=1}^{r_{ij}} n_\alpha^{ij}, \qquad n_1^{ii}=1$$

and

$$n_\alpha^{ii}=n_{\alpha'}^{ii}.$$

By (7.1) and (5.6),

(7.3) $$n_i n_\alpha^{ij}=n_j n_\alpha^{ji}.$$

We put $B_\alpha^{ij}=B_f$ for $f=f_\alpha^{ij}$, and

(7.4) $$a_{\alpha\gamma\beta}^{ijk}=a_{fgh}$$

for $f=f_\alpha^{ij}$, $g=f_\beta^{jk}$ and $h=f_\gamma^{ik}$, so that

(7.5) $$B_\alpha^{ij} B_\beta^{jk}=\sum_\gamma a_{\alpha\gamma\beta}^{ijk} B_\gamma^{ik}.$$

(This accounts for all the non zero intersection numbers. The arrangement of subscripts is because of our ultimate interest in the regular representation of C.)

The matrix B_α^{ij} has the form

$$B_\alpha^{ij} = X_i \left\{ \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ \hline 0 & M_\alpha^{ij} & 0 \\ \hline 0 & 0 & 0 \end{bmatrix}}^{X_j} \right.$$

and

(7.6) $$M_\alpha^{ij} M_\beta^{jk} = \sum_\gamma a_{\alpha\gamma\beta}^{ijk} M_\gamma^{ik}.$$

The following properties of the M_α^{ij} are immediate.

(7.7) $$M_1^{ii} = I \text{ and } (M_\alpha^{ij})^t = \begin{cases} M_\alpha^{ij} & \text{if } i \neq j \\ M_{\alpha'}^{ii} & \text{if } i = j. \end{cases}$$

(7.8) M_α^{ij} has row sum n_α^{ij} and column sum n_α^{ii}.

(7.9) $\sum_\alpha M_\alpha^{ij} = J$, where J has all entries 1.

Conversely, it is not difficult to see that a set

$$\{M_\alpha^{ij} \mid 1 \leq i,\ j \leq t,\ 1 \leq \alpha \leq r_{ij}\}$$

of (0, 1)-matrices satisfying (7.6) through (7.9), with the coefficients $a_{\alpha\gamma\beta}^{ijk}$ in (7.6) nonnegative rational integers, is associated with a coherent configuration.

Let $\bar{f} = f_\alpha^{ij} \cap (X_i \times X_j)$ and, using the notation of § 1, put

$$\mathfrak{G}_\alpha^{ij} = \begin{cases} \mathfrak{K}_{\bar{f}} & \text{if } i \neq j \\ \mathfrak{I}_{\bar{f}} & \text{if } i = j. \end{cases}$$

Then $\mathfrak{G}_\alpha^{ii}$ has adjacency matrix $\begin{pmatrix} 0 & M_\alpha^{ij} \\ M_\alpha^{ij} & 0 \end{pmatrix}$ or M_{ii}^α according as $i \neq j$ or $i = j$.

The regular representation of C is the isomorphism of C onto a subalgebra $\tilde{C}$ of $\mathrm{Mat}_K \mathcal{O}$ such that for $f \in \mathcal{O}$,

$$(\widetilde{\mathrm{char}\, f})(g, h) = a_{fhg} \qquad (f, g, h \in \mathcal{O}).$$

In terms of matrices, B_{ij}^{α} maps onto $\widetilde{B}_{\alpha}^{ij}$ of the form

$$\widetilde{B}_{\alpha}^{ij} = \begin{array}{c|c|c|c|c|c|c|}
 & & \mathcal{O}^{j1} & \mathcal{O}^{j2} & & \mathcal{O}^{jt} & \\
\hline
 & & & & & & \\
\hline
\mathcal{O}^{i1} & & M_{\alpha}^{ij1} & & & & \\
\hline
\mathcal{O}^{i2} & & & M_{\alpha}^{ij2} & & & \\
\hline
 & & & & \cdots & & \\
\hline
\mathcal{O}^{it} & & & & & M_{\alpha}^{ijt} & \\
\hline
 & & & & & & \\
\end{array}$$

where the blank entries are all zero,

(7.10) $$M_{\alpha}^{ijk} = (a_{\alpha\beta\gamma}^{ijk})_{\beta,\gamma}$$

and

(7.11) $$M_{\alpha}^{ijk} M_{\beta}^{ilk} = \sum_{\gamma} a_{\alpha\gamma\beta}^{ilk} M_{\gamma}^{ilk}\ .$$

We refer to the matrices M_{α}^{ijk} as the *intersection matrices* for $\mathfrak{g}$.

