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ON DIRECT DECOMPOSITIONS OF TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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1. In his book [5], I. Kaplansky asked: *if $G \oplus G$ and $H \oplus H$ are isomorphic, are G and H isomorphic*? B. Jónsson [4] has given an example of two torsion-free indecomposable abelian groups [1]) G and H of rank 2 which are not isomorphic, but their direct squares are isomorphic. Puzzled by this example, we have investigated the direct powers of torsion-free indecomposable groups, and have found that to every integer $m \geqq 2$, there exist two indecomposable groups, G and H, of rank 2, such that the direct sum of n copies of G is isomorphic to the direct sum of n copies of H exactly if m divides n (Theorem 1). Moreover, we can choose G such that the direct sum of m copies of G decomposes into the direct sum of m pairwise non-isomorphic groups (Theorem 2) [2]).

Any two direct decompositions of a torsion-free group of rank $\leqq 2$ are isomorphic: in fact, the group is either indecomposable or completely decomposable [3]). However, it is an open problem whether or not a torsion-free group of rank $m \geqq 3$ can have infinitely many, pairwise non-isomorphic direct decompositions. We make a step towards the solution in our Theorem 3 which asserts that to every $m \geqq 3$ and every

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1) All the groups in this note are additively written abelian groups. For the definition of rank and related concepts see e.g. Fuchs [2].

2) For another type of pathological direct decomposition of torsion-free groups of finite rank, see Corner [1].

3) See e.g. Fuchs [2].

large t there is a torsion-free group of rank m which has at least t non-isomorphic direct decompositions. This yields a negative solution to Problem 15 in Fuchs [3] [4]).

2. We start off with the construction of groups needed in the proofs.

Let $A_i (i=1, ..., m)$ be isomorphic groups of rank 1, and $B_i (i=1, ..., m)$ isomorphic groups of rank 1 such that

$$\text{Hom}(A_i, B_i) = 0 = \text{Hom}(B_i, A_i) \tag{1}$$

and for their endomorphism rings [5])

$$\text{End}\, A_i \cong Z \cong \text{End}\, B_i \tag{2}$$

holds. For instance, we can choose two disjoint, infinite sets of primes, say P_1 and P_2, and define A_i as a subgroup of the rational vector space with basis a_i, generated by $p_1^{-1}a_i$ for all $p_1 \in P_1$:

$$A_i = \langle p_1^{-1}a_i \mid p_1 \in P_1 \rangle,$$

and similarly,

$$B_i = \langle p_2^{-1}b_i \mid p_2 \in P_2 \rangle$$

for $i=1, .., m$. In view of (2), we can choose elements $a_i \in A_i$ and $b_i \in B_i$ such that $a_1, ..., a_m$ and $b_1, ..., b_m$ correspond to each other under isomorphisms of the A_i and the B_i, respectively, and $p \nmid a_i$, $p \nmid b_i$ for some prime p (which will be suitably chosen later on). Then the groups

$$X_i = \langle A_i \oplus B_i, p^{-1}(a_i + b_i) \rangle \qquad \text{for } i=1, ..., m \tag{3}$$

are isomorphic among themselves, and as standard arguments show, they are indecomposable.

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[4]) This problem asks for the existence of an integer $k(n)$ such that every torsion-free group of rank n has at most $k(n)$ non-isomorphic direct decompositions into indecomposable summands.

[5]) Z denotes the ring of integers.

Let $C_i \cong A_i (i=1, \ldots, m)$ and $D_i \cong B_i (i=1, \ldots, m)$ be other sets of groups, and let $c_i \in C_i$, $d_i \in D_i$ correspond to a_i, b_i under some isomorphisms. Then for any choice of $k_i = 1, \ldots, p-1$, we can form the groups

$$(4) \qquad Y_i = \langle C_i \oplus D_i,\ p^{-1}(c_i + k_i d_i) \rangle \qquad \text{for } i=1, \ldots, m.$$

These are again indecomposable. All X_i, Y_i are of rank 2.

