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PROJECTIVITIES OF FREE PRODUCTS

by CHARLES S. HOLMES *)

RÉSUMÉ. A projectivity of a group G onto a group H is an isomorphic mapping of the subgroup lattice $L(G)$ of G onto the subgroup lattice $L(H)$ of H. Considerable attention has been devoted to the following questions: under what conditions on G is any projectivity of G induced by some isomorphism of G, and when must G be determined by its lattice of subgroups in the sense that any group H with $L(H)$ isomorphic to $L(G)$ must itself be isomorphic to G? Theorems 4 and 6 of this paper are partial answers to these questions for the class of groups which have a non-trivial decomposition as a free product with amalgamated subgroup.

Let $G = A_C^* B$ $(A, B \neq C)$ be a free product of its subgroups A and B with amalgamated subgroup $C = A \cap B$. Suppose $[A : C] > 2$ or $[B : C] > 2$. *Theorem 4.* If C is normal in G, then G is determined by its lattice of subgroups. *Theorem* 6. If C is the center of G, then every projectivity of G is induced by a unique group isomorphism. *Corollary.* Any projectivity of a free product A^*B $(A, B \neq 1)$ is induced by a unique isomorphism. The proofs depend upon the fact that $A^*B(A, B \neq 1)$ contains a non-cyclic free group unless A and B are both of order two. The methods used are primarily extensions and refinements of those used by E. L. Sadovskii (Mat. Sbornik 21 (63) (1947), 63-82).

Introduction.

The collection of all subgroups of a group G forms a lattice $L(G)$; here the lattice meet, $H \wedge K$, of two subgroups H and K is their intersection $H \cap K$, while the lattice join, $H \vee K$, is the subgroup generated by the union $H \cup K$. Following Suzuki [4] we define a projectivity

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from a group G onto a group H to be an isomorphic mapping of the subgroup lattice $L(G)$ onto the subgroup lattice $L(H)$. For the definitions of concepts discussed informally in the introduction the reader is referred to the section, Definitions and Notations, of this paper.

Every isomorphism from a group G onto a group H clearly induces a projectivity from G onto H. Thus if G and H are isomorphic, $L(G)$ and $L(H)$ are isomorphic. Considerable attention has been devoted to the converse questions: under what conditions on G is any projectivity of G induced by some isomorphism of G, and when must G and H be isomorphic if $L(G)$ and $L(H)$ are isomorphic?

It is easy to see that non-isomorphic groups may have isomorphic lattices of subgroups; hence a group need not be determined by its lattice of subgroups and a projectivity need not be induced by an isomorphism. The simplest examples are the groups of prime order, all of which have the two element lattice as subgroup lattice. There are less trivial examples of groups which are not determined by their subgroup lattices. On the other hand several classes of groups have been shown to consist of groups which are determined by their lattices.

Sadovsky, for instance, has shown the following. If $G=A^*B$ is a free product, without amalgamation, of non-trivial groups A and B, then G is determined by its lattice of subgroups $L(G)$. He also speculated that any projectivity p of such a free product $G=A^*B$ is induced by an isomorphism, but left the question open. He made several contributions to the solution of this problem, two of which are useful here. First he showed that if an inducing isomorphism exists for a projectivity of a free product, then it must be unique. Sadovsky also showed that any projectivity of a free product $G=A^*B$ where A and B are both of order two is induced by an isomorphism. It is an immediate consequence of Theorem 6 in this paper that any projectivity of a free product $G=A^*B$ where A or B is of order greater than two is induced by an isomorphism of G; again A and B are non-trivial. Combining these results we have the following result. Any projectivity of a free product $G=A^*B$ with non-trivial factors A and B is induced by a unique isomorphism.

In this paper we partially answer the same questions for the more general class of free products $G=A^*_CB$ with amalgamated subgroup $C=A\cap B$, where C is a proper subgroup of both A and B. For the definition and elementary properties of free products and free groups we refer the reader to the Theory of Groups [2] by Kurosh. In Theorem 4 it is shown that any group $G = A_C^* B$ is determined by its lattice of subgroups $L(G)$, if C is a proper normal subgroup of A and B, and the index of C in A, $[A:C]$, or the index of C in B, $[B:C]$, is strictly greater than two. If in addition C is the center of G, every projectivity p of G is induced by a unique isomorphism of G (Theorem 6).

These results are certainly not the best possible. It is not difficult to see that the arguments given below permit us to draw the same conclusions with the conditions on C weakened in various significant ways. However, we have not stated any such extensions of our main result, partly because we have not been able to prove any that does not involve rather complicated and artificial conditions, and partly because we have no evidence that the conclusions do not hold quite generally with the conditions on C (except for C proper in A and B) simply dropped. It seems evident that our methods could not yield such a result without considerable revision and extension.

The methods used here are based upon the following two facts. The first fact is that any projectivity p from a group G onto a group H is fully determined by its action on the cyclic subgroups $[g]$ generated by the elements g of G, since every subgroup K of G is the join of the cyclic subgroups contained in K. The second fact (due to Baer [1]) is that the image of a cyclic subgroup must be cyclic. That is, for every element g of G there is an element h (not necessarily unique) such that $p[g] = [h]$. Combining the two results we see that the image of the subgroup $[g_1] \vee [g_2]$ under the projectivity p must be $p([g_1] \vee [g_2]) = [h_1] \vee [h_2]$ where $p[g_1] = [h_1]$ and $p[g_2] = [h_2]$. In this connection it is simple to observe that

$$[g_1] \vee [g_2] = [g_1 g_2] \vee [g_2] = [g_1] \vee [g_1 g_2];$$

hence that

$$[h_1] \vee [h_2] = [h] \vee [h_2] = [h_1] \vee [h]$$

where $p[g_1 g_2] = [h]$. This use of the join preserving property of the projectivity is one of the rare instances in which lattice theory enters into our arguments.

For the moment let $G=A^*B$ be a free product, without amalgamation, of non-trivial subgroups A and B, and let p be a projectivity from some group G' onto G. Let A' and B' be the subgroups of G' such that $pA'=A$ and $pB'=B$. Sadovsky found that he could pick elements v' in G' and v in G with $p[v']=[v]$ in such a way that the relation

$$R=\{(a', a) : a'\in A', a\in p[a'], p[a'v']=[av]\}$$

was an isomorphism from A' onto A. This isomorphism in turn induced the restriction of p on A'. In a similar way isomorphisms were determined from B' onto B which also induced the projectivity on B'. Sadovsky then showed that G' was a free product of A' and B', hence that G and G' were isomorphic. He left open the problem of extending the isomorphisms on the components to an isomorphism on G' inducing p.

Now let p be a projectivity of G' onto $G=A_C^*B$ where C is a proper normal subgroup of A and B, and $[A : C]$ or $[B : C]$ is strictly greater than two. Again let A' and B' be the subgroups of G' such that $pA'=A$, and $pB'=B$. In Theorem 3 of this paper it is shown that for any element g' in G' there is an isomorphism from $[g'] \vee C'$ onto $p[g'] \vee C$. Shortly thereafter it is shown that there are isomorphisms f_1 and f_2 which map A' onto A and B' onto B, respectively, such that f_1 and f_2 agree on C'. We define f to be the map pieced together from the isomorphisms between $[g'] \vee C_2$ and $p[g'] \vee C$. In Theorem 4 it is shown that G' is isomorphic to G by showing that f is the natural extension of f_1 and f_2 to all of G'.

1. Definitions and Notation.

If $g_1, ..., g_n$ are elements of the group G, we shall write $[g_1, ..., g_n]$ for the subgroup of G generated by $g_1, ..., g_n$. We shall also write $[g_1, ..., g_n, U_1, ..., U_m]$ for the subgroup generated by the elements g_i together with the subsets U_j of G. Thus if G_1 and G_2 are two subgroups of G, then $[G_1, G_2]=G_1 \vee G_2$. Throughout the paper we let 1 stand for the identity subgroup or the first natural number, as appropriate. If H is a subgroup of the group G, we write $H\leq G$. If X and Y are subsets of a set S, we write $X-Y$ for the complement of Y in X.

We record here the fundamental definitions of this subject and presume that the introduction provides ample discussion of these topics.

Definition. Let G and H be groups, and let $L(G)$ and $L(H)$ be the lattices of subgroups of G and H, respectively. A mapping p from $L(G)$ into $L(H)$ is a *projectivity* of G onto H if and only if p is a one to one mapping of $L(G)$ onto $L(H)$ and for any two subgroups G_1 and G_2 in G, $p(G_1 \cap G_2) = pG_1 \cap pG_2$ and $p[G_1, G_2] = [pG_1, pG_2]$. Two subgroup lattices $L(G)$ and $L(H)$ are said to be *isomorphic* if and only if there is a projectivity from G onto H.

Definition. A group G is *determined by* $L(G)$, its lattice of subgroups, if and only if any group H with $L(H)$ isomorphic to $L(G)$ must itself be isomorphic to G. A projectivity p from a group G onto a group H is said to be *induced by an isomorphism*, if there is an isomorphism φ from G onto H such that for any subgroup K in G, $pK = \{\varphi(k) \mid k \in K\}$.

We shall use the notation $G = A *_C B$ to mean that G is the free product of its subgroups A and B with amalgamated subgroup $C = A \cap B$. When C is the identity subgroup we shall write simply $G = A^* B$ and call G the free product of A and B, without amalgamation. In fact we shall never use this notation except under the assumption that C is properly contained in both A and B; that is, we assume that $A - C$ and $B - C$ are non-empty.

Definition. Let $G = A *_C B$. Then any expression $g_1 \ldots g_n$ with g_i alternately in $A - C$ and $B - C$ and with n a positive integer is said to be a *word* in G.

In a free product $G = A *_C B$ any word $g_1 \ldots g_n$ considered as a product in G is an element of the complement of C in G, $G - C$. Conversely every element in $G - C$ can be expressed as a word in G. For an element w in $G - C$ this representation $w = g_1 \ldots g_n$ as a word in G is not in general unique, but w does determine uniquely the *length* n and the double cosets $Cg_1C, \ldots, Cg_nC$.

In order to see this let $g_1 \ldots g_m$ and $h_1 \ldots h_n$ be two words which represent the same element in $G - C$, that is $g_1 \ldots g_m = h_1 \ldots h_n$. Then $h_1^{-1} \ldots h_1^{-1} g_1 \ldots g_m = 1$, where 1 stands for the identity element in G. Now $h_1^{-1} g_1$ must be in C, for otherwise the expression on the left is a word or can be written as one. That is if h_1 and g_1 are from the same group and $h_1^{-1} g_1$ is not in C, then $h_1^{-1}, h_1^{-1} g_1$, and g_1 are all from the same set $A-C$ or $B-C$ and

$$h_n^{-1} \ \dots \ h_2^{-1}(h_1^{-1}g_1)g_2 \ \dots \ g_m$$

is a word as written. In a similary way if h_1 and g_1 are from different groups then the expression

$$h_n^{-1} \ \dots \ h_1^{-1}g_1 \ \dots \ g_m$$

is a word as written. A simple induction shows that $h_i^{-1} \dots h_1^{-1}g_1 \dots g_i$ is in C, where i is less than or equal to m and n. Hence that $m=n$ and that h_i and g_i are both in the same set $A-C$ or $B-C$. Moreover the double cosets Ch_iC and Cg_iC are equal for each $i=1, \dots, m$, or simply $h_iC=g_iC$ when C is normal. When C is the identity subgroup of G or in other words when $G=A^*B$ is a free product without amalgamation, every non-identity element in G has a unique representation as a word in G.

Definition. Let $G=A_C^*B$. Suppose that g is an element of the complement of C in G and that $g=g_1 \dots g_n$ is any representation for g as a word in G. If g_1 is in $A-C$ or $B-C$, we say that g *begins* with an element of A or B, respectively. If g_n is in $A-C$ or $B-C$, we say that g *ends* with an element of A or B, respectively. We also say that the length $|g|$ of g is n, and take the length of any element in C to be zero.

The following results are simple consequences of these definitions. Let g and h be any two elements in the complement of C in $G=A_C^*B$. Then

1) $|gh| \leq |g|+|h|$

2) $|gh| < |g|+|h|$ if and only if g^{-1} and b both begin with elements from the same group A or B.

3) If $|g|=|h|$ and gh is not an element of C, then gh and g begin with elements of the same group while gh and h end with elements from the same group.