If we put

$$C^{ijk} = \langle M_{\alpha}^{ijk} \mid \alpha = 1,\ 2,\ \ldots,\ r_{ij} \rangle_K\ ,$$

then the restriction of the regular representation of C to C^{ij} gives a homomorphism

$$\begin{cases} C^{ij} \to C^{ijk} \\ B_{\alpha}^{ij} \to M_{\alpha}^{ijk} \end{cases}$$

which is the regular representation of C^{ii} in case $i=j=k$, and is bijective in case $k=i$ or $k=j$.

Now we summarize the properties of intersection numbers in terms of our present notation.

According to (5.10) and (5.11)

$$\sum_{\gamma} a_{\alpha\beta\gamma}^{ijk} = n_{\alpha}^{ij} \ \text{ and } \ \sum_{\alpha} a_{\alpha\beta\gamma}^{ijk} = n_{\gamma}^{kj}$$

that is

$$\text{(7.12)}\quad \begin{cases} M_\alpha^{ijk} \text{ has row sum } n_\alpha^{ij} \text{ and} \\ \sum_\alpha M_\alpha^{ijk} = \begin{bmatrix} n^{kj} \\ n^{kj} \\ \cdot \\ \cdot \\ \cdot \\ n^{kj} \end{bmatrix} \quad \text{where } n^{kj} = (n_1^{kj}, n_2^{kj}, \ldots). \end{cases}$$

By (5.4) and (5.5)

$$\text{(7.13)}\quad \begin{cases} a_{1\beta\gamma}^{iik} = \delta_{\beta\gamma} & \text{(i.e., } M_1^{iik} = I) \\ a_{\alpha\beta 1}^{ijj} = \delta_{\alpha\beta} & (i \neq j) \\ a_{\alpha\beta 1}^{iii} = \delta_{\alpha\beta'} & \\ a_{\alpha 1\gamma}^{iji} = \delta_{\alpha\gamma} n_\alpha^{ij} & \end{cases}$$

By (3.3)

$$\text{(7.14)}\quad a_{\alpha\beta\gamma}^{ijk} = \begin{cases} a_{\gamma\beta\alpha}^{kji} & (i \neq k) \\ a_{\gamma\beta'\alpha}^{iji} & (i = k). \end{cases}$$

By (2.8)

$$n_\beta^{kj} a_{\alpha\beta\gamma}^{ijk} = \begin{cases} n_\gamma^{kj} a_{\alpha\gamma\beta}^{jik} & (i \neq j) \\ n_\gamma^{ki} a_{\alpha'\gamma\beta}^{iik} & \end{cases}$$

so if we put $D^{lm} = \operatorname{diag}(n_1^{lm}, n_2^{lm}, \ldots)$

$$\text{(7.15)}\quad \begin{cases} D^{ki} M_\alpha^{ijk} = M_\alpha^{jik} D^{kj} & (i \neq j) \\ D^{ki} M_\alpha^{iik} = M_{\alpha'}^{iik} D^{ki}. \end{cases}$$

From (7.12) and (7.15) we get

$$\text{(7.16)}\quad n^{ik} M_\alpha^{ijk} = n_\alpha^{ji} n^{jk},$$

and in particular n^{ik} is a left eigenvector of M_α^{iik} with eigenvalue n_α^{ii}.