Assume there is an isomorphism

$$\varphi : X^* = X_1 \oplus \ldots \oplus X_m \to Y^* = Y_1 \oplus \ldots \oplus Y_m .$$

From (1) we conclude that $A = A_1 \oplus \ldots \oplus A_m$ and $B = B_1 \oplus \ldots \oplus B_m$ are fully invariant subgroups of X^*, thus φ must map A onto $C = C_1 \oplus \ldots \oplus C_m$ and B onto $D = D_1 \oplus \ldots \oplus D_m$:

$$(5) \qquad \varphi : a_i \mapsto \sum_{j=1}^{m} \alpha_{ij} c_j \text{ and } b_i \mapsto \sum_{j=1}^{m} \beta_{ij} d_j$$

where owing to (2) the α_{ij}, β_{ij} are integers such that the matrices $\| \alpha_{ij} \|$, $\| \beta_{ij} \|$ are invertible, i.e. the determinants

$$(6) \qquad \det(\alpha_{ij}) = \pm 1, \qquad \det(\beta_{ij}) = \pm 1.$$

Notice that, for every i,

$$(7) \qquad \varphi : a_i + b_i \mapsto \sum_{j=1}^{m} \alpha_{ij}(c_j + k_j d_j) + \sum_{j=1}^{m} (\beta_{ij} - k_j \alpha_{ij}) d_j .$$

An isomorphism must map an element divisible by p upon an element of the same sort, thus from the independence of the d_j we obtain

$$(8) \qquad \beta_{ij} - k_j \alpha_{ij} \equiv 0 \pmod{p}$$

for all i and j. From (6) and (8) it follows that

$$(9) \qquad k_1 \ldots k_m \equiv \pm 1 \pmod{p}.$$

Let us observe that (6) and (8) together guarantee that (5) and (7) yield an isomorphism between X^* and Y^*, since (6) implies that (5) induces an isomorphism between $A \oplus B$ and $C \oplus D$, while (8) implies, because of (7) and (6), that all $a_i + b_i$ are divisible by p if and only if all the $c_j+k_jd_j$ are divisible by p.

3. Now we are ready to prove:

THEOREM 1. *Given any integer $m\geqq 2$, there exist two torsion-free indecomposable groups X and Y, of rank 2 such that*

$$X\oplus \dots \oplus X\cong Y \oplus \dots \oplus Y \quad (n \text{ summands})$$

if and only if

$$n\equiv 0 \pmod{m}.$$

We choose X as in (3) and Y as in (4) with $k_1= \dots =k_m=k$ in order to have isomorphic groups Y_i. Our goal is to assure that the direct sums of m copies of X and Y, respectively, are isomorphic, but the same fails to hold for a smaller number of copies. In view of (9), this means

$$k^m\equiv -1 \pmod{p}, \tag{10}$$

but

$$k^n\not\equiv \pm 1 \pmod{p} \text{ for } n=1, \dots, m-1. \tag{11}$$

(We dropped the possibility $k^m\equiv 1$, because this can occur in the presence of (11) only for odd m). In order to get a solution for (10) and (11), let us choose the prime p so as to satisfy

$$p\equiv 1 \pmod{2m};$$

by Dirichlet's theorem on primes in arithmetic progressions, such a p does exist. If l is a primitive root mod p, then set $k=l^{\frac{p-1}{2m}}$; it is easily seen that both (10) and (11) hold.

Next we show α_{ij} and β_{ij} can be chosen so as to satisfy (6) and (8) with $k_j=k$ as chosen above. Let x satisfy $kx\equiv 1 \pmod{p}$ and choose, for instance, the lower triangular $m\times m$-matrix

$$
\| \alpha_{ij} \| = \left\| \begin{array}{ccccccc}
1 & & & & & & \\
x & 1 & & & & & \\
0 & x & 1 & & & & \\
x & 1 & x & 1 & & & \\
0 & x & 1 & x & 1 & & \\
 & & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & x & -1
\end{array} \right\|
$$