Now let $A:A$ designate the set of elements in $G-C$ beginning and ending with elements of A, and $A:B$ the set of elements beginning with elements of A and ending with elements of B. Also let $B:A$ and $B:B$ be defined similarly. It is easy to see that the set G is the disjoint union of C, $A:A$, $A:B$, $B:A$, and $B:B$.

2. Elementary Propositions.

In this section we isolate some rather technical propositions which are useful in the following section. Throughout the paper we shall reserve the word proposition for facts from group theory and the word lemma and theorem for the results about projectivities of groups.

PROPOSITION 1. Let G be a group, N a normal subgroup and V a non-trivial cyclic subgroup of G. Let U and W be cyclic subgroups of G. Suppose

1) $U \leq N$
2) $V \cap N = 1$, and
3) $[U, W] = [U, V] = [W, V]$.

Then for any generator w of W there is a unique u in N and there is a unique v in V such that

$$w = uv.$$

Moreover $[v] = V$ and $[u, v] = [U, V]$.

PROOF. Suppose w is some generator of W. From $W \leq [N, V]$ with N normal in G it follows that there are elements u in N and v in V such that $w = uv$. If also $w = u_1 v_1$ with u_1 in N and v_1 in V, then $u_1^{-1}u$ and $v_1 v^{-1}$ are equal and therefore equal to the identity element in G since $V \cap N = 1$. Hence u and v are unique in N and V respectively.

From $[U, W] = [U, V]$ and $U \leq N$ it follows that

$$V \leq [U, W] \leq [N, W] = [N, uv] = [N, v] = N[v].$$

Hence

$$V \leq [v],$$

for otherwise

$$V \cap N \neq 1.$$

Thus $[v]=V$. But then

$$[U, V]=[W, V]=[uv, v]=[u, v]$$

and the proposition is proved.

Corollary. If in Proposition 1 the subgroup N is in the center of G, then $[u]=U$.

Proof. Since N is in the center of G the subgroup $[U, V]$ is the direct product of U and V. Hence u is an element of U. Then from $[u, V]=[U, V]$ the conclusion follows.

It is easy to see that in Proposition 1 we could have concluded that there are unique elements u in U and v in V such that $w=vu$.

Proposition. 2. Let G be any group and U, V and W non-trivial cyclic subgroups of G. Suppose that

$$[U, W]=U^*V=[V, W].$$

Then there is a unique u in U and there is a unique v in V such that

$$W=[uv].$$

Moreover $[u]=U$ and $[v]=V$.

Proof. We assume that W is not in U or V, for otherwise $U^*V=U$ or $U^*V=V$. Let w be a generator of W. Suppose that $w \in U : U$. Then

$$|w|>1,$$

and all the elements of $[V, W]$ which begin and end with elements of U have length three or more. That is, $w \in U : U$ and $w \notin U$ imply the contradiction

$$U \not\subseteq [W, V]=[U, V].$$

Thus $w \notin U : U$. Similarly $w \notin V : V$.

Now $w \in U : V$ or $w \in V : U$, and w^{-1} is in the other. We may suppose that $w \in U : V$. Let

$$w = u_1 v_1, \dots, u_n v_n \qquad (1 \leq n),$$

$$w' = u_1 v_1, \dots, u_n .$$

If $1 < n$, then $w' \in U : U$ but $w' \notin U$ and we again have the contradiction

$$U \not\leq [V, w'] = [V, W] = [U, V].$$

Thus

$$n = 1 \text{ and } w = u_1 v_1 .$$

Since $U^* V$ is a free product, without amalgamation, there are no other elements $u_2 \in U$, $v_2 \in V$ such that $w = u_2 v_2$. Furthermore W is infinite cyclic and its only other generator is $w^{-1} = v_1^{-1} u_1^{-1} \in V : U$. So u_1 and v_1 are the only two elements of G such that $W = [u_1 v_1]$.

We write simply $w = uv$. The fact that $[u] = U$ and $[v] = V$ follows immediately from the relations

$$U \leq [W, V] = [u, V] \text{ and } V \leq [U, W] = [U, v].$$

Again it is easy to see that in Proposition 2 we could have concluded that there exist unique elements u in U and v in V such that $W = [vu]$.

PROPOSITION 3. Let $G = [t]^*[v]^*[x, y, z]$ where t and v are not of order two. Suppose that

$$((tx)^k (yv)^l)^j = (tz)^m v^n,$$

where $j, k, l, m, n = \pm 1$. If $y \neq 1$ or $m, n = 1$, then

$$j, k, l, m, n = +1 \text{ and } z = xy.$$

PROOF. Let $T = [t]$, $V = [v]$ and $X = [x, y, z]$. Let w be the element given by the equated expressions. The expression on the right shows that w ends with an element of V. Hence $j = 1$ and

$$(tx)^k (yv)^l = (tz)^m v^n.$$

Suppose first that $m, n=1$. Then $k=-1$ implies that w begins with x^{-1} or t^{-1} rather than t as it must. Hence $k=1$. Similarly $l=1$.

Suppose now that y is non-trivial. Since w ends with an element of V, $l=1$ and $n=1$. Thus

$$(tx)^k y=(tz)^m.$$

Let u be the element given by these expressions. Then either u does or does not end with an element of X. If u does end with an element of X, then $m=1$ and we are done. If u does not end with an element of X, then $k=1$ and x is the inverse of y. Now $m=1$, for otherwise t is its own inverse.

3. Functions Induced by a Projectivity.

Let p be a projectivity from a group G' onto a group G. Let v' be an element of G' and v an element of G such that $p[v']=[v]$. Let U' be a subgroup of G' and U the image of U' under p.

DEFINITION. Suppose that for every u' in U' there is a unique u in U such that $p[u'v']=[uv]$ (or $p[v'u']=[vu]$). Then the function f defined by $f(u')=u$ for all elements u' in U' is said to be the function on U' *induced by the projectivity p and the pair* (v', v) with v' and v on the right (left).

For a given function f from U' into U we write $f=f(.; v', v)$ ($f=f(v', v; .)$) if and only if f is the function induced by the projectivity p and the pair (v', v) with (v', v) on the right (left). Thus for any element u' in U',

$$f(u')=f(u'; v', v).$$

We also say that a function f from U' into U is *induced by the projectivity p* if and only if there is a pair of elements v' in G' and v in G such that $f=f(.; v', v)$ or $f=f(v', v; .)$. Because there is usually only one projectivity p under discussion, we sometimes simply say that the pair (v', v) induces the functions $f(v', v; .)$ and $f(.; v', v)$.

In this section we shall show that functions $f=f(.; v', v)$ do exist for certain subgroups U' and U and pairs v' and v. In fact if U' and U are the identity subgroups of G' and G, respectively, and if f is the function which maps U' onto U in the only possible way then $f=f(.;\ v',\ v)=f(v',\ v;\ .)$. In the more general situation in which U' and U are not trivial groups there may be non-trivial elements u in U such that $p[v']=[uv]$. For example suppose that v has order twelve in G and that $U\cap[v]=[v^4]$. If v' is any element of G such that $p[v']=[v]$, then $p[v']=[(v^4)v]$. That is the element v^4 also corresponds to the identity of G'. Thus in the following work we shall almost always assume that $[v]$ meets U in the identity subgroup.

For the remainder of this section let p be a projectivity of some group G' onto a group G. If H is any subgroup of G we write H' for the subgroup of G' such that $pH'=H$.

LEMMA 1. Let N be a normal subgroup of G and V an infinite cyclic subgroup of G such that $V\cap N=1$. Suppose v' and v are generators of V' and V, respectively. Then for each u' in N', there is a unique u in N such that $p[u'v']=[uv]$.

PROOF. Let u' be an element in N' and U the cyclic subgroup of N such that $p[u']=U$. Also let W be the cyclic subgroup of G such that $p[u'v']=W$. Since

$$[u',\ u'v']=[u',\ v']=[u'v',\ v'] \qquad \text{in } G',$$

we have $[U,\ W]=[U,\ V]=[W,\ V]$ in G. Thus Proposition 1 is applicable to the situation at hand.

By Proposition 1 the canonical homomorphism φ from G onto G/N maps W and V onto the same subgroup $\bar{V}$ of G/N. Since $V\cap N=1$, the restriction of φ to V is an isomorphism of V onto $\bar{V}$. However it is also true that $W\cap N=1$, since V is an infinite cyclic group. Hence the restriction of φ to W is an isomorphism of W onto $\bar{V}$. Therefore for any generator v of V there is a unique generator w of W such that $\varphi(w)=\varphi(v)$.

Now if v is the generator of V given in the lemma and w is the generator of W such that $\varphi(w)=\varphi(v)$, then $w=uv$ where u is unique in U since $N \cap V = 1$. Combining the uniqueness of w and the uniqueness of u we have that there is a unique u in N such that $p[u'v'] = [uv]$.

COROLLARY. If in Lemma 1 the subgroup N is in the center of G, then $[u] = U$.

PROOF. This follows immediately from the corollary to Proposition 1.

There is of course a « dual » result to Lemma 1 which asserts that under the conditions of Lemma 1 there is a unique element u in N such that $p[v'u'] = [vu]$. The proof uses the « dual » to Proposition 1 which was mentioned after the proof of that Proposition. It is easy to see that Lemma 1 and its dual show that for all u' in N', the conditions

$$f(u';\ v',\ v) = u \text{ if } p[u'v'] = [uv] \text{ and } u \in N$$

and

$$f(v',\ v;\ u') = u \text{ if } p[v'u'] = [vu] \text{ and } u \in N$$

define functions on N'. The corollary shows that if N is in the center of G these functions induce the projectivity which is the restriction of p to N'.

We return once more to the general situation where G' and G satisfy no special conditions. Let U be a subgroup of G. Suppose that p and $(t', t) \in G \times G$ induce the function $f(.;\ t', t)$ from U' into U and that p and $(v', v) \in G' \times G$ induce the function $f(.;\ v', v)$ from U' into U. It may happen (and frequently does) that $f(.;\ t', t) = f(.;\ v', v)$.

DEFINITION. Let Θ be a subset of $G' \times G$. Suppose that every pair $(t', t) \in \Theta$ (together with p) induces a function $f(.;\ t', t)$ ($f(t', t;\ .)$) from U' into U and that all of the maps $f(.;\ t', t)$ ($f(t', t;\ .)$) with $(t', t) \in \Theta$ are the same function f from U' into U. Then f is said to be the function from U' into U *induced by the projectivity p and the set Θ* with Θ on the right (left).

We write $f = f(.;\ \Theta)$ (or $f = f(\Theta;\ .)$) if and only if f is induced by p and the set Θ with Θ on the right (left).

4. An Extension of a Projectively Induced Function.

Throughout this section let p be a projectivity from some group G' onto a group G, let $\bar{G}$ be the quotient of G with respect to a normal subgroup N, and let φ be the canonical homomorphism from G onto $\bar{G}=G/N$. In studying subgroups of the three groups G', G, and $\bar{G}$, we shall use the notation, H', H, and $\bar{H}$ for subgroups of G', G, and $\bar{G}$, respectively, which are related by $pH'=H$ and $\varphi H=\bar{H}$.

LEMMA 2. Let N be a normal subgroup of G, V a non-trivial cyclic subgroup of G such that $V\cap N=1$, and U a non-trivial cyclic subgroup of G such that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$ in $\bar{G}$. Let u' and v' be generators of U' and V', respectively. Then there is a unique element u in $[U, N]$ and a unique element v in V such that $p[u'v']=[uv]$. Moreover $[v]=V$, $[u, v]=[U, V]$ and $[u, N]=[U, N]$.

PROOF. Let $p[u'v']=W$. Then from the equations

$$[u', u'v']=[u', v']=[u'v', v']$$

which hold in G', it follows that

$$[U, W]=[U, V]=[W, V] \text{ in } G,$$

and

$$[\bar{U}, \bar{W}]=\bar{U}^*\bar{V}=[\bar{W}, \bar{V}] \text{ in } \bar{G}.$$

By Proposition 2 there are unique elements $\bar{u}$, $\bar{v}$, $\bar{w}$, in $\bar{G}$ such that $\bar{U}=[\bar{u}]$, $\bar{V}=[\bar{v}]$, $\bar{W}=[\bar{w}]$, and $\bar{w}=\bar{u}\bar{v}$. Since $V\cap N=1$, φ maps V isomorphically onto $\bar{V}$. It is also true that φ maps W isomorphically onto $\bar{W}$, since $\bar{W}$ is an infinite cyclic group. Now take elements $w\in W$ and $v\in V$, so that $\varphi(w)=\bar{w}$ and $\varphi(v)=\bar{v}$.