For convenience of reference we note explicitly certain of the simplifications in notation which can be made in case $\mathfrak{G}$ is transitive. Here we drop the superscripts, writing we drop the superscripts, writing $\mathcal{O}=\{f_1, f_2, \ldots, f_r\}$ with $f_1=I_X$ and $f_\alpha^{\cup}=f_{\alpha'}$, and $m_\alpha=n_{f_\alpha}$, so that

$$m_1=1, \qquad \sum_\alpha m_\alpha=n, \qquad m_{\alpha'}=n_\alpha. \tag{7.17}$$

We write B_α for B_{f_α} and M_α for the corresponding intersection matrix, so that

$$B_\alpha B_\beta=\sum_\gamma a_{\alpha\gamma\beta}B_\gamma, \tag{7.18}$$

$$M_\alpha M_\beta=\sum_\gamma a_{\alpha\gamma\beta}M_\gamma, \tag{7.19}$$

and

$$M_\alpha=(a_{\alpha\beta\gamma})_{\beta\gamma} \tag{7.20}$$

where $a_{\alpha\beta\gamma}=a_{f_\alpha f_\beta f_\gamma}$. We have

$$\begin{cases} M_\alpha \text{ has row sum } m_\alpha, \text{ and} \\ \sum_\alpha M_\alpha=\begin{bmatrix} m \\ m \\ \cdot \\ \cdot \\ m \end{bmatrix} \text{ where } m=(m_1, m_2, \ldots, m_r), \end{cases} \tag{7.21}$$

$$\begin{cases} a_{1\beta\gamma}=\delta_{\beta\gamma} \qquad \text{(i.e., } M_1=I) \\ a_{\alpha\beta 1}=\delta_{\alpha\beta'} \\ a_{\alpha 1\gamma}=\delta_{\alpha\gamma}m_\alpha, \end{cases} \tag{7.22}$$

$$a_{\alpha\beta\gamma}=a_{\gamma\beta'\alpha}, \tag{7.23}$$

and

$$m_\beta a_{\alpha\beta\gamma}=m_\gamma a_{\alpha'\gamma\beta}, \tag{7.24}$$

i.e.,

(7.25) $M_{\alpha'}=DM_\alpha{}^tD^{-1}$ where $D=$ diag $(1, m_2 \ \ldots, m_r)$.

In particular, since $m_\beta=m_{\beta'}$,

(7.26) $$a_{\alpha\beta\beta'}=a_{\alpha'\beta'\beta}\,.$$

Under the regular representation, B_α maps onto M_α so

(7.27) *If $F(t)\in\mathbf{C}[t]$ is such that $F(M_\alpha)=0$, and if θ is a root of $F(t)$ of multiplicity m, then*

$$\mu_\theta=\frac{\text{trace } f_\theta(M_\alpha)}{f_\theta(\theta)},\ \textit{where}\ f_\theta(t)=\frac{f(t)}{(t-\theta)^m},$$

is a rational integer.

In fact, μ_θ is the multiplicity of θ as an eigenvalue of B_α.

8. Coherent partitions.

In this section $\mathfrak{C}=(X, \mathcal{O})$ denotes a coherent configuration. We refer to a pair $(x, y)\in f$ as an *f-edge.* A partition $\mathfrak{S}$ of X is *coherent* if, for $f\in\mathcal{O}$, S, $T\in\mathfrak{S}$ and $x\in S$, the number of f-edges (x, y) with $y\in T$ is independent of the choice of $x\in S$. This number is

$$\sum_{y\in T}(\text{char } f)\ (x, y)=|\, f(x)\cap T\,|.$$

In particular, $\mathfrak{S}$ is coherent if and only if each block of each matrix B_φ, $\varphi\in C$, when blocked according to $\mathfrak{S}$, has constant row (and column) sum.

Let $\mathfrak{S}$ be a coherent partition of X and let $S\in\mathfrak{S}$. Then $S\cap X_i\neq\emptyset$ for some i, and if $x\in S\cap X_i$ and $y\in S$, then the number of E_i-edges from x to S, namely 1, is equal to the number from y to S. Hence $y\in X_i$ and $S\subseteq X_i$. In particular $\{S\in\mathfrak{S}\mid S\subseteq X_i\}$ is a partition of X_i and is clearly coherent for $\mathfrak{C}^i$.

The following simple fact is useful.