where in the main diagonal we have 1's except for the last entry and, below the diagonal, we have alternately x and 0 in the first column, and x and 1 in the other columns. Then $\det(\alpha_{ij}) = -1$. In accordance with (8), we multiply $\| \alpha_{ij} \|$ by k and from each entry we subtract a multiple of p to get a matrix $\| \beta_{ij} \|$ with determinant $+1$:

$$
\| \beta_{ij} \| = \left\| \begin{array}{ccccccc}
k & & & & & & y \\
1 & k & & & & & \\
0 & 1 & k & & & & \\
1 & k & 1 & k & & & \\
0 & 1 & k & 1 & k & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & k & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & -k
\end{array} \right\|
$$

which is obtained from $\| \alpha_{ij} \|$ by replacing 1 by k and x by 1 throughout and by putting $y = (-1)^{m+1}(k^m + 1)$ in the upper right corner. Consequently, we can choose α_{ij}, β_{ij} and k such that $\det(\alpha_{ij}) = -1$, $\det(\beta_{ij}) = 1$ and $\beta_{ij} - k\alpha_{ij} \equiv 0 \pmod{p}$. We conclude that the direct sum of m copies of X is isomorphic to the direct sum of m copies of Y, while (11) ensures that the same fails to hold for n copies if n is smaller than m, and hence if it is not divisible by m. This completes the proof of the theorem.

It is worth while noticing that our method yields a somewhat better result for odd integres m. Namely, to any odd m and any integer t, there exist t groups of rank 2 such that any two of them behave like X and Y in Theorem 1 (in this case we replace p by a product of primes). For even integers m we could construct t non-isomorphic groups such that the direct sums of m copies are isomorphic (but m is not necessarily the smallest number with this property).

4. Let us point out that for the direct sum of m copies of X in the last theorem we might have a completely different sort of decomposition:

THEOREM 2. *Given an integer $m \geqq 2$, there exist pairwise non-isomorphic torsion-free indecomposable groups $X, Y_1, \ldots, Y_m$ of rank 2 such that*

$$X \oplus \ldots \oplus X \cong Y_1 \oplus \ldots \oplus Y_m \quad (m \text{ summands}),$$

but $X \oplus \ldots \oplus X$ (n summands) is not isomorphic to the direct sum of n groups

$$Y_{i_1}, \ldots, Y_{i_n} \text{ with } 1 \leqq i_1 < \ldots < i_n \leqq m \text{ and } n < m.$$

Choose $X \cong X_i$ in (3) and the Y_i as in (4). In this case we choose incongruent k_i so as to satisfy

$$k_1 \ldots k_m \equiv -1 \pmod p, \tag{12}$$

but

$$k_{i_1} \ldots k_{i_n} \not\equiv \pm 1 \pmod p \qquad \text{for } 1 \leqq n < m. \tag{13}$$

For instance, if we let $k_1 = k$, $k_2 = k^3$, ..., $k_m = k^{3^{m-1}}$ for a k satisfying

$$k^{\frac{1}{2}(3^m - 1)} \equiv -1 \pmod p$$

such that no power of k with a smaller exponent is $\equiv \pm 1 \pmod p$, then (12) and (13) will be fulfilled. As in the proof of Theorem 1, we can find a prime p and a k with this property. We choose $\| \alpha_{ij} \|$ as earlier, but the x's in the jth column will be replaced by $x^{3^{j-1}}$, while in the earlier form of $\| \beta_{ij} \|$, the k's in the jth column ought to be replaced by $k^{3^{j-1}}$, and $y=(-1)^{m+1}(k^{\frac{1}{2}(3^m-1)}+1)$. This completes the proof of Theorem 2.