Let $u=wv^{-1}$. Then $\varphi(u)=\bar{w}\bar{v}^{-1}=\bar{u}$, and u is an element of $[U, N]$. Obviously $w=uv$ and $W=[uv]$. It is also easy to see that there are no elements $u_0\in[U, N]$ and $v_0\in V$ such that $w^{-1}=u_0v_0$. For otherwise $[\bar{U}, \bar{V}]$ is not a free product without amalgamation. Thus w is the only generator of W which can be written as a product of an element in $[U, N]$ and an element in V. From $V\cap[U, N]=1$ it follows that u and v are the only elements in $[U, N]$ and V, respectively, such that

$w=uv$. Therefore u and v are the unique elements of $[U, N]$ and V, respectively, such that $p[u'v']=[uv]$.

Since φ maps V isomorphically onto $\bar{V}$ and since $\bar{v}=\varphi(v)$ generates $\bar{V}$, it is easy to see that v generates V. That is $[v]=V$. Thus $[U, V]=[W, V]=[uv, v]=[u, v]$. Now let u_0 be a generator of U and let $\bar{u}_0=\varphi(u_0)$ be the corresponding generator of $\bar{U}$. Since $\bar{u}$ is also a generator of $\bar{U}$, there is an integer k such that

$$\bar{u}_0=\bar{u}^k.$$

Thus there is an element n in N such that $u_0=u^k n$.
Hence

$$[U, N]=[u_0, N]\leq[u, N].$$

As we have already seen u is an element of $[U, N]$ so that

$$[u, N]\leq[U, N].$$

Thus

$$[u, N]=[U, N].$$

Corollary. If in Lemma 2 it is also assumed that N is in the center of G, then also $[u]=U$.

Proof. Let u_0 be a generator of U. From Lemma 2 we have that

$$[u_0, v]=[u, v] \text{ and } [u_0, N]=[u, N].$$

Thus

$$u_0=u^k n$$

where k is some integer and n is an element of $[u, v]\cap N$. Since N is in the center of G,

$$[u, v]\cap N=[u]\cap N.$$

Therefore

$$u_0\in[u].$$

In a similar way it can be shown that $u\in[u_0]$. Hence $[u]=U$.

As in the case of Lemma 1 there is a dual result to Lemma 2 which asserts that under the conditions of Lemma 2 there are unique u and v (not necessarily the same elements as in Lemma 2) in $[U, N]$ and V, respectively, such that

$$p[v'u'] = [vu].$$

As in Lemma 2 these elements u and v also satisfy the condition $[v] = V$, $[u, v] = [U, V]$, and $[u, N] = [U, N]$.

If in Lemma 2 we had also assumed that V had infinite order, then the pair (v', v) determined in the Lemma and the projectivity p would induce a function $f(.; v', v)$ from N' into N. We would like to extend this function from N' to all of $[U', N']$. More specifically we would like to have our extended function be the relation f defined by $f(u') = u$ if and only if $p[u'v'] = [uv]$ for all u' in $[U', N']$. However, there may be a u' in $U' - N'$ such that

$$p[u'v'] = [uv^{-1}]$$

for some u in $[U, N]$. In that case we have by Lemma 2 that there is no u in $[U, N]$ such that $p[u'v'] = [uv]$, and the relation f is not a function from $[U', N']$ into $[U, N]$. In the remainder of this section we show how to pick v' and v so that this difficulty can be avoided.

LEMMA 3. (Sadovsky).

a) If u' and v' are non-trivial elements of G' such that $p[u', v'] = p[u']^{*}p[v']$, then there is a unique element u in $p[u']$ and a unique element v in $p[v']$ such that $p[u'v'] = [uv]$. The elements u and v generate $p[u']$ and $p[v']$, respectively.

b) If in part *a*) it is also assumed that $p[u']$ and $p[v']$ are infinite cyclic or equivalently that $p[u', v']$ is a non-cyclic free group, then $p[(u')^{m}(v')^{n}] = [u^{m}v^{n}]$ for all $m, n = \pm 1$.

PROOF.

a) In Lemma 2, let $N = 1$. Given u' and v', take $U = p[u']$ and $V = p[v']$. Then the hypotheses of the lemma are fulfilled, and the conclusion follows.

b) Let $U=[u]$, $V=[v]$, and $W=p[u'(v')^{-1}]$. By part *a*) W must be one of the subgroups $[uv]$, $[u^{-1}v]$, $[uv^{-1}]$, $[u^{-1}v^{-1}]$. But $V\cap[W, uv]=p([v']\cap[u'(v')^{-1}, u'v'])\neq 1$, while

$$1=V\cap[uv, uv]=V\cap[u^{-1}v, uv]=V\cap[u^{-1}v^{-1}, uv].$$

Hence $W=[uv^{-1}]$. The other cases, i.e. $m=-1$, $n=+1$ and m, $n=-1$, can be verified in a similar way.

In Lemma 4 we use two facts from group theory without proof. First if $G=[u_1, u_2]^*[t]^*[v]$ where u_1, u_2, t, v are non-trivial elements of G with u_1, u_2 not necessarily distinct, then $H=[u_1t, u_2v]$ is a non-cyclic free subgroup of G. Second let $G=[a, b]$ and let $\bar{G}=[\bar{a}, \bar{b}]$ be the image of G under some homomorphism φ where $\bar{a}=\varphi(a)$ and $\bar{b}=\varphi(b)$. If $\bar{G}$ is a non-cyclic free group then G is also a non-cyclic free group.

Lemma 4. Let N be a normal subgroup of G, V a non-trivial subgroup of G such that $V\cap N=1$, and U a non-trivial subgroup of G such that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$. Let t' and v' be two elements of G' such that $p[t', v']$ is a non-cyclic free subgroup of V. Let u' be an element of $[U', N']$. Then there are unique u in $[U, N]$, unique t in $p[t']$, and unique v in $p[v']$ such that

$$p[u't']=[ut],\ p[u'v']=[uv],\ \text{and}\ p[t'v']=[tv].$$

Proof. First let u' be an element of the complement of N' in $[U', N']$. By Lemma 2 we know there are unique u_1 in $[p[u'], N]$ and t in $p[t']$ such that $p[u't']=[u_1t]$. Similarly there are unique u_2 in $[p[u'], N]$, and v in $p[v']$ such that $p[u'v']=[u_2v]$. We shall show that $u_1=u_2$ and that $p[t'v']=[tv]$.

The group $[t, v]$ is a free subgroup of V since $p[t', v']=[t, v]$. Because $V\cap N=1$ the canonical homomorphism φ from G onto $\bar{G}$ maps $[t, v]$ isomorphically onto $[\bar{t}, \bar{v}]$, where $\bar{t}=\varphi t$, $\bar{v}=\varphi v$. Hence $[\bar{t}, \bar{v}]$ is a free subgroup of $\bar{V}$. From this fact and from the fact that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$ in $\bar{G}$ it follows that $[\bar{u}_1\bar{t}, \bar{u}_2\bar{v}]$ is a free subgroup of $\bar{G}$, where $\bar{u}_1=\varphi u_1$ and $\bar{u}_2=\varphi u_2$. But $[\bar{u}_1\bar{t}, \bar{u}_2\bar{v}]$ is the image of $[u_1t, u_2v]$ under the canonical homomorphism φ. Therefore $[u_1t, u_2v]$ is a free subgroup of G.

We are now able to apply Lemma 3 b to the cyclic group generated by $(u't')^{-1}u'v'=(t')^{-1}v'$. Thus there are integers k, $l=\pm 1$ such that

$$p[(u't')^{-1}(u'v')]=[(t^{-1}u_1^{-1})^k(u_2v)^l],$$

and integers m, $n=\pm 1$ such that

$$p[(t')^{-1}v']=[(t^{-1})^m v^n].$$

But then $(t^{-1}u_1^{-1})^k(u_2v)^l$ and $(t^{-1})^m v^n$ must both generate the same infinite cyclic group in G. Therefore there is an integer $j=\pm 1$ such that

$$((t^{-1}u_1^{-1})^k(u_2v)^l)^j=(t^{-1})^m v^n.$$

Mapping both sides of this equation by φ into $\bar{G}$ we have

$$((\bar{t}^{-1}\bar{u}_1^{-1})^k(\bar{u}_2\bar{v})^l)^j=(\bar{t}^{-1})^m\bar{v}^n.$$

By Proposition 3

$$j,\ k,\ l,\ m,\ n=+1 \text{ and } \bar{u}_1=\bar{u}_2\,.$$

Therefore $t^{-1}u_1^{-1}u_2v=t^{-1}v$ in G, and consequently $u_1=u_2$ in G. Also since m, $n=+1$, $p[(t')^{-1}v']=[t^{-1}v]$, and by Lemma 3 b $p[t'v']=[tv]$.

Now let u' be an element of N'. By Lemma 3 b pick elements t and v in G so that $p[t']=[t]$, $p[v']=[v]$, and $p[t'v']=[tv]$. By Lemma 1 there are unique u_1 and u_2 in N such that

$$p[u't']=[u_1t] \text{ and } p[u'v']=[u_2v].$$

We shall show that $u_1=u_2$.

Again $[t, v]$ and $[\bar{t}, \bar{v}]$ are non-cyclic subgroups of V and $\bar{V}$, respectively. Since $\varphi(u_1t)=\bar{t}$ and $\varphi(u_2v)=\bar{v}$, we have that $[u_1t, u_2v]$ is a non-cyclic free subgroup of G. As in the previous case we apply Lemma 3 b to the cyclic subgroup generated by $(u't')^{-1}u'v'=(t')^{-1}v'$. Thus there are integers k, $l=\pm 1$ such that

$$p[(u't')^{-1}(u'v')]=[(t^{-1}u_1^{-1})^k(u_2v)^l].$$

Since t and v were already selected so that $p[t'v']=[tv]$, we have by Lemma 3 b that

$$p[(t')^{-1}v']=[t^{-1}v].$$

Thus there is an integer $j=\pm 1$ such that

$$((t^{-1}u_1^{-1})^k(u_2v)^l)^j=t^{-1}v$$

in G. Mapping both sides of this equation by φ into $\bar{G}$ we have

$$(\bar{t}^{-k}\bar{v}^l)^j=\bar{t}^{-1}\bar{v}.$$

Since $\bar{t}$ and $\bar{v}$ are a system of free generators for a free group, we have that

$$j,\ k,\ l=+1.$$

Hence $t^{-1}u_1^{-1}u_2v=t^{-1}v$ and $u_1=u_2$.

In addition to being unique in $[p[u'], N]$ the element u is unique in all of $[U, N]$. That is, there is no element u_1 in $[U, N]$ such that $u_1\notin[p[u'], N]$ and $[uv]=[u_1v]$. For if there were the elements $\bar{u}_1=\varphi(u_1)$, $\bar{u}=\varphi(u)$ and $\bar{v}=\varphi(v)$ would have to satisfy the conditions $\bar{u}_1\neq\bar{u}$ and $[\bar{u}\bar{v}]=[\bar{u}_1\bar{v}]$ in $\bar{U}^*\bar{V}$. But this is clearly impossible in a free product without amalgamation.

Thus we see that for each element u' in $[U', N']$ there is a unique element u in $[U, N]$ such that $p[u'v']=[uv]$. Therefore, for all u' in $[U', N']$, the condition

$$f(u';\ v',\ v)=u \text{ if } p[u'v']=[uv] \text{ and } u\in[U,\ N]$$

defines a function $f(.;\ v',\ v)$ from $[U', N']$ into $[U, N]$. In a similar way it can be shown that there exist analogous functions $f(.;\ t',\ t)$, $f(v',\ v;\ .)$ and $f(t',\ t;\ .)$ from $[U', N']$ into $[U, N]$. From Lemma 4 and its « dual » it follows that

$$f(.;\ t',\ t)=f(.;\ v',\ v)$$

and that

$$f(t',\ t;\ .)=f(v',\ v;\ .).$$

In the next section we show that all four functions are equal. It is also shown there that the function induced by the projectively p and the pairs (t', t) and (v', v) is an isomorphism from $[U', N']$ onto $[U, N]$.