(8.1) *Let $\mathfrak{S}$ be a coherent partition of X, and, for $\varphi\in C$, S, $T\in\mathfrak{S}$ and $x\in S$, put*

$$\varphi_{\mathfrak{S}}(S, T)=\sum_{y\in T}\varphi(x, y).$$

Then $\beta_{\mathfrak{S}} : \varphi \mapsto \varphi_{\mathfrak{S}}$ *is a K-algebra homomorphism of C onto a subalgebra* $C_{\mathfrak{S}}$ *of* $\mathrm{Mat}_K \mathfrak{S}$.

PROOF. $\beta_{\mathfrak{S}}$ is well-defined because of the coherence of $\mathfrak{S}$, and $\beta_{\mathfrak{S}}$ is clearly K-linear. Take $\varphi, \psi \in C$, $S, T \in \mathfrak{S}$, and $x \in S$, then

$$(\varphi\psi)_{\mathfrak{S}}(S, T) = \sum_{y\in T} (\varphi\psi)(x, y) = \sum_{y\in T} \sum_{z\in X} \varphi(x, z)\psi(z, y)$$

$$= \sum_{U\in\mathfrak{S}} \sum_{z\in U} \{\varphi(x, z) \sum_{y\in T} \psi(z, y)\} = \sum_{U\in\mathfrak{S}} \sum_{z\in U} \varphi(x, z)\psi \quad (U, T)$$

$$= \sum_{U\in\mathfrak{S}} \varphi_{\mathfrak{S}}(S, U)\psi_{\mathfrak{S}}(U, T) = (\varphi_{\mathfrak{S}}\psi_{\mathfrak{S}})(S, T).$$

Hence $(\varphi\psi)_{\mathfrak{S}} = \varphi_{\mathfrak{S}}\psi_{\mathfrak{S}}$, proving (7.1).

The matrix of $\varphi_{\mathfrak{S}}$ is obtained from B_φ by blocking according to $\mathfrak{S}$ and replacing each block by its row sum.

Certain coherent partitions are always available, namely, choose $x_k \in X_k$ and let $\mathfrak{S}_k = \{g(x_k) \mid g \in \sum_{j=1}^{t} \mathcal{O}^{kj}\}$. We know that $\mathfrak{S}_k$ is a partition of X. For $f \in \mathcal{O}$ and $g, h \in \sum_j \mathcal{O}^{kj}$, the number of f-edges from $x \in g(x_k)$ to $h(x_k)$ is

$$| f(x) \cap h(x_k) | = a_{fh^\cup g^\cup}$$

which is independent of the choice of $x \in g(x_k)$. Hence $\mathfrak{S}_k$ is coherent, and we see that in terms of matrices in the notation of § 7, $\beta_{\mathfrak{S}_k}$ maps

$$B_\alpha^{ij} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & M_\alpha^{ij} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

onto

$$\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \tilde{M}_\alpha^{ijk} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

with $\tilde{M}_\alpha^{ijk} = (\mu_{\alpha\beta\gamma}^{ijk})_{\beta,\gamma}$, where

$$\mu^{ijk}_{\alpha\beta\gamma} = \begin{cases} a^{ijk}_{\alpha\beta\gamma} \text{ if } i \neq k \text{ and } j \neq k \\ a^{iij}_{\alpha\beta\gamma'} \text{ if } i \neq k \text{ and } j = k \\ a^{iji}_{\alpha\beta'\gamma} \text{ if } i = k \text{ and } j \neq k \\ a^{iii}_{\alpha\beta'\gamma'} \text{if } i = j = k \end{cases}$$

that is, if P_i is the permutation matrix of the pairing $\tau(i) : \alpha \to \alpha'$, then

$$\tilde{M}^{ijk}_{\alpha} = \begin{cases} M^{ijk}_{\alpha} \quad \text{ if } i \neq k \text{ and } j \neq k \\ M^{iji}_{\alpha} P_j \text{ if } i \neq k \text{ and } j = k \\ P_i M^{ijk}_{\alpha} \quad i = k \text{ and } j \neq k \\ P_i M^{iii}_{\alpha} P_i \text{ if } i = j = k. \end{cases}$$