REMARK 1. It is straightforward to show that our choice of k_i implies that

$$Y_{i_1}\oplus \dots \oplus Y_{i_n} \cong Y_{j_1}\oplus \dots \oplus Y_{j_n}$$

with

$$1 \leqq i_1 < \dots < i_n \leqq m, \quad 1 \leqq j_1 < \dots < j_n \leqq m$$

can hold only if

$$i_1 = j_1, \ \dots, \ i_n = j_n .$$

REMARK 2. The fact that the summands in Theorem 1 and 2 are of rank 2 is not essential. In fact, X_i in (3) can be replaced by

$$\langle A_i \oplus B_i \oplus \dots \oplus E_i, \ p^{-1}(a_i+b_i), \ \dots, \ p^{-1}(a_i+e_i) \rangle,$$

and Y_i in (4) by

$$\langle C_i \oplus D_i \oplus \dots \oplus F_i, \ p^{-1}(c_i+k_i d_i), \ \dots, \ p^{-1}(c_i+k_i f_i) \rangle$$

where every pair among A_i, B_i, ..., E_i satisfies (1) and (2).

5. We proceed to the problem of constructing a group of rank $m \geqq 3$ which has many direct decompositions. It is obvious that it suffices to consider the rank 3 case only.

THEOREM 3. *To any given integer t, there exists a torsion-free group of rank 3 which has at least t non-isomorphic direct decompositions.*

Let $A_1 \cong B_1$ and C be rank 1 groups whose endomorphism rings are isomorphic to Z and $\mathrm{Hom}(A_1, C)=0=\mathrm{Hom}(C, A_1)$. We choose elements $a_1 \in A$, $b_1 \in B$ and $c \in C$ which are not divisible by a given prime p (to be specified later) such that $a_1 \leftrightarrow b_1$ under some isomorphism.

Put

$$G=A_1\oplus H_1 \text{ with } H_1=\langle B_1\oplus C,\ p^{-1}(b_1+c)\rangle$$

where H_1 is indecomposable of rank 2.

If α_i, β_i, γ_i, δ_i are integers such that $\alpha_i\delta_i-\beta_i\gamma_i=1$, and if we put

$$a_i=\alpha_i a_1+\beta_i b_1,\quad b_i=\gamma_i a_1+\delta_i b_1 \qquad (i=2, \dots, t),$$

then $A_1\oplus B_1=A_i\oplus B_i$ where A_i, B_i are the pure subgroups generated by a_i, b_i, and $a_1\leftrightarrow a_i$, $b_1\leftrightarrow b_i$ under suitable isomorphisms $A_1\cong A_i$, $B_1\cong B_i$. We wish to choose the A_i, B_i so that, for some integer $1<k_i<p$, we have

$$G=A_i\oplus H_i \text{ with } H_i=\langle B_i\oplus C,\ p^{-1}(k_i b_i+c)\rangle \tag{14}$$

with indecomposable H_i. Then

$$k_i b_i+c=k_i\gamma_i a_1+(k_i\delta_i-1)b_1+(b_1+c)$$

must be divisible by p, thus

$$k_i\gamma_i\equiv 0 \text{ and } k_i\delta_i\equiv 1 \pmod p. \tag{15}$$

It is easily seen that conversely, (15) implies that (14) holds. If to any $1<k_i<p$ we choose $\gamma_i=p$, $\alpha_i=k_i$ and β_i, δ_i so as to satisfy $k_i\delta_i-p\beta_i= =1$, then (15) will be satisfied. Notice that an isomorphism $\varphi_{ij}: H_i\to H_j$ must map B_i upon B_j and C upon itself such that $b_i\mapsto\pm b_j$ and $c\mapsto\pm c$. Consequently, φ_{ij} maps $k_i b_i+c$ upon $\pm(k_i b_j\pm c)$. Since divisibility by p is preserved under isomorphism and $p\nmid b_j$, we conclude that φ_{ij} can exist only if $k_i\equiv\pm k_j \pmod p$.

Therefore, if we choose

$$(k_1=1),\ k_2=2,\ \dots,\ k_t=t \text{ and } p>2t+1,$$

then $k_i\not\equiv\pm k_j \pmod p$ for $i\neq j$, and thus the groups $H_1, \dots, H_t$ are pairwise non-isomorphic and indecomposable, establishing the existence of at least t non-isomorphic decompositions for G.

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