Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 43 (1970), p. 269-276

<http://www.numdam.org/item?id=RSMUP_1970__43__269_0>

BILINEAR GENERATING FUNCTIONS FOR LAGUERRE AND LAURICELLA POLYNOMIALS IN SEVERAL VARIABLES

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1. Erdélyi [1] has defined the Laguerre polynomial in k variables by means of

$$(1.1)\qquad (1-u_1-\dots-u_k)^{-\alpha-1}\exp\left\{-\frac{x_1u_1+\dots+x_ku_k}{1-u_1-\dots-u_k}\right\}=$$

$$=\sum_{n_1,\dots,n_k=0}^{\infty} L^{(\alpha)}_{n_1,\dots,n_k}(x_1,\dots,x_n)u_1^{n_1}\dots u_k^{n_k}.$$

This is equivalent to

$$(1.2)\qquad L^{(\alpha)}_{n_1,\dots,n_k}(x_1,\dots,x_k)=$$

$$=\frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1!\dots n_k!}\sum_{r_j=0}^{n_j}(-1)^{r_1+\dots+r_k}\binom{n_1}{r_1}\dots\binom{n_k}{r_k}\frac{x_1^{r_1}\dots x_k^{r_k}}{(\alpha+1)_{r_1+\dots+r_k}}.$$

He obtained the following bilinear generating function.

$$(1.3)\qquad \sum_{n_1,\dots,n_k=0}^{\infty}\frac{n_1!\dots n_k!}{\Gamma(\alpha+n_1+\dots+n_k+1)}u_1^{n_1}\dots u_k^{n_k}\cdot$$

$$\cdot L^{(\alpha)}_{n_1,\dots,n_k}(x_1,\dots,x_k)L^{(\alpha)}_{n_1,\dots,n_k}(y_1,\dots,y_k)=$$

$$=\frac{(u;\ x,\ y)^{-\frac{1}{2}\alpha}}{(1-U)^{\alpha+1}}\exp\left(-\frac{X+Y}{1-U}\right)I_\alpha(2\sqrt{(u;\ x,\ y)}),$$

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*) Supported in part by NSF grant GP-7855.

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where

(1.4) $$U=\sum_{j=1}^{k} u_j\,,\qquad X=\sum_{j=1}^{k} u_j x_j\,,\qquad Y=\sum_{j=1}^{k} u_j y_j\,,\qquad W=\sum_{j=1}^{k} u_j x_j w_j$$

and

(1.5) $$(u;\ x,\ y)=(1-U)^{-2}(W-UW+XY).$$

(There is a slight error in the definition of $(u;\ x,\ y)$ as given in [1]). For $k=1$, (1.3) reduces to the Hardy-Hille formula [2, p. 101]:

(1.6) $$\sum_{n=0}^{\infty}\frac{n!u^n}{\Gamma(\alpha+n+1)}\,L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)=$$

$$\frac{(xyu)^{-\frac{1}{2}\alpha}}{1-u}\exp\left(-\frac{(x+y)u}{1-u}\right)I_\alpha\left(\frac{2\sqrt{xyu}}{1-u}\right).$$

The object of the present paper is to give an elementary proof of (1.3) and indeed of the following more general bilinear formula.

(1.7) $$\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{(\alpha)_{n_1+\cdots+n_k}}{n_1!\ \ldots\ n_k!}\,u_1^{n_1}\ \ldots\ u_k^{n_k}\cdot$$

$$\cdot F_A(-n_1\,,\ \ldots,\ -n_k\,;\ \gamma;\ \alpha;\ y_1\,,\ \ldots,\ y_k)=$$

$$\cdot F_A(-n_1\,,\ \ldots,\ -n_k\,;\ \beta;\ \alpha;\ x_1\,,\ \ldots,\ x_k)\cdot$$

$$=(1-U)^{-\alpha+\beta+\gamma}(1-U+X)^{-\beta}(1-U+Y)^{-\gamma}\cdot$$

$$\cdot F\left[\beta,\ \gamma;\ \alpha;\ \frac{(1-U)W+XY}{(1-U+X)(1-U+Y)}\right].$$

where

(1.8) $$F_A(-n_1\,,\ \ldots,\ -n_k\,;\ \beta;\ \alpha;\ x_1\,,\ \ldots,\ x_k)=$$

$$=\sum_{r_j=0}^{n_j}(-1)^{r_1+\cdots+r_k}\binom{n_1}{r_1}\ \ldots\ \binom{n_k}{r_k}\frac{(\beta)_{r_1+\cdots+r_k}}{(\alpha)_{r_1+\cdots+r_k}}\,x_1^{r_1}\ \ldots\ x_k^{r_k}\,.$$

For $k=1$, (1.7) reduces to

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} u^n F(-n, \beta; \alpha; x) F(-n, \gamma; \alpha; y) = \tag{1.9}$$

$$= (1-x)^{-\alpha+\beta+\gamma}(1-u+xu)^{-\beta}(1-u+yu)^{-\gamma} \cdot$$

$$\cdot F\left[\beta, \gamma; \alpha; \frac{xyu}{(1-u+xu)(1-u+yu)}\right],$$

a formula due to Weisner [3].