From (8.2) we see that the statements (2.1) through (2.4) concerning connectedness, etc., as applied to the graph $\mathcal{G}^{ij}_{\alpha}$ hold with the intersection matrix M^{iji}_{α} in place of M^{ij}_{α}. (Here M^{ij}_{α} plays the role of the matrix B_f of § 2.) More precisely,

(8.3) *If $i \neq j$, then $\mathcal{G}^{ij}_{\alpha}$ is connected if and only if M^{ij}_{α} is indecomposable under permutation equivalence.*

(8.4) *Assuming that $\mathcal{G}^{ij}_{\alpha}$ is connected for some $i \neq j$, then* (2.2) *holds with M^{iji}_{α} in place of M^{ij}_{α} .*

(8.5) *The following are equivalent:*

a) *$\mathcal{G}^{ii}_{\alpha}$ is connected,*

b) *M^{iii}_{α} is irreducible under permutation similarity, and*

c) *M^{iii}_{α} is indecomposable under permutation similarity.*

(8.6) *Assume that $\mathcal{G}^{ii}_{\alpha}$ is connected. Then* (2.4) *holds with M^{iii}_{α} in place of M^{ii}_{α}.*

9. Automorphisms, refinements and coarsenings.

An automorphism of a configuration $\mathfrak{g} = (X, \mathcal{O})$ is a permutation of the elments of X which permutes the elements of $\mathcal{O}$ under componentwise action on $X \times X$. An automorphism is *strict* if the permutation induced on $\mathcal{O}$ is trivial. The strict automorphisms of $\mathfrak{g}$ constitute a normal subgroup Aut $*\mathfrak{g}$ of the group Aut $\mathfrak{g}$ of all automorphisms of $\mathfrak{g}$.

We say that a group G is *represented* in the (strict) automorphism group of $\mathfrak{g}$ if G acts on X in such a way that the permutations of X associated with the elements of G are (strict) automorphisms of $\mathfrak{g}$, or, what is the same thing, if there is a homomorphism of G into (Aut $*\mathfrak{g}$) Aut $\mathfrak{g}$. The configuration $\mathfrak{g}$ is realized by the action of a group G on X, i.e., $\mathcal{O}=\mathcal{O}_G(X)$, if and only if G is represented in the strict automorphism group of $\mathfrak{g}$ and rank $_G X=r$ (where r is the rank of $\mathfrak{g}$).

Suppose given configurations $\mathfrak{g}_1=(X, \mathcal{O}_1)$ and $\mathfrak{g}_2=(X, \mathcal{O}_2)$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are partitions of $X\times X$. We call $\mathfrak{g}_1$ a refinement of $\mathfrak{g}_2$ if $\mathcal{O}_1$ is a *refinement* of $\mathcal{O}_2$; if in addition, $\mathfrak{g}_1$ is (semi) coherent, we say that $\mathfrak{g}_1$ is a (*semi*) *coherent refinement* of $\mathfrak{g}_2$. It is clear that

(9.1) *If $\mathfrak{g}_1$ is a coherent refinement of a coherent configuration $\mathfrak{g}$, the the orbits of $\mathfrak{g}_1$ constitute a coherent partition of $\mathfrak{g}$ (in the sense of* § 8).

An immediate but useful remark is studying automorphism groups of configurations is

(9.2) *It a group G is represented in the strict automorphism group of a configuration $\mathfrak{g}=(X, \mathcal{O})$, then the configuration $(X, \mathcal{O}_G(X))$ is a coherent refinement of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is coherent, then the G-orbits in X constitute a coherent partition of X.*

If $\mathfrak{g}_1$ is a refinement of $\mathfrak{g}_2$, then $\mathfrak{g}_2$ will be called a *coarsening* of $\mathfrak{g}_1$; if, in addition, $\mathfrak{g}_2$ is (semi) coherent it will be called a (*semi*) *coherent coarsening of* $\mathfrak{g}_1$.

