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STUDIES OF SOME ITEMS OF THE LATTICE THEORY IN RELATION TO THE HILBERT - HERMITE SPACE

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Introduction.

The present paper discusses some items in the theories of lattices, measure theory and separable complete Hilbert-Hermite space. These items seem to be useful not only as mathematical theories, but also as mathematical tools for the general theories of Quantum Mechanics. In this respect the paper may be considered as a continuation of the book by the author: « Mathematical Apparatus for Quantum theories. Springer Verlag 1966, X+952 pp. ».

The paper contains three Chapters, α, β, γ, and the last chapter γ is composed of three sections.

The part of the paper are:

α) Study of the Cantor-Mac Neille's measure - extension device for Boole'an lattices.

β) A study in the cartesian product of abstract measured Boole'an lattices.

γ) Contains three Sections:

Section 1. A special metric topology on the lattice of all closed subspaces in the separable and complete Hilbert-Hermite space.

Section 2. Measure - topologies on a geometrical tribe of spaces in the Hilbert-Hermite space.

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Section 3. Linearization of geometrical tribes of spaces in the Hilbert-Hermite space.

We shall continuously refer to various items in the quoted book of the author. The references to that book are indicated by capital letters accompanied (or not) by numbers eg. [B.2.13], [AI.3.0], [DI].

References to items of the present paper are indicated by α, β, γ and some numbers, eg. [α.2.3], [.β12.1], [γ], [γ.4].

All quotation-codes have the parenthesis [ ]. The purpose of this way of notation is to facilitate the author in referring to his previous papers.

CHAPTER α.

STUDY OF THE CANTOR-MAC NEILLE'S MEASURE-EXTENSION DEVICE FOR BOOLEAN LATTICES

α.**1.** For terminology and references concerning tribes, we refer, in general, to the chapter [A] of our book: « Mathematical apparatus for quantum-theories, Springer-Verlag, 1966, X+pp. 952 ».

However, in this respect, we shall recall some items.

By a *tribe* (*Boolean lattice*) we understand any complementary and distributive lattice. Its elements will be called *somata*, (sing.: *soma*).

A *lattice* is an *ordering* ($\leq$), (partial ordering), for which the *union* (*join, sum*), $a+b$, and the *intersection*, (*meet, product*), $a\cdot b$ of two somata a, b are always meaningful, [A.1.3].

The lattice is said to be *complementary*, whenever it possesses the « smallest » soma, the *zero*, *null* 0, and also the « greatest » soma, the *unit*, 1, and when for every soma a there exists another one co a, the *complement* of a, such that

$$a+\text{co}\, a=1,\quad a\cdot\text{co}\, a=0;\ \text{if}\ a\leq b,\ \text{then}\ \text{co}\, b\leq\text{co}\, a.$$

The lattice is said *distributive*, whenever

$$(a+b)\cdot c=a\cdot c+b\cdot c.$$

For a lattice we define the *difference* $a-b$ by $a\cdot \mathrm{co}\, b$ and the *algebraic addition* by

$$a\dot{+}b =_{df} (a-b)+(b-a),$$

called usually « symmetric difference ».

In discrimination we call $a+b$ *somatic sum.*

The tribe is said to be *trivial,* when it contains only one soma $O=1$. The « smallest », non trivial tribe is that containing only two somata O and 1.

α.1.1. By a *measure* (*finitely additive*) *ond the tribe* $\mathcal{T}$ we understand any non-negative-valued function $\mu(\dot{a})$ of the varisble some [1]), defined for all somata $a\cdot$, and having the properties

1) $0 \leq \mu(a) < +\infty$, for all a,
2) if $a\cdot b=O$, then $\mu(a+b)=\mu(a)+\mu(b)$,
3) $\mu(O)=0$,
4) if $a=b$, then $\mu(a)=\mu(b)$.

We call 2) *additivity* (*finite additivity*).

The measure is said to be *trivial,* whenever $\mu(a)=0$ for all a.

The measure is called *effective* whenever the following is true:

« if $\mu(a)=0$, then $a=O$ ».