2. It will be convenient to define

$$\Phi(n; r, s) = \sum \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \binom{n_1}{s_1} \cdots \binom{n_k}{s_k} \cdot \tag{2.1}$$

$$\cdot \frac{u_1^{n_1} \cdots u_k^{n_k}}{n_1! \cdots n_k!} x_1^{r_1} \cdots x_k^{r_k} y_1^{s_1} \cdots y_k^{s_k},$$

where the summation is ower all nonnegative n_j, r_j, s_j such that

$$n_1 + \dots + n_k = n, \; r_1 + \dots + r_k = r, \; s_1 + \dots + s_k = s. \tag{2.2}$$

Thus Φ $(n; r, s)$ is homogeneous of weight n in $u_1, \dots, u_k$, of weight r in $x_1, \dots, x_k$ and of weight s in $y_1, \dots, y_k$. Alternatively we may define $\Phi(n; r, s)$ by means of

$$\sum_{n, r, s=0}^{\infty} \Phi(n; r, s) = \exp(U+X+Y+W), \tag{2.3}$$

where U, X, Y, W are defined by (1.4).

We rewrite (1.3) in the form

$$\sum_{n_1, \dots n_k=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1! \cdots n_k!} L_{n_1, \dots, n_k}^{(\alpha)}(x_1, \dots, x_k) \cdot \tag{2.4}$$

$$\cdot L_{n_1, \dots, n_k}^{(\alpha)}(y_1, \dots, y_k) =$$

$$= (1-U)^{-\alpha-1} \exp\left(-\frac{X+Y}{1-U}\right) \sum_{n=0}^{\infty} \frac{(u; x, y)^n}{n!(\alpha+1)_n}.$$

Making use of (1.2) and (2.2), it is easily seen that the left hand side of (2.4) is equal to

$$\sum_{n,\, r,\, s=0}^{\infty} (-1)^{r+s}\Phi(n;\ r,\ s)\frac{(\alpha+1)_n}{(\alpha+1)_r(\alpha+1)_s}. \tag{2.5}$$

Hence that part of the left hand side of (2.4) that is of weight r in x_1 ..., x_k and of weight s in y_1, ..., y_k is equal to

$$\sum_{n=0}^{\infty} \Phi(n;\ r,\ s)\frac{(\alpha+1)_n}{(\alpha+1)_r(\alpha+1)_s}. \tag{2.6}$$

We set the right hand side of (2.4) equal to

$$\sum_{n,\, r,\, s=0}^{\infty} \psi(n;\ r,\ s), \tag{2.7}$$

where $\psi(n;\ r,\ s)$ is homogeneous of degree n in u_1, ..., u_k, of weight r in x_1, ..., x_k and of weight s in y_1, ..., y_k. Then clearly (2.4) is equivalent to

$$\sum_{n=0}^{\infty} \Phi(n;\ r,\ s)\frac{(\alpha+1)_r(\alpha+1)_s}{(\alpha+1)_n} = \sum_{n=0}^{\infty} \psi(n;\ r,\ s) \quad (r,\ s=0,\ 1,\ 2,\ \ldots). \tag{2.8}$$

The right hand side of (2.4) is equal to

$$(1-U)^{-\alpha-1} \sum_{r,\, s=0}^{\infty} (-1)^{r+s}\frac{X^r Y^s}{r!s!}(1-U)^{-r-s}\cdot \tag{2.9}$$

$$\cdot\sum_{j=0}^{\infty} \frac{[(1-U)W+XY]^j}{j!(\alpha+1)_j}(1-U)^{-2j}.$$

Since $(1-U)W+XY$ is of weight one in x_1, ..., x_k and of weight one in y_1, ..., y_k, it follows from (2.9) that

$$\sum_{n=0}^{\infty} \psi(n;\ r,\ s)= \tag{2.10}$$

$$=(-1)^{r+s}(1-U)^{-\alpha-r-s-1} \sum_{j=0}^{\min(r,\, s)} \frac{X^{r-j}Y^{s-j}[(1-U)W+XY]^j}{j!(r-j)!(s-j)!(\alpha+1)_j}.$$