(9.3) *If $\mathfrak{g}$ is a transitive coherent configuration such that its centralizer ring $C_Z(\mathfrak{g})$ is commutative, then $\mathfrak{g}_1=(X, \mathcal{O}_1)$, where $\mathcal{O}_1=\{f\cup f^{\cup}\mid f\in\mathcal{O}\}$, is a coherent coarsening of $\mathfrak{g}$.*

The proof of (9.3) is a straightforward application of the conditions *a*) through *e*) given at the end of § 4, together with the fact that the assumed commutativity and (3.3) imply that $a_{f^{\cup}g^{\cup}h^{\cup}}=a_{fgh}$ for $f, g, h\in\mathcal{O}$. We can replace coherent by semi coherent in (9.3) if we assume commutativity of $R(\mathfrak{g})$ instead of $C_Z(\mathfrak{g})$.

As a sort of dual to (9.2) we have

(9.4) *Let G be represented in the automorphism group of a transitive coherent configuration $\mathcal{G}=(X, \mathcal{O})$. Then $\mathcal{G}_1=(X, \mathcal{O}_1)$, where $\mathcal{O}_1=\{\bigcup_{\sigma\in G} f^\sigma \mid f\in\mathcal{O}\}$, is a coherent coarsening of $\mathcal{G}$.*

The proof is similar to that of (9.3), using the fact that for $\sigma\in \operatorname{Aut}\mathcal{G}$, and $f, g, h\in\mathcal{O}$, $a_{f^\sigma g^\sigma h^\sigma}=a_{fgh}$. Coherent can be replaced by semi coherent in (9.4).

10. Primitivity and connectedness.

In this section $\mathcal{G}=(X, \mathcal{O})$ denotes a semi coherent configuration. We call $\mathcal{G}$ *primitive* if the only equivalence relations in R are I_X and $X\times X$; this implies that $\mathcal{G}$ is transitive. We call $\mathcal{G}$ *imprimitive* if it is transitive and not primitive. These definitions are consistent with the meaning of these terms in the theory of G-spaces in case $\mathcal{G}$ is realized by a group.

(10.1) *If $\mathcal{G}$ is coherent and transitive, then for $f\in\mathcal{O}$, the kernel E_f of f is an equivalence relation.*

PROOF. Since R is closed under union and composition, E_f is in R. Hence $E_f=f_1+f_2+\ \dots\ +f_n$, $f_i\in\mathcal{O}$, so $|E_f(x)|$ is independent of $x\in X$. If $y\in E_f(x)$, then $E_f(y)\subseteq E_f(x)$ and hence $E_f(y)=E_f(x)$. So E_f is symmetric and hence an equivalence relation.

(10.2) *If $\mathcal{G}$ is coherent, then $\mathcal{G}$ is primitive if and only if the graphs $\mathcal{S}_f$, $f\in\mathcal{O}-\{I_X\}$, are all connected.*

PROOF. Assume that $\mathcal{G}$ is primitive and take $f\in\mathcal{O}-\{I_X\}$. Then $E_f\neq I_X$ so $E_f=X\times X$ which means that $\mathcal{S}_f$ is connected. On the other hand, suppose that E is an equivalence relation in R and assume that $E\neq I_X$. Then $E=f_1+f_2+\ \dots\ +f_m$ with $f_i\in\mathcal{O}$ and, say, $f_1\neq I_X$. Then $E_{f_1}\subseteq E$, so that if $\mathcal{S}_{f_1}$ is connected, $X\times X=E_{f_1}=E$.

Let E be an equivalence relation on X and let $\pi: X\to X/E$ be the natural map. For $f\in \operatorname{Rel} X$ put $f/E=\pi^{\cup}f\pi$, so that $f/E\in \operatorname{Rel} X/E$.

(10.3) *If $\mathcal{G}=(X, \mathcal{O})$ is semicoherent and transitive, and if E is an equivalence relation in $R(\mathcal{G})$, then $\mathcal{G}/E=(X/E, \mathcal{O}/E)$, where $\mathcal{O}/E=\{f/E \mid f\in\mathcal{O}\}$, is semicoherent and transitive.*

PROOF. To show that $\mathfrak{G}/E$ is semi coherent we verify the conditions 1), 2), 3) and 4′) of § 4.