5. Isomorphisms Induced by a Projectivity.

Throughout this section let p be a projectivity from some group G' onto a group G, N a normal subgroup of G, and φ the canonical homomorphism from G onto $\bar{G}=G/N$. As in Section 4 if H is a subgroup of G we write H' for the subgroup of G' such that $pH'=H$ and $\bar{H}$ for the subgroup of $\bar{G}$ such that $\varphi H=\bar{H}$.

LEMMA 5. Suppose there is a non-cyclic free subgroup V of G such that $V \cap N=1$. Then N' is a normal subgroup of $[N', V']$.

PROOF. Let u' and v' be elements of N' and V', respectively. We wish to show that $v'^{-1}u'v'$ is an element of N'. Let t' be one of the numerous elements of V' such that $p[t', v']$ is a non-cyclic free subgroup of V. For such t', u', and v' we shall show that the subgroup $H'=[t'u'v', t'v']$ meets $[v', N']$ in N', that is

$$[t'u'v', t'v'] \cap [v', N'] \leq N'.$$

But $(v')^{-1}u'v'=(t'v')^{-1}(t'u'v')$ is an element of the intersection on the left, and consequently an element of N'.

By Lemma 3 there are unique elements t and v in G such that $p[t']=[t]$, $p[v']=[v]$ and $p[t'v']=[tv]$. By Lemma 1 there is a unique u in N such that $p[t'u']=[tu]$. We let $H=pH'$, $\bar{H}=\varphi H$, $\bar{t}=\varphi t$, and $\bar{v}=\varphi v$. We first show that $\bar{H} \cap [\bar{t}]$ or $\bar{H} \cap [\bar{v}]$ must be the identity subgroup of $\bar{G}$. Then we show that $\bar{H}=[\bar{t}\bar{v}]$ and that

$$\bar{H} \cap [\bar{t}]=1=\bar{H} \cap [\bar{v}].$$

From this we conclude that both $H' \cap [v', N']$ and $H' \cap [t', N']$ are contained in N'.

Since $[tu, v]$ and $[t, v]$ have the same image under φ in $\bar{G}$, it follows that $[tu, v]$ is a non-cyclic free subgroup of V. By Lemma 3 there are integers $k,\ l=\pm 1$ such that $p[t'u'v']=[(tu)^k v^l]$. Thus $H=[(tu)^k v^l,\ tv]$ and $\bar{H}=[\bar{t}^k \bar{v}^l,\ \overline{tv}]$. The four possibilities for $\bar{H}$ are

$$\bar{H}_1=[\overline{tv},\ \overline{tv}]=[\overline{tv}]$$

$$\bar{H}_2=[\bar{t}(\bar{v})^{-1},\ \overline{tv}]=[\bar{v}^2,\ \overline{tv}]$$

$$\bar{H}_3=[(\bar{t})^{-1}\bar{v},\ \overline{tv}]=[\bar{t}^2,\ \overline{tv}]$$

$$\bar{H}_4=[\overline{vt},\ \overline{tv}].$$

It is easy to see that

$$1=\bar{H}_1\cap[\bar{t}]=\bar{H}_2\cap[\bar{t}]=\bar{H}_3\cap[\bar{v}]=\bar{H}_4\cap[\bar{v}].$$

Hence $\bar{H}\cap[\bar{t}]=1$ or $\bar{H}\cap[\bar{v}]=1$. Consequently $H\cap[t,N]$ or $H\cap[\bar{v},N]$ is contained in N. But this means that $H'\cap[t',\ N']$ or $H'\cap[v',\ N']$ is contained in N'.

Up to this point we have shown that $t'u'(t')^{-1}$ or $(v')^{-1}u'v'$ is an element of N'. It remains to be shown that both elements are in N'. We suppose without loss of generality that $t'u'(t')^{-1}=u'_1$ in N'. Then $t'u'v'=u'_1t'v'$. By Lemma 1 there is a unique element u_1 in N such that

$$pH'=p[u'_1t'v',\ t'v']=[u_1tv,\ tv]=H.$$

Hence $\bar{H}=[\overline{tv}]$ and

$$1=\bar{H}\cap[\bar{v}]=\bar{H}\cap[\bar{t}].$$

But this means that both $H'\cap[t',\ N']$ and $H'\cap[v',\ N']$ are in N'. Hence $(v')^{-1}u'v'$ must be an element of N', for all v' in V'.

Corollary. With the same notation as in the proof of Lemma 5 we have that for each element u' in N' there is a unique element u in N such that $p[t'u']=[tu]$, $p[u'v']=[uv]$, and $p[t'u'v']=[tuv]$.

Proof. Let u' be some element of N'. We have already seen that there are unique elements u_1 and u_2 in N such that $p[t'u']=[tu_1]$ and $p[u'v']=[u_2v]$. By Lemmas 5 and 1 there is a unique element u_3 in N such that

$$p[t'u'v']=p[(t'u't'^{-1})t'v']=[(tu_3t^{-1})tv]=[tu_3v].$$

In this corollary we show that $u_1=u_2=u_3$.

We show first that $u_1=u_3$. Because $[tu_1, v]$ is a non-cyclic free subgroup of G, there are by Lemma 3 two integers k, $l=\pm 1$ such that

$$[p(t'u')v']=[(tu_1)^k v^l].$$

Hence there is an integer $j=\pm 1$ such that

$$(tu_1)^k v^l=(tu_3 v)^j.$$

If $\bar{t}=\Phi(t)$ and $\bar{v}=\Phi(v)$ in $\bar{G}$ this equation implies that

$$\bar{t}^k\bar{v}^l=(\bar{t}\bar{v})^j.$$

By the uniqueness of representation of elements in a free group j, k, $l=+1$. Therefore $u_1=u_3$.

A similar argument shows that $u_2=u_3$.

We are now able to prove our first major result.

THEOREM 1. Let p be a projectivity from a group G' onto a group G. Let N be a normal subgroup of the group G and N' the subgroup of G' such that $pN'=N$. Suppose there is a non-cyclic free subgroup V of G such that $V\cap N=1$. Then N' is isomorphic to N.

PROOF. Let t' and v' be in V' such that $p[t', v']$ is a non-cyclic free group and let t and v be elements of V such that $p[t']=[t]$, $p[v']=[v]$, and $p[t'v']=[tv]$. We shall show that the function $f=f(t', t; .)$ from N' into N induced by p and the pair (t', t) with (t', t) on the left is an isomorphism from N' onto N.

Accordingly let u'_1, u'_2 be two elements in N' and let $u'=u'_1u'_2$. If $u=f(u')$, $u_1=f(u'_1)$ and $u_2=f(u'_2)$, we have by Lemma 1, that $p[t'u'_1]=[tu_1]$ and by the Corollary to Lemma 5 that $p[u'_2v']=[u_2v]$, and $p[t'u'v']=[tuv]$. Let $\bar{t}=\varphi(t)$ and $\bar{v}=\varphi(v)$. As before the fact that $[\bar{t}, \bar{v}]$ is a free subgroup of $\bar{G}$ implies that $[tu_1, u_2v]$ is a free subgroup of V. Thus by Lemma 3 there are integers k, $1=\pm 1$ such that $p[t'u'_1u_2v']=[(tu_1)^k(u_2v)^1]$. From this fact it follows that there is an integer j such that in G

$$(tu_1)^k(u_2v)^l=(tuv)^j.$$

Consequently in $\bar{G}$

$$\bar{t}^k \bar{v}^l = (\bar{t}\bar{v})^j.$$

Again by uniqueness of representation in a free group it follows that j, k, $l = +1$ and $u = u_1 u_2$. Thus f is a homomorphism because

$$f(u'_1 u'_2) = u = u_1 u_2 = f(u'_1) f(u'_2).$$

To show that the mapping f is one to one and onto, we show that it has an inverse. Since N' is normal in $[N', V']$ and $N' \cap V'$ is the identity subgroup of G', we have by Lemma 1 that for each element u in N there is a unique element u' in N' such that $p^{-1}[tu] = [t'u']$. It is easy to see that the function g defined by $g(u) = u'$ for each element u in N is the inverse of f.

COROLLARY 1. Here the notation is the same as in Theorem 1 and its proof; in particular $f = f(t', t; .)$ is the function which was just shown to be an isomorphism from N' onto N. Let

$$\Theta = \{(t', t), (v', v), (t'^{-1}, t^{-1}), (v'^{-1}, v^{-1})\}.$$

Then

$$f(\Theta; .) = f = f(.; \Theta).$$

That is, f is the function induced by p and the set Θ with Θ on the left or with Θ on the right.

PROOF. By Lemma 1 the projectivity p and any pair $(x', x) \in \Theta$ induce the functions $f(x', x; ,)$ and $f(.; x', x)$ from N' into N. By the remarks following Lemma 4 we have that

$$f(t', t; .) = f(v', v; .)$$
$$f(.; t', t) = f(.; v', v).$$

However by the Corollary to Lemma 5

$$f(t', t; .) = f(.; v', v).$$

Hence (t', t) and (v', v) induce the function f from N' into N on the left and on the right. The fact that (t'^{-1}, t^{-1}) and (v'^{-1}, v^{-1}) induce the function f follows from the fact that $p[t'^m v'^n]=[t^m v^n]$ for all $m, n=\pm 1$.

COROLLARY 2. Here again the notation is the same as in Theorem 1 and its proof. Suppose now, however, that N is in the center of G. Then the isomorphism $f=f(t',t; .)$ induces p from N' onto N. That is, for every subgroup $K'\leq N'$

$$pK'=fK'.$$

PROOF. By the Corollary to Lemma 1.

We are now ready to fulfill the promises made at the end of Section 4.

THEOREM 2. Let p be a projectivity from G' onto G. Let N be a normal subgroup of G and U a subgroup of G. Suppose that there is a non-cyclic free subgroup V of G such that $V\cap N=1$ and $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$ in $\bar{G}=G/N$. Then $[U', N']$ is isomorphic to $[U, N]$.

PROOF. Let t' and v' be two elements in G' such that $p[t', v']$ is a non-cyclic free subgroup of V. Pick t and v in G so that $p[t']=[t]$, $p[v']=[v]$, and $p[t'v']=[tv]$. Let $f(t', t; .)$ be the function from $[U', N']$ into $[U, N]$ induced by p and (t', t) with (t', t) on the left, and let $f(.; v', v)$ be the function from $[U', N']$ into $[U, N]$ induced by p and (v', v) with (v', v) on the right. We first show that

$$f(t', t; .)=f(.; v', v)$$

by showing that for each u' in $[U', N']$ there is a unique u in N such that $p[t'u']=[tu]$, $p[u'v']=[uv]$, and $p[t'u'v']=[tuv]$. By the Corollary to Lemma 5 this result is true for all u' in N'.

Thus let u' be an element of the complement of N' in $[U', N']$, and let u_1 and u_2 be the unique elements of $[U, N]$ such that $p[t'u']=$ $=[tu_1]$ and $p[u'v']=[u_2v]$. Then by Lemma 3 *b* there are unique integers $k, l, m, n=\pm 1$ such that

$$p[t'(u'v')]=[t^k(u_2v)^l] \text{ and } p[(t'u')v']=[(tu_1)^m v^n],$$

since $[t, u_2v]$ and $[tu_1, v]$ are non-cyclic free subgroups of G. But then there is an integer $j=\pm 1$ such that

$$(t^k(u_2v)^l)^j=(tu_1)^m v^n.$$

Mapping into $\bar{G}$ by φ and applying Proposition 3 we have that

$$j,\ k,\ l,\ m,\ n=+1, \text{ and } u_1=u_2.$$

Letting $u=u_1=u_2$, we have the preliminary result.