In the next place, (2.8) is equivalent to

$$\sum_{n,\,r,\,s=0}^{\infty} \Phi(n;\; r,\; s)(\alpha+1)_n = \sum_{n,\,r,\,s=0}^{\infty} \psi(n;\; r,\; s)(\alpha+1)_r(\alpha+1)_s. \tag{2.11}$$

But, by (2.1) and (2.2),

$$\sum_{n,\,r,\,s=0}^{\infty} (-1)^{r+s}\Phi(n;\; r,\; s)(\alpha+1)_n$$

$$= \sum_{n_j=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1!\dots n_k!} u_1^{n_1} \dots u_k^{n_k}\cdot$$

$$\cdot \sum_{r_j,\, s_j=0}^{n_j} (-1)^{r_1+\dots+r_k+s_1+\dots+s_k} \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \binom{n_1}{s_1} \dots \binom{n_k}{s_k}$$

$$x_1^{r_1} \dots x_k^{r_k} y_1^{s_1} \dots y_k^{s_k} = \sum_{n_j=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1!\dots n_k!} u_1^{n_1} \dots u_k^{n_k}(1-x_1)^{n_1} \dots (1-x_k)^{n_k}$$

$$(1-y_1)^{n_1} \dots (1-y_k)^{n_k} = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} \{\sum_{j=1}^{k} u_j(1-x_j)(1-y_j)\}^n.$$

$$= \{1-U+X+Y-W\}^{-\alpha-1}. \tag{2.12}$$

On the other hand, by (2.10),

$$\sum_{n,\,r,\,s=0}^{\infty} \Psi(n;\; r,\; s)(\alpha+1)_r(\alpha+1)_s =$$

$$=(1-U)^{-\alpha-1} \sum_{r,\,s=0}^{\infty} (-1)^{r+s}(\alpha+1)_r(\alpha+1)_s(1-U)^{-r-s}\cdot$$

$$\cdot \sum_{j=0}^{\min(r,\,s)} \frac{X^{r-j}Y^{s-j}[(1-U)W+XY]^j}{j!(r-j)!(s-j)!(\alpha+1)_j} =$$

$$=(1-U)^{-\alpha-1} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!} [(1-U)W+XY]^j(1-U)^{-2j}\cdot$$

$$\cdot \sum_{r,\,s=0}^{\infty} (-1)^{r+s} \frac{(\alpha+j+1)_r(\alpha+j+1)_s}{r!\, s!} X^rY^s(1-U)^{-r-s} =$$

$$=(1-U)^{-\alpha-1} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!} [(1-U)W+XY]^j(1-U)^{-2j}\cdot$$

$$\cdot\left(1+\frac{X}{1-U}\right)^{-\alpha-j-1}\left(1+\frac{Y}{1-U}\right)^{-\alpha-j-1}-$$

$$=(1-U)^{\alpha+1}(1-U+X)^{-\alpha-1}(1-U+Y)^{-\alpha-1}\sum_{j=0}^{\infty}$$

$$\frac{(\alpha+1)_j}{j!}\quad\frac{[(1-U)W+XY]^j}{(1-U+X)^j(1-U+Y)^j}$$

$$=(1-U)^{\alpha+1}\{(1-U+X)(1-U+Y)-(1-U)W+XY\}^{-\alpha-1}=$$

$$=(1-U)^{\alpha+1}\{(1-U)^2+(1-U)(X+Y)-(1-U)W\}^{-\alpha-1}=$$

$$=(1-U+X+Y-W)^{-\alpha-1},$$

so that

$$\sum_{n,\,r,\,s=0}^{\infty}\psi(n;\ r,\ s)(\alpha+1)_r(\alpha+1)_s=(1-U+X+Y-W)^{-\alpha-1}. \tag{2.13}$$

In view of (2.12) and (2.13), (2.11) is satisfied. This completes the proof of (2.4) and therefore of (1.3).