1) If $(S, T)\in X/E\times X/E$, take $x\in S$ and $y\in T$. Then $(x, y)\in f$ for some $f\in\mathcal{O}$, so $(S, T)\in f/E$. Suppose that $f/E\cap g/E\neq\emptyset$, f, $g\in\mathcal{O}$. Then $f\cap EgE\neq\emptyset$, so $f\subseteq EgE$ (since $EgE\in R$). Hence $EfE\subseteq E^2gE^2=EgE$, and similarly $EgE\subseteq EfE$, so that $EfE=EgE$ and hence $f/E=g/E$. Hence $\mathcal{O}/E$ is a partition of $X/E\times X/E$ and 1) holds.

2) Since $\mathfrak{G}$ is transitive, $I_X\in\mathcal{O}$ and hence $I_{X/E}=I_X/E\in\mathcal{O}/E$.

3) $(f/E)^{\cup}=f^{\cup}/E$ so 3) holds.

4′) Let $R/E=\{f/E\mid f\in R\}$, then R/E is the $\boldsymbol{F}_2$-subspace of $\mathrm{Rel}\,(X/E)$ spanned by $\mathcal{O}/E$. To verify 4′) it suffices to show that R/E is closed under composition. If f, $g\in R$, then

$$(f/E)(g/E)=\pi^{\cup}f\pi\pi^{\cup}g\pi=\pi^{\cup}fEg\pi=fEg/E$$

which is in R/E as required.

Since $I_{X/E}\in\mathcal{O}/E$, $\mathfrak{G}/E$ is transitive.

In case $\mathfrak{G}$ is realized by a G-space X, so that $\mathcal{O}=\mathcal{O}_G(X)$, then a necessary and sufficient condition for the imprimitivity of $\mathfrak{G}$ in the sense defined here is that X be an imprimitive G-space. In this case $\mathfrak{G}/E$ (as well as $\mathfrak{G}$) is coherent since $\mathcal{O}/E=\mathcal{O}_G(X/E)$.

(10.4) *Let* $\mathfrak{G}_i=(X_i, \mathcal{O}_i)$, $i=1, 2$, *be semi coherent and let* $f\in\mathrm{Rel}\,(X_1, X_2)$, $f\neq\emptyset$, *be such that* $f^{\cup}R_1f\subseteq R_2$ *and* $fR_2f^{\cup}\subseteq R_1$, *where* $R_i=R(\mathfrak{G}_i)$. *Then* f *induces an isomorphism of* $\mathfrak{G}_1/L_f$ *onto* $\mathfrak{G}_2/R_f$.

PROOF. Let $\pi_L: X_1\to X_1/L_f$ and $\pi_R: X_2\to X_2/R_f$ be the natural maps. By the Fundamental Theorem on Relations,

$$\bar{f}=\pi_L{}^{\cup}f\pi_R: X_1/L_f\to X_2/R_f$$

is a bijection. If $g\in\mathcal{O}_1$, then

$$\begin{aligned}(\pi_L{}^{\cup}g\pi_L)^{\bar{f}}&=(\bar{f})^{-1}(\pi_L{}^{\cup}g\pi_L)\bar{f}\\&=\pi_R{}^{\cup}f^{\cup}\pi_L\pi_L{}^{\cup}g\pi_L\pi_L{}^{\cup}f\pi_R\\&=\pi_R{}^{\cup}(f^{\cup}L_fgL_ff)\pi_R\,.\end{aligned}$$

Hence $(\mathcal{O}_1/L_f)^f \subseteq \mathcal{O}_2/R_f$ and the reverse inequality holds similarly.

(10.4) Applies, for example, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are the full subconfigurations on orbits X_1 and X_2 of a semi coherent configuration $\mathcal{G}$ and $f \in R^{12} - \{\emptyset\}$.

As a corollary to (10.4) we remark

(10.5) *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ and f be as in* (10.4) *and assume that $\mathcal{G}_1$ is primitive and $\mathcal{G}_2$ is transitive. Then either $\mathcal{H}_f$ is connected or f induces an isomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_2/R_f$.*

PROOF. Since $\mathcal{G}_1$ is primitive, ether $L_f = X \times X$ or $L_f = I_X$. In the first case, $\mathcal{H}_f$ is connected since $\mathcal{G}_2$ is transitive. In the second case f induces an isomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_2/R_f$ by (10.4).

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