α.1.1.1. If, in a tribe $\mathcal{T}$, we keep the multiplication, but change the addition $a+b$ into $a\dot{+}b$, we reorganize the tribe into a commutative ring with unit 1. We call this ring « *Stone's ring* ».

α.1.1.2. Remark. Given a Stone's ring[2]) (called also Boolean ring), i.e. a ring with $a\cdot a=a$, and we change its addition $a+b$ into

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[1]) To emphasize that the letter a denotes a variable quantity, we shall use the dot placed over the letter: $\dot{a}$.

If M is a correspondence (relation), $\mathrm{G}\, M$ will denote its domain and $\mathrm{D}\, M$ its range.

The symbol $=_{df}$ means definition.

[2]) G. Birkhoff, Lattice Theory, American Math. Soc. Colloquium Publications. Vol. XXV. Revised edition, p. 154.

$a+b=_{df}a+a\cdot b+b$, we get a tribe with ordering ($\leqq$), defined by

$$a\leq b\cdot =_{df}\cdot a\cdot b=a.$$

α.1.2. Thus there is a $1\rightarrow 1$ correspondence between tribes and Stone's rings.

α.1.3. Since $\mathfrak{T}$ can be conceived as ring, we can speak of *ideals* in it.

α.2. If we define the *distance between the somata a, b* of the tribe $\mathfrak{T}$, by

$$|\, a,\ b\,|_{\mu}=_{df}\ \mu(a\dot{+}b),$$

we get a kind of a *set-topology*, whose « points » are somata of $\mathfrak{T}$, and which may be complete or not.

Now, using something like Cauchy's fundamental sequences, known in the Cantor's theory of irrational numbers, we can extend the tribe $\mathfrak{T}$ together with the measure μ.

We call it *Cantor-Mac Neille-s-extension of the measured tribe* [3]).

α.3. In the just quoted paper the said extension in given by the statements of main theorems, but there are no explicite proofs. The definitions are given but for proofs sometimes short indications only, though some results are far from being obvious and requiring only straight forward reasoning.

Since the Mac Neille's extension does not coincide with the generalized Lebesgue's one [A1.6], it seems to be in order to supply the theory with explicite proofs. We shall show at the end of the present paper, that even, if we deal with simple Boolean lattices, whose somata are proint-sets, the Mac Neille's extension can introduce new elements, (sometimes), which are no sets at all.

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3) H. Mac Neille, Proceedings of the National Academy of Sciences, Vol. 24, n. 4, pp. 188-193 (1938).

α.4. In the present paper, the results of Mac Neille's are completed and everywhere clarity and logical precision are aimed at. Especially « identifications », commonly used of entities having different logical type, will be avoided, as logically incorrect [4]).

α.4.1. We start with recallection of auxiliaries, mainly dealing with properties of the algebraic addition $a \dot{+} b$ of somata of a tribe.

α.5. We have

$$a \dot{+} b =_{\mathrm{df}} (a-b)+(b-a).$$

The algebraic addition is commutative and associative. We have the distributive law:

$$(a \dot{+} b)\cdot c = a\cdot c \dot{+} b\cdot c.$$

We also have

$$1 \dot{+} a = \mathrm{co}\, 0,$$

$$a+b = a \dot{+} a\cdot b \dot{+} b,$$

$$a \dot{+} a = 0,$$

$$\mathrm{co}\, a \dot{+} \mathrm{co}\, b = a \dot{+} b,$$

$$(a+b) \dot{+} (a_1+b_1) \leq (a \dot{+} a_1)+(b \dot{+} b_1),$$

$$a\cdot b \dot{+} a_1\cdot b_1 \leq (a \dot{+} a_1)+(b \dot{+} b_1),$$

$$(a-b) \dot{+} (a_1-b_1) \leq (a \dot{+} a_1)+(b \dot{+} b_1),$$

$$(a \dot{+} b) \dot{+} (a_1 \dot{+} b_1) = (a \dot{+} a_1) \dot{+} (\dot{+} b_1):$$

α.5.1. Supposing $\mu(a)$ is a non trivial measure on $\mathcal{T}$, we have the properties, stated in [α.1.1]. We also have:

α.5.2. For any a, b we have $\mu(a+b) \leq \mu(a)+\mu(b)$.