3. We shall now prove (1.7). By (1.8) the left hand side of (1.7) is equal to

$$\sum_{n_j=0}^{\infty}\frac{(\alpha)_{n_1+\dots+n_k}}{n_1!\ \dots\ n_k!}u_1^{n_1}\ \dots\ u_k^{n_k}\cdot$$

$$\sum_{r_j,s_j}^{\infty}(-1)^{r_1+\dots+r_k+s_1+\dots+s_k}\binom{n_1}{r_1}\dots\binom{n_k}{r_k}\binom{n_1}{s_1}\dots\binom{n_k}{s_k}\cdot$$

$$=\frac{(\beta)_{r_1+\dots+r_k}(\gamma)_{s_1+\dots s_k}}{(\alpha)_{r_1+\dots+r_k}(\alpha)_{s_1+\dots s_k}}x_1^{r_1}\ \dots\ x_k^{r_k}y_1^{s_1}\ \dots\ y_k^{s_k}=$$

$$=\sum_{n,\,r,\,s=0}^{\infty}(-1)^{r+s}(\alpha)_n\Phi(n;\ r,\ s)\frac{(\beta)_r(\gamma)_s}{(\alpha)_r(\alpha)_s},$$

where $\Phi(n;\ r,\ s)$ is defined by (2.1).

Now by (2.8) we have

$$\sum_{n=0}^{\infty}\Phi(n;\ r,\ s)\frac{(\alpha)_n}{(\alpha)_r(\alpha)_s}=\sum_{n=0}^{\infty}\Psi'(n;\ r,\ s)\qquad(r,\ s=0,\ 1,\ 2,\ \dots), \tag{3.1}$$

where $\psi'(n;\ r,\ s)$ is the result of replacing $\alpha+1$ by α in $\Psi(n;\ r,\ s)$. Consequently the left member of (1.7) is equal to

$$\sum_{n,r,s=0}^{\infty} \Psi'(n;\ r,\ s)(\beta)_r(\gamma)_s\,. \tag{3.2}$$

Then, by (2.10), this sum is equal to

$$(1-U)^{-\alpha}\sum_{r,s=0}^{\infty}(-1)^{r+s}(\gamma)_s(1-U)^{-r-s}\cdot$$

$$\cdot\sum_{j=0}^{\min(r,s)}\frac{X^{r-j}Y^{s-j}[(1-U)W+XY]^j}{j!(r-j)!(s-j)!(\alpha)_j}=$$

$$=(1-U)^{-\alpha}\sum_{j=0}^{\infty}\frac{(\beta)_j(\gamma)_j}{j!(\alpha)_j}[(1-U)W+XY]^j(1-U)^{-2j}\cdot$$

$$\cdot\sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{(\beta+j)_r(\gamma+j)_s}{r!s!}X^rY^s(1-U)^{-r-s}=$$

$$=(1-U)\sum_{j=0}^{\infty}\frac{(\beta)_j(\gamma)_j}{j!(\alpha)_j}[(1-U)W+XY]^j(1-U)^{-2j}\cdot$$

$$\cdot\left(1+\frac{X}{1-U}\right)^{-\beta-j}\left(1+\frac{Y}{1-U}\right)^{-\gamma-j}=$$

$$=(1-U)^{-\alpha+\beta+\gamma}(1-U+X)^{-\beta}(1-U+Y)^{-\gamma}\sum_{j=0}^{\infty}\frac{(\beta)_j(\gamma)_j}{j!(\alpha)_j}$$

$$\left\{\frac{(1-U)W+XY}{(1-U+X)(1-U+Y)}\right\}^j=(1-U)^{-\alpha+\beta+\gamma}(1-U+X)^{-\beta}(1-U+Y)^{-\gamma}$$

$$F\left[\beta,\ \gamma;\ \alpha;\ \frac{(1-U)W+XY}{(1-U+X)(1-U+Y)}\right].$$

This completes the proof of (1.7).

4. Exactly as in proving (1.7), we can establish the following more general result.

$$\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{(\alpha)_{n_1+\ldots+n_k}}{n_1!\ \ldots\ n_k!}u_1^{n_1}\ \ldots\ u_k^{n_k}\cdot \tag{4.1}$$

$$\cdot F_A(-n_1, \ldots, -n_k; \beta; \beta'; x_1, \ldots, x_k) F_A(-n_1, \ldots, -n_k; \gamma; \gamma'; y_1, \ldots, y_k) =$$

$$= (1-U)^{-\alpha} \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j (\gamma)_j}{j! (\beta)_j (\gamma')_j} [(1-U)W + XY]^j (1-U)^{-2j} \cdot$$

$$\cdot F\left[\alpha+j,\ \beta+j;\ \beta'+j;\ -\frac{X}{1-U}\right] F\left[\alpha+j,\ \gamma+j;\ \gamma'+j;\ -\frac{Y}{1-U}\right].$$

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Manoscritto pervenuto in redazione il 3 novembre 1969.