We can now show that the function $f=f(t', t;\ .)=f(.;\ v', v)$ is a homomorphism. Let u'_1 and u'_2 be arbitrary elements in U'. If u'_1 and u'_2 are both elements of N', then $f(u'_1u'_2)=f(u'_1)f(u'_2)$ by Theorem 1. Thus we may suppose that u'_1 or u'_2 is not in N'. Let $u'=u'_1u'_2$ and $u=f(u'_1u'_2)$. Then as we have just seen

$$p[t'u'_1u'_2v']=[tuv].$$

By Lemma 3 b there are integers $k,\ l=\pm 1$ such that

$$p[(t'u'_1)(u'_2v')]=[(tu_1)^k(u_2v)^l].$$

By mapping into $\bar{G}$ and applying Proposition 3 again we have that

$$u=u_1u_2.$$

The homomorphism f is also one to one and onto. By Lemma 2 if u' is in the complement of N' in $[U', N']$, then $[p[u'], v]=[f(u'), v]$. Since $[p[u'], v]\neq[v]$, $f(u')$ cannot be the identity element in G. Since the restriction of f to N' is an isomorphism, the kernel of f must be the identity subgroup of G'. Hence f must be one to one. To show that f maps $[U', N']$ onto $[U, N]$ it suffices to show that for any $u\in[U, N]-N$ there is a preimage for u under f. Let u' be an element of $[U', N']$ such that $p[u']=[u]$, and let $f(u')=u_0$. By Lemma 2

$$[u, N]=[u_0, N].$$

Hence there is an integer k and an element n in N such that $u=u_0{}^k n$. By Theorem 1 there is an n' in N' such that $f(n')=n$. Thus $(u')^k n'$ is the preimage of u under f.

COROLLARY 1. Here the notation is the same as in Theorem 2 and its proof; in particular $f=f(t', t; .)$ is the function which was just shown to be an isomorphism from $[U', N']$ onto $[U, N]$. Let

$$\Theta=\{(t', t), (v', v), (t'^{-1}, t^{-1}), (v'^{-1}, v^{-1})\}.$$

Then

$$f(\Theta; .)=f=f(.; \Theta).$$

PROOF. In Theorem 2 it was shown that

$$f(t', t; .)=f(.; v', v).$$

From the remarks following Lemma 4 we have that

$$f(v', v; .)=f(t', t; .)=f(.; v', v)=f(.; t', t).$$

Since $p[t'^m v'^n]=[t^m v^n]$ for all $m, n=\pm 1$, the pairs (t'^{-1}, t^{-1}) and (v'^{-1}, v^{-1}) also induce the isomorphism f from $[U', N']$ onto $[U, N]$.

COROLLARY 2. Here again the notation is the same as in Theorem 2 and its proof. Suppose, however, that N is in the center of G. Then the isomorphism $f=f(t', t; .)$ induces p from $[U', N']$ onto $[U, N]$. That is, for every subgroup $K' \leq [U', N']$

$$pK'=fK'.$$

PROOF. By the Corollary to Lemma 2.

6. Free Subgroups of $G=A_C^* B$

Throughout this section let $G=A_C^* B$ be a free product of its subgroups A and B with amalgamated subgroup $C=A \cap B$ normal in G. Let φ be the canonical homomorphism from G onto the quotient group $G/C=\bar{G}$. For any subgroup H of G we write $\bar{H}$ for the image of H under φ, i.e. $\bar{H}=\varphi H$. It is easy to see that $\bar{G}$ is a free product, without amalgamation, of its subgroups $\bar{A}$ and $\bar{B}$. That is $\bar{G}=\bar{A}^* \bar{B}$. It is also true that any expression

$$g=g_1 g_2 \ \dots \ g_n$$

for an element g as a word in G will be mapped by φ to

$$\varphi(g)=\varphi(g_1)\varphi(g_2)\ \dots\ \varphi(g_n)$$

which is the expression for $\varphi(g)$ as a word in $\bar{G}$. Hence the length of an element g in G is the length of $\varphi(g)$ in $\bar{G}$. Also if $g \in A : A$ $(B : B)$ in G, then $\varphi(g) \in \bar{A} : \bar{A}$ $(\bar{B} : \bar{B})$ in $\bar{G}$.

Now let U be a subgroup of G which except for elements of C is contained in $A : A$. Also let V be a subgroup of G which except for elements of C is contained in $B : B$. Then the non-trivial elements of $\bar{U}$ and $\bar{V}$ are contained in $\bar{A} : \bar{A}$ and $\bar{B} : \bar{B}$, respectively. It is an elementary consequence of the fact that $\bar{G}$ is a free product, without amalgamation, of its subgroups $\bar{A}$ and $\bar{B}$ and of the definition of free product, without amalgamation, that $[\bar{U}, \bar{V}]$ is also a free product, without amalgamation, of $\bar{U}$ and $\bar{V}$. That is, $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$.

We use the following definition to show similar results.

DEFINITION. Let l and r be elements of G. We write $l \mid r$ for the set of all elements u in G which have a factorization $u=lxr$ in G such that $|u|=|l|+|x|+|r|$. In addition we say that a set U is a *factor set* if and only if there are elements l and r in G such that $U=l \mid r$.

Factor sets, of course, are infinite sets. For example, if the elements l and r begin and end with elements of A, then $l \mid r=l(B : B)r$. Here the set on the right is the set of all products lur with $u \in B : B$. In a similar way other choices of l and r yield other sets X such that $l \mid r=lXr$. If S is a set of elements in G, we write S^{-1} for the set consisting of the inverses of all elements in S. It is easy to show that $(l \mid r)^{-1}=r^{-1} \mid l^{-1}$.

Consider for a moment the following basic example. Let l and r be two elements of G such that $|l|=|r|$ and rl is not an element of C. It is obvious that l and r are not elements of C. As we saw in Section 1, rl and r begin with elements from the same group A or B while rl and l end with elements from the same group A or B. If v is an element of the set $l \mid r$ and h is the element of G such that $v=lhr$, then

$$v^2=(l)h(rl)h(r); \text{ and}$$
$$|v^2|=|l|+|h|+|rl|+|h|+|r|,$$

since the length function is additive over the factorization of ν^2 given by the parentheses. In a similar way it can be shown that for any positive integer k,

$$|\nu^k|=|l|+|r|+k|h|+(k-1)|rl|.$$

Hence any element ν in $l|r$ has infinite order and $[\nu]\cap C=1$.

The next proposition is a simple, but important, extension of the idea in the above example. We remind the reader that if H is a subgroup of G, then $\bar{H}=\varphi H$.

Proposition 4. Let l, r, k and s be elements of G. Let $L=l|r$ and $K=k|s$. Let U and V be subgroups of G which are contained in $L\cup L^{-1}\cup C$ and $K\cup K^{-1}\cup C$, respectively. If

1) $|l|=|r|=|k|=|s|$, and

2) rk, sl, rs^{-1}, and $l^{-1}k$ are not elements of C, then

$$[\bar{U}, \bar{V}]=\bar{U}^*\bar{V} \text{ in } \bar{G}.$$

Proof. We need only show that any product

$$g=g_1 \ \dots \ g_n$$

with the elements g_i alternately in $U-C$ and $V-C$ has length strictly greater than one in G. From the fact that $U-C$ and $V-C$ are contained in $L\cup L^{-1}$ and $K\cup K^{-1}$, respectively, it follows that for any i

$$g_i=l_ih_ir_i$$

where l_i is one of the elements l, k, r^{-1}, s^{-1}; r_i is one of the elements r, s, l^{-1}, k^{-1}; and $|g_i|=|l_i|+|h_i|+|r_i|$.
Therefore

$$g=(l_1)h_1(r_1l_2)h_2 \dots h_{n-1}(r_{n-1}l_n)h_n(r_n).$$

Since the elements g_i and g_{i+1} are alternately in $U-C$ and $V-C$ and since conditions 1) and 2) hold, it follows that (r_il_{i+1}) and r_i begin with elements from the same group A or B while (r_il_i) and l_i end with elements from the same group A or B. Therefore the length function is additive over the factorization of g given by the parentheses. Hence the length of g must be greater than 1 in G.

Another way of saying that the product gh of two elements g and h in G is not in C is to say that the cosets $g^{-1}C$ and hC are distinct. Because of this fact the following notation can more economically express lengthy conditions similar to 1) and 2) in Proposition 4.

DEFINITION. Let $g_1, \ldots, g_n$ be n elements in G. We write $\wedge\{g_1, \ldots, g_n\}$ if and only if 1) all the g_i have the same length and 2) all the cosets g_iC are distinct.

Our next proposition provides us with a tool for constructing free groups in $G=A{*}_C B$.

PROPOSITION 5. Let $l_1, \ldots, l_n$ and $r_1, \ldots, r_n$ be elements of G such that

$$\wedge\{l_1, r_1^{-1}, l_2, r_2^{-1}, \ldots, l_n, r_n^{-1}\}.$$

Then any set $v_1, \ldots, v_n$ with $v_i \in l_i \mid r_i$ is a system of free generators of a free group V such that $V \cap C = 1$.

PROOF. As we saw earlier each v_i has infinite order, since by the given condition $|l_i| = |r_i|$ and $r_i l_i$ is not an element of C. We now show that V is the free product, without amalgamation, of the cyclic subgroups generated by the elements $v_1, \ldots, v_n$. Let

$$g = g_1 \quad \ldots \quad g_n$$

be any product such that g_j is a non-trivial element of one of the subgroups $[v_j]$ and no two adjacent elements g_j and g_{j+1} are from the same group. It suffices to show that g has length greater than one in G. Now if $g_j \in [v_i]$, then $g_j = k_j h_j s_j$ where k_j is one of the elements l_i or r_i^{-1}, s_j one of the elements r_i or l_i^{-1}, and $|g_j| = |k_j| + |h_j| + |s_j|$. We have that

$$g = (k_1)h_1(s_1 k_2)h_2 \quad \ldots \quad h_{n-1}(s_{n-1}k_n)h_n(s_n).$$

From the fact that no two adjacent elements g_j and g_{j+1} are from the same group and the condition $\wedge\{l_1, r_1^{-1}, \ldots, l_n, r_n^{-1}\}$ it follows that

$s_jk_{j+1} \notin C$ and that $|s_j| = |k_{j+1}|$. We now have that s_jk_{j+1} and s_j *begin* with elements from the same group, and that s_jk_{j+1} and k_{j+1} end with elements from the same group. Therefore the length function is additive over the factorization of g given by the parentheses. Consequently the length of g must be greater than one in G, as was to be shown.

Again let $G = A \ast_C B$ with C normal in G and let $\bar{A}$ and $\bar{B}$ be the images of A and B under the canonical homomorphism φ from G onto G/C. When C has index two in A and B, then there is only one non-trivial element in $\bar{A}$ and $\bar{B}$ and there are exactly two words of any given length in $\bar{G}$. Apart from this case, i.e. when $[A : C] > 2$ or $[B : C] > 2$, there are least two words of length two in $\bar{A} : \bar{B}$ and hence at least 2^n of length $2n$. Thus for any positive integer n there are n-distinct elements in $\bar{G}$ which have the same length and which begin with elements of $\bar{A}$. Of course there is a similar set of n elements beginning with elements of $\bar{B}$. Hence there are n elements $g_1, \ldots, g_n$ in G beginning with elements of A (or B) such that $\wedge\{g_1, \ldots, g_n\}$. Proposition 6 is a simple consequence of this fact.

PROPOSITION 6. Suppose $[A : C] > 2$ or $[B : C] > 2$. Then the following are true.

a) There are non-cyclic free subgroups S and T of G such that $S \cap C = 1 = T \cap C$, $[\bar{A}, \bar{S}] = \bar{A}^* \bar{S}$, and $[B, \bar{T}] = \bar{B}^* \bar{T}$.

b) For any cyclic subgroup U of G which is not contained in C, there is a non-cyclic free subgroup V of G such that $V \cap C = 1$ and $[\bar{U}, \bar{V}] = \bar{U}^* \bar{V}$.

PROOF.

a) Let b be any element of $B - C$. Let l, r^{-1}, k, and s^{-1} be any set of elements in G beginning with elements of A such that $\wedge\{l, r^{-1}, k, s^{-1}\}$. As we have seen such sets do exist in G, when $[A : C] > 2$ or $[B : C] > 2$. The crucial fact here is that $\wedge\{bl, b^{-1}r^{-1}, bk, b^{-1}s^{-1}\}$. Then by Proposition 5 any subgroup $S = [t, v]$ where $t \in bl \mid rb$ and $v \in bk \mid sb$ is a non-cyclic free subgroup of G which meets C in the identity subgroup. That is, $S \cap C = 1$. It is easy to see that S is contained in $(B : B) \cup C$. Hence $[\bar{A}, \bar{S}] = \bar{A}^* \bar{S}$.