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[4]) The content of the present paper constitutes a part of my lectures at the Mathematical Institute of the University in Naples, in 1965.

α.5.3. For any *a, b* we have

$$|\mu(a)-\mu(b)|\leq\mu(a\dot{+}b).$$

α.5.4. We also have:

$$\mu(\text{co }a\dot{+}\text{co }a_1)=\mu(a\dot{+}a_1),$$

$$\mu[(a+b)\dot{+}(a_1+b_1)]\leq\mu(a\dot{+}a_1)+\mu(b\dot{+}b_1),$$

$$\mu[a\cdot\dot{+}a_1\cdot b_1]\leq\mu(a\dot{+}a_1)+\mu(b\dot{+}b_1),$$

$$\mu[(a-b)\dot{+}(a_1-b_1)]\leq\mu(a\dot{+}a_1)+\mu(b\dot{+}b_1),$$

$$\mu[(a\dot{+}b)\dot{+}(a_1\dot{+}b_1)]\leq\mu(a\dot{+}a_1)+\mu(b\dot{+}b_1).$$

α.6. DEFINITION. By the *distance between the somata a, b* we understand the number

$$|a, b|_\mu =_{df}\mu(a\dot{+}b).$$

α.6.1.We have

$$|a, b|_\mu\geq 0.$$

The notion $|a, b|_\mu$ is *invariant with respect to the equality* $(=)$ of somata of $\mathfrak{T}$. This means that if $a=a_1$, $b=b_1$, then

$$|a, b|_\mu=|a_1, b_1|_\mu=0,$$

We have

$$|a, b|_\mu=|b, a|_\mu,$$

$$|a, b|_\mu\leq|a, c|_\mu+|c, b|_\mu,$$

$$|a, a|_\mu=0.$$

The above yields the following theorem:

α.6.2. THEOREM. The following are equivalent:

I. $|a, b|_\mu = 0$ implies $a = b$;

II. The measure $\mu(\dot{a})$ is effective, which means that:

$$\text{« if } \mu(a)=0, \text{ then } a=0 \text{.}$$

α.6.3. THEOREM. The notion of distance organizes the tribe $\mathfrak{T}$ into a metric space, whenever the measure $\mu(\dot{a})$ is effective.

This topology may be neither complete, nor separable, but is satisfies the «first Hausdorff condition» of countability.

α.7. Now even in the general case of measure, a kind of topology can be introduced in $\mathfrak{T}$; this in the following way:

The collection of all somata p, with $\mu(p)=0$, is and ideal J in D$\mathfrak{T}$, [A.9] [5]), [α.1.1.].

α.7.1. The ideal J reorganizes $\mathfrak{T}$ into a new ordering on $\mathfrak{T}$, denoted by $\mathfrak{T}_J$, or by

$$a \leq^J b, \text{ (or } a \leq^\mu b\text{)},$$

and defined by the condition

$$a - b \in J.$$

On $\mathfrak{T}_J$ we get a new equality of somata

$$a =^J b, \text{ (or } a =^\mu b\text{)},$$

defined by

$$\mu(a \dot{+} b) = 0,$$

which is equivalent to

$$a \dot{+} b \in J.$$

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[5]) On a tribe $\mathfrak{T}$, the ideal J is defined as any not empty subset of $\mathfrak{T}$, satisfying the condition [A.9.1.]:

1) if $a, b \in J$, then $a + b \in J$,

2) if $a \in J$, $b \leq a$, then $b \in J$.

The ordering $\mathfrak{T}_J$ is a finitely additive tribe with $(=^J)$ as governing equality in it, [A].

α.7.2. The ordering $(\leq^J)$ generates the new operations $a+^J b$, $a \cdot^J b$, $\mathrm{co}^J b$ which are invariant with respect to that equality.

We have

$$a+b=^J a+^J b$$

$$a \cdot b=^J a \cdot^J b,$$

$$\mathrm{co}\, a=^J \mathrm{co}^J a,$$

so the operations on $\mathfrak{T}_J$ can be replaced by the analogous operations in $\mathfrak{T}$. This can be done in any respective formula, but only the symbols $(=)$, $(\leq)$, must be replaced by $(=^J)$, $(\leq^J)$.

