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ON A FIXED POINT THEOREM IN METRIC SPACE

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Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction mapping if there exists k, $0 \leq k < 1$, such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all x, y in X.

The well-known Banach Contraction Principle states that a contraction mapping on a complete metric space X has a unique fixed point.

Recently Maia [4] proved the following theorem:

Let X have two metrics d and δ such that

(1) $d(x, y) \leq \delta(x, y)$ for all x, y in X.

(2) X be complete with respect to d.

(3) $T: X \to X$ be a mapping continuous with respect to d.

and

(4) $T: X \to X$ be contraction with respect to δ.

Then there exists a unique fixed point of T in X.

The aim of this note is to get the same conclusion under much less restricted condition. Also, we do not need the continuity of T with respect to d on X, just the continuity at a point will serve the purpose.

THEOREM. Let X have two metrics d and δ, and the following

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conditions are fulfilled.

(1) $d(x, y) \leq \delta(x, y)$ for all x, y in X,

(2) $T : X \rightarrow X$ is a contraction mapping with respect to δ, i.e. $\delta(Tx, Ty) \leq k\delta(x, y)$ for all x, y in X.

(3) T is continuous at $p \in X$ with respect to d.

and

(4) There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ convering to p in metric d.

Then T has a unique fixed point.

PROOF. Given $x_0 \in X$. We define

$$x_{n+1} = Tx_n \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

i.e. $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, ..., $x_n = T^n x_0$.

It follows from [4] that $\{x_n\}$ is a Cauchy sequence with respect to δ. Since $d(x, y) \leq \delta(x, y)$ for all x, y in X, it follows that $\{x_n\}$ is a Cauchy sequence with respect to d. Since the subsequence $\{x_{n_i}\}$ of the Cauchy sequence $\{x_n\}$ converges to p, therefore $\{x_n\}$ converges to p under d i.e. $\lim_{n \rightarrow \infty} x_n = p$.

Since T is continuous at p therefore

$$Tp = T \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = p.$$

Thus T has a fixed point.

We need to prove that T has a unique fixed point .Let x and y be two different fixed points. Then

$$\begin{aligned} \delta(x, y) &= \delta(Tx, Ty) \\ &\leq k\delta(x, y). \end{aligned}$$

Since $0 \leq k < 1$, we get a contradiction and therefore T has a unique fixed point .

It can be easily seen that condition (4) does not imply the completeness of X, with respect to d. For example, let $X=[0, 1)$ and $T: X \to X$ be given by $Tx=\frac{4}{x}$. d is the usual metric. In this case the sequence of iterates has a convergent subsequence but the space is not complete.

REMARK. The theorem will remain true if condition (2) is replaced by the following condition:

(2′) There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ is a Cauchy sequence with respect to δ.

This condition is so general that results given by several mathematicians can be easily unified.

It can be easily seen that the sequence of iterates of the mappings given below is always a Cauchy sequence .

EXAMPLES.

(1) $\delta(Tx, Ty) \leq k\delta(x, y)$ for all $x, y \in X$ and $0 \leq k < 1$.

(2) $\delta(Tx, Ty) \leq \alpha\{\delta(x, Tx)+\delta(y, Ty)\}$ for all $x, y \in X$ and $0 \leq \alpha < \frac{1}{2}$. This mapping has been given by Kannan [3].

(3) Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$.

This mapping has been studied by Meir and Keeler [5].

The mappings considered by Edelstein [2], Rakotch [6] and Boyd and Wong [1] satisfy above condition given in example (3), and therefore the sequence of iterates will be Cauchy sequence.

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