The proof of the existence of the subgroup T mentioned in the statement of the proposition is the dual of the result just shown.

b) The cyclic subgroup U must be contained in one and only one of the following sets

$$(A:A)\cup C,\ (B:B)\cup C,\ \text{and}\ (A:B)\cup(B:A)\cup 1$$

according to which sets $A:A$, $B:B$ or $A:B$ contain a generator for U. If U is in $(A:A)\cup C$, then the subgroup S constructed in part *a*) does the job because it is also true that $[\bar{U}, \bar{S}]=\bar{U}^*\bar{S}$. Similarly if U is contained in $(B:B)\cup C$, then the subgroup T from *a*) is satisfactory.

It remains to show the result for the case in which U has a generator u in $A:B$. Suppose that

$$u=a_1\ \dots\ b_n$$

is some representation for u as a word in G. By hypothesis $[A:C]>2$ or $[B:C]>2$. Suppose without loss of generality that $[B:C]>2$. Then there are two elements b and d in $B-C$ such that b_nb and b_nd^{-1} are not elements of C. Since da_1 and $a_1^{-1}b$ are also not in C and since a_1, b_n, b and d all have length one, it is possible to apply Proposition 4 to the sets $L=a_1\,|\,b_1$ and $K=b\,|\,d$ and to the group U which is contained in $L\cup L^{-1}\cup C$. That is, if V is any subgroup contained in $(b\,|\,d)\cup(d^{-1}\,|\,b^{-1})\cup C$, then $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$. If l, r, k, and s are as in part *a*) and if $t\in bl\,|\,rd$ and $v\in bk\,|\,sd$, then $V=[t, v]$ is a non-cyclic free subgroup of G (contained in $(b\,|\,d)\cup(b\,|\,d)^{-1}\cup C$) such that $V\cap C=1$ and $[\bar{U}, \bar{V}]=[\bar{U}^*V]$.

It is not hard to see that we have proved much more than we started to prove. The following corollary sums up what has been shown. Again suppose $[A:C]>2$ or $[B:C]>2$.

COROLLARY. Let U be a subgroup of G such that either *a*) U is contained in $(A:A)\cup C$ or $(B:B)\cup C$, or *b*) U is a cyclic subgroup of G with a generator in $A:B$. Then there are sets $L=l\,|\,r$ and $K=k\,|\,s$ with $\wedge\{l, r^{-1}, k, s^{-1}\}$, which have the following property. Any subgroup $V=[t, v]$ generated by $t\in L\cup L^{-1}$ and $v\in K\cup K^{-1}$ satisfies the condition $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$.

Our next proposition and its corollary embody the basic idea underlying our main theorem.

PROPOSITION 7. Suppose $[A : C]>2$ or $[B : C]>2$. Let $g_1, \dots, g_n$ be any n elements (not necessarily distinct) in the complement of C in G. Then there are n elements $h_1, \dots, h_n$ such that

1) $|g_ih_i|=|g_i|+|h_i|$ for all i, and

2) $\wedge\{g_1h_1, \dots, g_nh_n\}$.

PROOF. First suppose that all of the g_i begin with elements of A. Take $k_1, \dots, k_n$ so that all of the numbers $|g_ik_i|=|g_i|+|k_i|$ are equal. Then either all of the elements g_ik_i end with elements of A or all of the elements g_ik_i end with elements of B. Suppose without loss of generality that all of the g_ik_i end with elements of B. Let $r_1, \dots, r_n$ be a set of n elements in G such that

a) all of the r_i begin with elements of A, and

b) $\wedge\{r_1, \dots, r_n\}$.

We saw that such sets do exist in the remarks preceding Proposition 6. Now take $h_i=k_ir_i$. Since k_i ends with an element of B and r_i begins with an element of A,

$$|h_i|=|k_i|+|r_i|.$$

Consequently

$$|g_ih_i|=|g_ik_ir_i|=|g_i|+|k_i|+|r_i|=|g_i|+|h_i|.$$

Since the factors $\varphi(r_i)$ of the elements $\varphi(g_ik_ir_i)$ are all distinct in $\bar{G}$, the elements $\varphi(g_ik_ir_i)$ are themselves all distinct in G. Hence

$$\wedge\{g_1k_1r_1, \dots, g_nk_nr_n\}.$$

Or

$$\wedge\{g_1h_1, \dots, g_nh_n\}.$$

The proof is the very same if all of the elements g_i begin with elements of B. In the general case suppose that the first m of the g_i begin with elements of A and the rest of the g_i begin with elements of B. First take $k_1, \ldots, k_m$ and $k_{m+1}, \ldots, k_n$ so that

$$\wedge\{g_1k_1, \ldots, g_mk_n\}, \quad \wedge\{g_{m+1}k_{m+1}, \ldots, g_nk_n\},$$

and

$$|g_ik_i| = |g_i| + |k_i|$$

for all i. Now pick r_i so that all of the numbers $|g_ik_ir_i| = |g_ik_i| + |r_i|$ are equal. Let $h_i = k_ir_i$. Then

$$\wedge\{g_1h_1, \ldots, g_mh_m\}$$

and

$$\wedge\{g_{m+1}h_{m+1}, \ldots, g_nh_n\}.$$

However, since all of the elements g_ih_i have the same length and since $g_1h_1, \ldots, g_mh_m$ begin with elements of A while $g_{m+1}h_{m+1}, \ldots, g_nh_n$ begin with elements of B, we have

$$\wedge\{g_1h_1, \ldots, g_mh_m, \ldots, g_nh_n\}.$$

COROLLARY. Suppose $[A : C] > 2$ or $[B : C] > 2$. Let $l_1 \mid r_1, \ldots, l_n \mid r_n$ be n factor sets (not necessarily distinct). Then there are elements $k_1, \ldots, k_n$ and $s_1, \ldots, s_n$ such that

1) $l_ik_i \mid s_ir_i \subseteq l_i \mid r_i$ for all i, and
2) $\wedge\{l_1k_1,\ r_1^{-1}s_1^{-1}, \ldots, l_nk_n,\ r_n^{-1}s_n^{-1}\}$.

PROOF. Consider the collection $g_1 = l_1$, $g_2 = r_1^{-1}$, $g_3 = l_2, \ldots, g_{2n-1} = l_n$, $g_{2n} = r_n^{-1}$. By Proposition 7 there are elements $h_1, \ldots, h_{2n}$ such that $|g_ih_i| = |g_i| + |h_i|$ and $\wedge\{g_1h_1, \ldots, g_{2n}h_{2n}\}$. Take $k_j = h_i$ where $j = (i+1)/2$ when i is odd and take $s_j = h_i$ where $j = i/2$ when i is even. Then

$$|s_jr_j| = |r_j^{-1}s_j^{-1}| = |r_j^{-1}| + |s_j^{-1}| = |r_j| + |s_j|,$$

$$|l_jk_j| = |l_j| + |k_j|,$$

and

$$\wedge\{l_1k_1,\ r_1^{-1}s_1^{-1}, \ldots, l_nk_n,\ r_n^{-1}s_n^{-1}\}.$$

From

$$| s_j r_j | = | s_j | + | r_j |$$

and

$$| l_j k_j | = | l_j | + | k_j |$$

it follows that

$$l_j k_j \mid s_j r_j \subseteq l_j \mid r_j .$$

7. Projectivities onto a Free Product $G = A^*_C B$.

In this section let p be a projectivity from a group G' onto the group $G = A^*_C B$. We assume that C is a proper normal subgroup of A and B and that $[A : C] > 2$ or $[B : C] > 2$. We write A', B' and C' for the subgroups of G' such that $pA' = A$, $pB' = B$, and $pC' = C$. Applying our previous results to this situation we have the following theorem.

THEOREM 3.

a) A', B', and C' are isomorphic to A, B, and C, respectively.

b) If U is any cyclic subgroup of G and U' the subgroup of G' such that $pU' = U$, then $[U', C']$ is isomorphic to $[U, C]$.

PROOF. By Proposition 6 there are non-cyclic free subgroups S, T, and V such that

$$1 = S \cap C = T \cap C = V \cap C,$$

and

$$[\bar{A}, \bar{S}] = \bar{A}^* \bar{S}, \ [\bar{B}, \bar{T}] = \bar{B}^* \bar{T}, \ [\bar{U}, \bar{V}] = \bar{U}^* \bar{V}.$$

By Theorem 1 the existence of any one of these subgroups S, T, or V guarantees that C' is isomorphic to C. From the fact that

$$A' = [A', C'], \ B' = [B', C'], \ A = [A, C] \text{ and } B = [B, C]$$

and from Theorem 2 it follows that A', B' and $[U', C']$ are isomorphic to A, B, and $[U, C]$, respectively.

Even though Theorem 3 has some interest of its own, it lacks sufficient detail to be useful in the proof of the main theorem. At the same time Theorem 3 only uses part of what we have shown. For example, by the Corollary to Proposition 6 there are very many non-cyclic free subgroups $V=[t, \nu]$ in G such that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$, where U is a cyclic subgroup of G. Hence by the Corollary to Theorem 2 there are many different pairs (t', t) inducing isomorphisms from $[U', C']$ onto $[U, C]$. In Lemmas A, B, C, and D we shall show that most of these pairs (t', t) induce the same isomorphism.

Again let p, G', and G satisfy the conditions given in the introduction. Throughout this section if H is a subgroup of G we write H' for the subgroup of G' such that $pH'=H$ and we write $\bar{H}$ for the subgroup of $\bar{G}=G/C$ which is the image of H under the canonical homomorphism from G onto $\bar{G}$. As one might expect the *range* of a set Θ of ordered pairs in $G' \times G$ will be taken to be the set of all $t \in G$ such that there is a $t' \in G'$ with $(t', t) \in \Theta$.

Definition. We say that two subsets Θ and Φ of $G' \times G$ are *compatible* if and only if the ranges of Θ and Φ are factor sets and for any $(t', t) \in \Theta$ and $(\nu', \nu) \in \Phi$ the following hold:

a) $V=[t, \nu]$ is a non-cyclic free group such that $V \cap C=1$, and

b) $p[t']=[t]$, $p[\nu']=[\nu]$, and $p[t'\nu']=[t\nu]$.

We shall say that a subset Θ of $G' \times G$ is *stable* if there is a subset Φ of $G' \times G$ such that Θ and Φ are compatible. It is easy to see that Θ and Φ are compatible if and only if Φ and Θ are compatible. If we define

$$\Theta^{-1}=\{(t'^{-1}, t^{-1}) : (t', t) \in \Theta\}$$

it is also easy to see that Θ^{-1} and Φ are compatible when Θ and Φ are compatible. Further if Θ_0 and Φ_0 are subsets of compatible sets Θ and Φ, respectively, and if Θ_0 and Φ_0 have ranges which are factor sets, then Θ_0 and Φ_0 are compatible.

Now let Θ and Φ be compatible sets with respective ranges $l \mid r$ and $k \mid s$. Let U be a subgroup of G such that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$ for any subgroup $V=[t, \nu]$ with $t \in l \mid r$ and $\nu \in k \mid s$. If $(t', t) \in \Theta$ and $(\nu', \nu) \in \Phi$, then by the Corollary to Theorem 2 the pair (t', t) induces the isomorphism $f(t', t; .)$ and $f(.; t', t)$ from $[U', C']$ onto $[U, C]$ while the pair (v', v) induces the isomorphism $f(v', v; .)$ and $f(.; v', v)$ from $[U', C']$ onto $[U, C]$. By the same corollary all of these isomorphisms are the same. Moreover if (t'_1, t_1) and (t'_2, t_2) are any elements of Θ then the isomorphism $f(t'_1, t_1; .)$ and $f(t'_2, t_2; .)$ are equal since for any $(v', v)\in\Phi$

$$f(t'_1, t_1; .)=f(v', v; .)$$

and

$$f(t'_2, t_2; .)=f(v', v; .).$$

Thus all of the pairs $(t', t)\in\Theta$ induce the same isomorphism from $[U', C']$ onto $[U, C]$. Similarly all of the pairs $(v', v)\in\Phi$ induce the same isomorphism from $[U', C']$ onto $[U, C]$. Thus we have shown that Θ and Φ induce the isomorphism

$$f(\Theta; .)=f(\Phi; .)=f(.; \Theta)=f(.; \Phi)$$

from $[U', C']$ onto $[U, C]$. In a similar way its can be shown that if Θ and Φ are compatible sets, then Θ and Φ induce the isomorphism

$$f(\Theta; .)=f(\Phi; .)=f(.; \Theta)=f(.; \Phi)$$

from C' onto C. Because the isomorphisms induced by Θ and Φ are the same on the left and the right we speak simply of the isomorphism determined by Θ and Φ.

DEFINITION. We say that two subsets Θ and Φ of $G'\times G$ *determine an isomorphism* f from $[U', C']$ onto $[U, C]$ if and only if

1) Θ and Φ are compatible sets with respective ranges $l\,|\,r$ and $k\,|\,s$,

2) $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$ for any subgroup $V=[t, v]$ with $t\in l\,|\,r$ and $v\in k\,|\,s$ or

2') U is a subgroup of C, and

3) f is the isomorphism from $[U', C']$ onto $[U, C]$ induced by p and Θ with Θ on the left or right.

The following lemma gives non-trivial examples of functions determined by sets of pairs.

LEMMA A. Let U be a subgroup of G not contained in C. Suppose that either $a)$ U is cyclic or $b)$ is contained in $(A:A)\cup C$ or $(B:B)\cup C$. Then there is an isomorphism f from $[U', C']$ onto $[U, C]$ which is determined by compatible sets Θ and Φ in $G' \times G$.

PROOF. Let U be a subgroup of G satisfying the hypothesis. By the remarks preceding the definition of an isomorphism determined by two sets Θ and Φ in $G' \times G$ it suffices to show that there exist two compatible sets Θ and Φ with respective ranges $l \mid r$ and $k \mid s$ such that $[\bar{U}, \bar{V}] = \bar{U}^* \bar{V}$ for any subgroup $V = [t, v]$ with $t \in l \mid r$ and $v \in k \mid s$. By the Corollary to Proposition 6 there are factor sets $l \mid r$ and $k \mid s$ in G such that $\wedge \{l, r^{-1}, k, s^{-1}\}$ and $[\bar{U}, \bar{V}] = \bar{U}^* \bar{V}$ for any subgroup $V = [t, v]$ with $t \in l \mid r$ and $v \in k \mid s$. By Proposition 5 any such subgroup $V = [t, v]$ must be a non-cyclic free group meeting C in the identity subgroup.

It is fairly easy to see that $l \mid r$ and $k \mid s$ would serve admirably as the ranges of appropriate compatible sets Θ and Φ. Thus we set out to construct compatible sets with these ranges. We first pick $t_0 \in l \mid r$ and $v_0 \in k \mid s$. It is a simple consequence of Lemma 3 b that there are unique generators t'_0 and v'_0 of $p^{-1}[t_0]$ and $p^{-1}[v_0]$, respectively, such that

$$p[t'_0 v'_0] = [t_0 v_0].$$

We now define

$$\Theta = \{(t', t) \in G' \times G : t \in l \mid r,\ p[t'] = [t] \text{ and } p[t'v'_0] = [tv_0]\}$$

and

$$\Phi = \{(v', v) \in G' \times G : v \in k \mid s,\ p[v'] = [v] \text{ and } p[t'_0 v'] = [t_0 v]\}.$$

There is a problem. It is conceivable that $p[t'v'_0] = [tv_0^{-1}]$ or $p[t'v'_0] = [t^{-1}v_0^{-1}]$ for some $t \in l \mid r$ and some generator t' of $p^{-1}[t]$. However, this is impossible. If u' an element of $U' - C'$, there is by Lemma 4 an element u in $[p[u'], C]$ such that $p[u't'_0] = [ut_0]$ and $p[u'v'_0] = [uv_0]$. By Lemma 2 we have that u and v_0 are the only elements of $[p[u'], C]$ and $[v_0]$, respectively such that $p[u'v'_0] = [uv_0]$. But either equation

$$p[t'v'_0]=[tv_0^{-1}] \text{ or } p[t'v']=[t^{-1}v_0^{-1}]$$

would imply (by Lemma 4 again) that there is an element u_1 in $[p[u'], C]$ such that $p[u'v'_0]=[u_1v_0^{-1}]$. A plain contradiction. By Lemma 3 *b* we have that for any $t\in l\,|\,r$ and $v\in k\,|\,s$ there are unique generators t' and v' of $p^{-1}[t]$ and $p^{-1}[v]$, respectively, such that $p[t'v'_0]=[tv_0]$ and $p[t'_0v']=[t_0v]$. Applying Lemma 4 twice we have first that

$$p[u't']=[ut] \text{ and } p[u'v']=[uv],$$

and then that

$$p[t'v']=[tv]$$

for every $(t', t)\in\Theta$ and $(v', v)\in\Phi$. Thus the sets Θ and Φ satisfy all of the conditions for compatible sets.

Our next lemma plays an important role in the proof of the main theorem. Again we assume that p is a projectivity from some group G' onto $G=A_C^*B$ as in the introduction to this section. Let x' and y' be elements of G'. Also let f be an isomorphism from $[x', C']$ onto $[p[x'], C]$ which is determined by compatible sets Σ and Θ in $G'\times G$. Similarly let h be an isomorphism from $[y', C']$ onto $[p[y'], C]$ which is determined by compatible sets Π and Φ in $G'\times G$. If $x=f(x')$ and $y=h(y')$, then we know that x and y are the only elements of $[p[x'], C]$ and $[p[y'], C]$, respectively, such that

$$p[t'x']=[tx], \text{ for every } (t', t)\in\Theta,$$

and

$$p[y'v']=[yv], \text{ for every } (v', v)\in\Phi.$$

We also have

LEMMA B. There are stable subsets Θ_1 and Φ_1 contained in Θ and Φ, respectively, such that

$$p[t'x'y'v']=[txyv]$$

for every

$$(t',\ t)\in\Theta_1 \text{ and } (v',\ v)\in\Phi_1 .$$

PROOF. In Theorem 2 we saw that

$$p[t'x's']=[txs]$$

for arbitrary pairs $(t',\ t)\in\Theta$ and $(s',\ s)\in\Sigma$. Similarly

$$p[u'y'v']=[uyv]$$

for any pair $(u',\ u)\in\Pi$ and any pair $(v',\ v)\in\Phi$. Now if some set $tx,\ yv,\ s,\ u$ is a system of free generators of the subgroup $[tx,\ yv,\ s,\ u]$, then by Lemmas 3 and 4 we must have

$$p[t'x'y'v']=[txyv].$$

Thus we are led to construct stable subsets Σ_1, Θ_1, Π_1, and Φ_1 of Σ, Θ, Π, and Φ, respectively, such that the subgroup $[tx,\ yv,\ s,\ u]$ is a free group of rank 4 for any $t,\ v,\ s,\ u$ in the respective ranges of Θ_1, Φ_1, Σ_1, Π_1.

Let the set Θ, Φ, Σ, and Π have respective ranges $l\mid r$, $k\mid s$, $m\mid n$, and $o\mid q$. We may suppose that rx and r begin with an element from the same group A or B, because there is always a factor set $l\mid r_0\leq l\mid r$ such that r_0x and r_0 begin with an element from the same group A or B. Similarly we suppose that yk and k end with an element from the same group A or B. Consider the array

$$(l,\ x^{-1}r^{-1},\ yk,\ s^{-1},\ m,\ n^{-1},\ o,\ q^{-1}).$$

By the Corollary to Proposition 7 there are elements l_0, r_0, k_0, s_0, m_0, n_0, o_0, q_0 in G such that

$$\wedge\{ll_0,\ x^{-1}r^{-1}r_0^{-1},\ ykk_0,\ s^{-1}s_0^{-1},\ mm_0,\ n^{-1}n_0^{-1},\ oo_0,\ q^{-1}q_0^{-1}\}$$

and

$$ll_0\mid r_0rx\subseteq l\mid rx,\ ykk_0\mid s_0s\subseteq yk\mid s,\ mm_0\mid n_0n\subseteq m\mid n$$

and

$$oo_0 \mid q_0q \subseteq o \mid q.$$

Since rx and r begin with an element from the same group

$$ll_0 \mid r_0r \subseteq l \mid r.$$

For similar reasons

$$kk_0 \mid s_0s \subseteq k \mid s.$$

Now take $l_1 = ll_0$, $r_1 = r_0r$, $k_1 = kk_0$, $s_1 = s_0s$, $m_1 = mm_0$, $n_1 = n_0n$, $o_1 = oo_0$ and $q_1 = q_0q$. Let Θ_1, Φ_1, Σ_1, and Π_1 be the stable subsets of Θ, Φ, Σ, and Π, respectively, with respective ranges $l_1 \mid r_1$, $k_1 \mid s_1$, $m_1 \mid n_1$ and $o_1 \mid q_1$. Since

$$\wedge \{l_1, x^{-1}r_1^{-1}, yk_1, s_1^{-1}, m_1, n_1^{-1}, o_1, q_1^{-1}\},$$

we have by Proposition 5 that the subgroup $[tx, yv, s, u]$ is a free group of rank 4 for any t, v, s, u in $l_1 \mid r_1$, $k_1 \mid s_1$, $m_1 \mid n_1$, and $o_1 \mid q_1$, respectively.

The following statement of this result though more technical is frequently useful. The proof has already been given.

COROLLARY. The set Θ contains a stable subset Θ_1 with range $l_1 \mid r_1$ and the set Φ contains a stable subset Φ_1 with range $k_1 \mid s_1$ such that

$$\wedge \{l_1, x^{-1}r_1^{-1}, yk_1, s_1^{-1}\}$$

and that

$$p[t'x'y'v'] = [txyv]$$

for every $(t', t) \in \Theta_1$ and $(v', v) \in \Phi_1$.

In particular this corollary is used in the proof of Lemma C.

LEMMA C. Let U be a subgroup of G. Let f and h be isomorphisms from $[U', C']$ onto $[U, C]$ which are determined by pairs of compatible sets in $G' \times G$. Then

$$f = h.$$

PROOF. Let Θ be one of a pair of compatible sets which determine f and let Φ be one of a pair of compatible sets which determine h. Let u' be an element of $[U', C']$ and let $u_1=f(u')$, $u_2=h(u')$. It suffices to show that

$$u_1=u_2\,.$$

We know that

$$p[t'u']=[tu_1],\text{ for all }(t',\ t)\in\Theta$$

and

$$p[u'v']=[u_2v],\text{ for all }(v',\ v)\in\Phi.$$

By the Corollary to Lemma B we know that there are stable subsets Θ_1 of Θ and Φ_1 of Φ with respective ranges $l_1\,|\,r_1$ and $k_1\,|\,s_1$ such that

$$\wedge\{\ l_1\,,\ u_1^{-1}r_1^{-1},\ k_1\,,\ s_1^{-1}\}$$

and that

$$p[(t'u')v']=[tu_1v]$$

for all $(t',\ t)\in\Theta_1$ and $(v',\ v)\in\Phi_1$. In this application of Lemma B the elments u'_1 and u_1 correspond to the elements x' and x, respectively, while the identity elements of G' and G correspond to the elements y' and y, respectively.

Now Θ_1 is still one of a pair of compatible sets which determine f. Similarly Φ_1 is still one of a pair of compatible sets determining h. Applying the Corollary to Lemma B once more to Θ_1 and Φ_1 we have that there are stable subsets Θ_2 of Θ_1 and Φ_2 of Φ_1 with respective ranges $l_2\,|\,r_2$ and $k_2\,|\,s_2$ such that

$$\wedge\,\{l_2\,,\ r_2^{-1},\ u_2k_2\,,\ s_2^{-1}\}$$

and that

$$p[t'(u'v')]=[tu_2v]$$

for every $(t',\ t)\in\Theta_2$ and $(v',\ v)\in\Phi_2$. Since Θ_2 is a subset of Θ_1 and is a subset of Φ_1 we still have

$$p[t'u'v']=[tu_1v]$$

for all $(t',\ t)\in\Theta_2$ and $(v',\ v)\in\Theta_2$.

Now let t be an element of $l_2 \mid r_2$ and v an element of $k_2 \mid s_2$. Since $\wedge \{l_2, r_2^{-1}, u_2k_2, s_2^{-1}\}$, it follows that $[t, u_2v]$ is a non-cyclic free group. Thus the group generated by either tu_1v or tu_2v must be an infinite cyclic subgroup of G. Hence there is an integer $j = \pm 1$, such that

$$(tu_1v)^j = tu_2v.$$

However from the conditions placed on k_1, r_1 and k_2, r_2 we have that both tu_1v and tu_2v are elements of $l_2 \mid s_2$. Since $s_2^{-1} \mid l_2^{-1}$ and $l_2 \mid s_2$ are disjoint, it is impossible for j to be minus one. Therefore

$$tu_1v = tu_2v,$$

and consequently

$$u_1 = u_2,$$

as was to be shown.

We can now speak of the isomorphism from A' onto A (B' onto B) which is determined by a pair of compatible sets. We also have the following important definition.

DEFINITION. Let g' be an element of G'. Let h be the isomorphism from $[g', C']$ onto $[p[g'], C]$ which is determined by a pair of compatible sets. We define f to be the mapping from G' onto G such that $f(g') = h(g')$ for each g' in G'.

We write f_1 for the restriction of f to A' and f_2 for the restriction of f to B'. It is easy to see that f_1 (f_2) is the isomorphism from A' onto A (B' onto B) determined by a pair of compatible sets. More generally if U is any conjugate of A or B in G, then the restriction of f to U' is the isomorphism from U' onto U determined by a pair of compatible subsets.

In the next lemma a pair of compatible sets are constructed which determine both f_1 and f_2.

LEMMA D. Let U be any cyclic subgroup of G. Let h be the isomorphism from $[U', C']$ onto $[U, C]$ which is determined by a pair of sets in $G' \times G$. Then there is a pair of compatible sets which determine the isomorphism h, f_1 and f_2.

PROOF. In Lemma A we saw that there were two compatible sets Θ and Φ contained in either $G' \times (A : A)$ or $G' \times (B : B)$ which determined the isomorphism h from $[U', C']$ onto $[U, C]$. Suppose for instance that Θ and Φ are in $G' \times (A : A)$. Then Θ and Φ also determine the isomorphism f_2 from B' onto B. We shall now show that compatible subsets Θ_1 and Φ_1 of Θ and Φ, respectively, can be selected so that Θ_1 and Φ_1 determine the isomorphism f_1 from A' onto A. Since f_2 and h are induced by each pair in Θ in Φ, the sets Θ_1 and Φ_1 will still determine h and f_2.

Let a_1, a_2, a_3, a_4 be four elements in $A-C$ such that Θ is contained in $G' \times (a_1 \mid a_2)$ and Φ is contained in $G' \times (a_3 \mid a_4)$. As we saw in the Corollary to Proposition 7, there are elements l, r, k, and s in G such that $\wedge \{l, r^{-1}, k, s^{-1}\}$ and that $a_1 l \mid r a_2$ and $a_2 k \mid s a_4$ are contained in the ranges of Θ and Φ, respectively. We see that $l \mid r$ and $k \mid s$ are contained in $B : B$. Thus any subgroup $V = [t, v]$ with $t \in l \mid r$ and $v \in k \mid s$ is a non-cyclic free subgroup of G and $[\bar{A}, \bar{V}] = \bar{A}^* \bar{V}$. For this reason the subgroup $T = [a_1 t a_2, a_3 v a_4]$ satisfies the condition $[\bar{A}, \bar{T}] = \bar{A}^* \bar{T}$. But this shows that all subgroups $T = [t_1, v_1]$ with $t_1 \in a_1 l \mid r a_2$ and $v_1 \in a_3 k \mid s a_4$ satisfy the condition $[\bar{A}, \bar{T}] = \bar{A}^* \bar{T}$. Now let Θ_1 be the subset of Θ with range $a_1 l \mid r a_2$ and let Φ_1 be the subset of Φ with range $a_3 k \mid s a_4$. Certainly Θ_1 and Φ_1 are compatible. Hence Θ_1 and Φ_1 must determine the isomorphism from A' onto A. Thus the compatible sets Θ_1 and Φ_1 determine f_1, f_2, and h.

8. Main Theorems.

Throughout this section we let G be a free product of its subgroups A and B with amalgamated subgroup $C = A \cap B$ properly contained in A and B. We also suppose that $[A : C] > 2$ or $[B : C] > 2$. That is, the index of C in A is greater than two or the index of C in B is greater than two. We now have the following fundamental theorem.

THEOREM 4. If C is a normal subgroup of G, then G is determined by its lattice of subgroups.

PROOF. Let p be a projectivity from some group G' onto G. As we saw in the previous section there is one and only one isomorphism f_1 from $A' = p^{-1}A$ onto A which is determined by a pair of compatible sets in $G' \times G$. Similarly there is one and only one isomorphism f_2 from

$B'=p^{-1}B$ onto B which is determined by a pair of compatible sets in $G'\times G$. By Lemma C, f_1 and f_2 must agree on $C'=A'\cap B'$; by Theorem 1, f_1 and f_2 must map C' onto C. We also take f to be the map defined in Section 7. It should be recalled that f_1 and f_2 are the restrictions of f to A' and B', respectively.

We show that G' is isomorphic to G by showing that G' is the free product of A' and B' with amalgamated subgroup C'. Specifically we show that

$$f(w')=f(w'_1) \ \dots \ f(w'_n)$$

where

$$w'=w'_1 \ \dots \ w'_n$$

is any product of n factors w'_i with the w'_i alternately in $A'-C'$ and $B'-C'$. The desired conclusion is an easy consequence of this result, as we see now. Since w'_i is in A' or B', $f(w'_i)$ is the image of w'_i under the appropriate map f_1 or f_2. Hence $f(w')$ is a word of length n in G and thereby a non-trivial element of G. From the fact that f restricted to $[w', C']$ is an isomorphism, it follows that w' cannot be the identity element of G'. By definition G' must be a free product of its subgroups A' and B' with amalgamated subgroup $C'=A'\cap B'$.

The proof proceeds by induction on n, the number of factors in the product

$$w'=w'_1w'_2 \ \dots \ w'_n .$$

If $n=1$, then $w'=w'_1$ and $f(w')=f(w'_1)$. Assuming the result for all products with $k-1$ factors, we then show that

$$f(w')=f(w'_1)f(w'_2) \ \dots \ f(w'_k)$$

for some product

$$w'=w'_1w'_2 \ \dots \ w'_k$$

with k factors w'_i alternately in $A'-C'$ and $B'-C'$. First consider the factorization

$$w'=x'y'$$

where $x'=w'_1$ and $y'=w'_2 \;\; ...,\; w'_k$. If h is the isomorphism from $[w', C']$ into G determined by a pair of sets in $G'\times G$, then by Lemma D we can take Θ to be one of a pair of compatible sets which determine f_1, f_2, and h. We now have that for all $(t', t)\in\Theta$

$$p[t'x']=[tx] \text{ and } p[t'w']=[tw]$$

where $w=f(w')$ and $x=f(w'_1)$. Also let Φ be one of a pair of compatible sets which determine the isomorphism from $[y', C']$ into G. Thus for all $(v', v)\in\Phi$,

$$p[y'v']=[yv]$$

where $y=f(w'_2) \;\; ... \;\; f(w'_k)$ by the induction assumption.

In Lemma B we saw that there were stable subsets Θ_1 and Φ_1 of Θ and Φ, respectively, with respective ranges $l_1 \mid r_1$ and $k_1 \mid s_1$ such that

a) $\wedge\{l_1,\; x^{-1}r_1^{-1},\; yk_1,\; s_1^{-1}\}$, and

b) $p[t'x'y'v']=[txyv]$ for all $(t', t)\in\Theta_1$ and $(v'_1 v)\in\Phi_1$.

Now Θ_1 is one of a pair of compatible sets which determine f_1, f_2, and h. Similarly Φ_1 is also one of a pair of compatible sets which determine the isomorphism from $[y', C']$ into G. So let Θ_2 and Φ_2 be stable subsets of Θ_1 and Φ_1, respectively, with respective ranges $l_2 \mid r_2$ and $k_2 \mid s_2$ such that

i) $\wedge\{l_2,\; w^{-1}r_2^{-1},\; k_2,\; s_2^{-1}\}$, and

ii) $p[t'w'v']=[twv]$ for all $(t', t)\in\Theta_2$ and $(v', v)\Phi_2$.

As in the proof of Lemma C the conditions *a*) and *i*) guarantee that

$$twv=txyv.$$

Hence

$$w=xy=f(w'_1)f(w'_2) \;\; ... \;\; f(w'_k)$$

as was to be shown.

Since G' is a free product of its subgroups A' and B', we know that there is one and only one homomorphism from G' into G which extends f_1 and f_2. From the fact that

$$f(w')=f(w'_1) \ \dots \ f(w'_n)$$

for any product $w'=w'_1 \ \dots \ w'_n$ with the n-factors w'_i alternately in $A'-C'$ and $B'-C'$ it follows that f is the homomorphism which extends f_1 and f_2. From the same fact it also follows that f maps G' onto G, that the kernel of f is trivial, and consequently that f is an isomorphism from G' onto G.

In our last theorem we shall show that f induces p when C is in the center of G. For the moment we content ourselves with pointing out the following fact.

THEOREM 5. With the same notation as in Theorem 4 and its proof, we have that the isomorphism f induces the projectivity p on all subgroups H' of G' which have trivial intersection with C' i.e. $pK'=fK'$ for all subgroups K' of H'. Moreover f is the only isomorphism of G' mapping A' onto A and B' onto B for which this is true.

PROOF. To see this let U' be a cyclic subgroup of G' which has trivial intersection with C'. As usual let $U=pU'$ and let $\bar{U}$ be the image of U under the canonical homomorphism from G onto G/C. Let Θ and Φ be compatible sets which determine an isomorphism h from $[U', C']$ onto $[U, C]$. Of course h is just the restriction of f to $[U', C']$. Also let $V=[t, v]$ with t and v in the respective ranges of Θ and Φ. Because Θ and Φ are compatible sets which determine h, we have that V is a non-cyclic free group, that $V \cap C=1$, and that $[\bar{U}, \bar{V}]=\bar{U}^*\bar{V}$.

From the fact that $U \cap C=1$ and $V \cap C=1$ it follows that $\bar{U}$ and $\bar{V}$ are isomorphic to U and V, respectively. Thus the subgroup $[U, V]$ has a homomorphic image $[\bar{U}, \bar{V}]$ which is isomorphic to the free product, without amalgamation, of the groups U and V. This means that $[U, V]$ must itself be the free product of its subgroups U and V, i.e. $[U, V]=U^*V$. By Theorem 2 and its Corollaries the restriction of f to U induces the projectivity p from U' onto U. Thus $pU'=fU'$ for every cyclic subgroup U' of G' with $U' \cap C'=1'$. If H' is a subgroup of G' which has trivial intersection with C', then p is induced by f on every cyclic subgroup of H'. Consequently p is induced by f on H'.

Suppose h is some other isomorphism having this property and that $hA'=A$ and $hB'=B$. Then for all a' in $A'-C'$ and b' in $B'-C'$

$$p[a'b']=[f(a')f(b')]=[h(a')h(b')].$$

For fixed a', b' there is a c in C such that

$$f(a')=h(a')c \text{ and } f(b')=c^{-1}h(b'),$$

since $f(a')f(b')=h(a')h(b')$. Letting a' and then b' vary separately over the elements of $A'-C'$ and $B'-C'$, respectively, we see that

$$f(a')=h(a')c \text{ for all } a' \text{ in } A'-C'$$

and

$$f(b')=c^{-1}h(b') \text{ for all } b' \text{ in } B'-C'.$$

Now either $[A':C']>2$ or $[B':C']>2$. Suppose without loss of generality that $[A':C']>2$ and that a'_1, a'_2 are two elements of $A'-C'$ such that $a'_1a'_2$ is in $A'-C'$. Then

$$f(a'_1a'_2)=f(a'_1)f(a'_2)=h(a'_1)ch(a'_2)c$$

and

$$f(a'_1a'_2)=h(a'_1a'_2)c=h(a'_1)h(a'_2)c.$$

It is easy to see that c must be the identity in G and that f and h must be the same isomorphism.

From Theorem 5 it follows that it there is an isomorphism which induces the projectivity p, it must be f. The following theorem shows that under certain conditions f does induce p. We remind the reader that throughout this section G has been assumed to be a free product of its subgroups A and B with amalgamated subgroup $C=A\cap B$ where

a) A, $B\neq C$ and

b) $[A:C]>2$ or $[B:C]>2$.

THEOREM 6. If C is the center G, then every projectivity of G is induced by a unique isomorphism.

PROOF. Let p be a projectivity from some group G' onto G. From the Corollaries to Lemma 1 and 2 it follows that f induces p on all of the cyclic subgroups of G'. That is $pU'=fU'$ for any cyclic subgroup U' of G'. Hence $pH'=fH'$ for any subgroup H' of G'. By definition p is induced by f.

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