α.7.3. The measure $\mu(\dot{a})$ is invariant with respect to $(=^J)$, i.e. if $a=^J b$, then $\mu(a)=\mu(b)$.

α.7.4. The measure μ is e f f e c t i v e on $\mathfrak{T}_J$, i.e. if $\mu(a)=0$, then $a=^J O$.

Indeed, let $\mu(a)=0$. We have $a=a\dot{+}0$; hence $\mu(a\dot{+}O)=0$; hence $a=^J O$, [α.7.1].

α.7.5. The distance $|a, b|_\mu$ is invariant with respect to $(=^J)$, i.e. if $a=^J a_1$, $b=^J b_1$, then

$$|a, b|_\mu=|a_1, b_1|_\mu .$$

α.7.6. Thus the equality $(=^J)$ reorganizes $\mathfrak{T}$ into a *metric space.* We sometimes call $\mathfrak{T}_J$ *tribe* $\mathfrak{T}$ *modulo J.*

$\mathfrak{T}_J$ has the same somata as $\mathfrak{T}$; only the ordering $(\leq)$ and the governing equality $(=)$ on $\mathfrak{T}$ are replaced by $(\leq^J)$ and $(=^J)$ respectively.

α.8. Thus the presence of measure μ allows to organize the tribe $\mathfrak{T}$ into a metric space, and the measure μ, which may be not effective, can be changed into an effective one just by a suitable change of the governing equality on $\mathfrak{T}$, and whithout changing the somata.

The presence of measure allows to perform another important change of the tribe: it allows to amplify the tribe and extend the measure to an effective one.

This will be performed by introducing so called μ-fundamental sequences of somata of $\mathcal{T}$.

α.8.0. HYPOTHESIS. We admit, in the sequel, that $\mathcal{T}$ is a finitely additive, non trivial tribe, and that $\mu(\dot{a})$ is a finitely additive, non trivial measure on $\mathcal{T}$, (which may be not effective).

α.8.1. DEFINITION. An infinite sequence $\{a_n\}$ of somata of $\mathcal{T}$ will be termed *μ-fundamental sequence,* whenever for every $\varepsilon>0$ there exists an index N, such that if $n\geq N$, $m\geq N$, we have

$$|a_n, a_m|_\mu \leq \varepsilon.$$

α.8.2 DEFINITION. An infinite sequence $\{a_n\}$ of somata of $\mathcal{T}$ will be termed *μ-null sequence,* whenever

$$\lim_{n\to\infty} \mu(a_n)=0.$$

The following, rather obvious theorems, hold true:

α.8.3. THEOREM. The notion of fundamental sequence and that of a null-sequence are both invariant with respect to the μ-equality, $(\leq^\mu)$ i.e. $(\leq^J)$. This means that if $\{a_n\}$ is a fundamental [null] sequence and

$$a_n =^\mu a'_n \text{ for } n=1, 2, ..., \text{ then}$$

$\{a'_n\}$ is also a μ-fundamental [null] sequence.

α.8.4. THEOREM. A μ-null-sequence is also a μ-fundamental sequence.

α.8.5. THEOREM. The constant sequence $\{a, a, ..., a, ...\}$ is a μ-fundamental sequence.

α.8.6. THEOREM. A subsequence $\{a_{k(\dot{n})}\}$ of a μ-fundamental [null]-sequence is also a μ-fundamental [null]-sequence.

α.8.7. THEOREM. If, given a μ-fundamental [null]-sequence, we perform on it a finite number of arbitrary changes, then the sequence remains to be μ-fundamental [null].

α.8.8. THEOREM. If $\{a_n\}$ is a μ-fundamental [null] sequence, then $\{\mathrm{co}\, a_n\}$ is also a μ-fundamental [null] sequence.

α.8.9. THEOREM. If $\{a_n\}$, $\{b_n\}$ are both μ-fundamental [null] sequences, then

$$\{a_n+b_n\},\ \{a_n\cdot b_n\},\ \{a_n-b_n\},\ \{a_n\dot{+}b_n\}$$

are also all μ-fundamental [null] sequences.

α.9. DEFINITION. The μ-fundamental sequences $\{a_{\dot{n}}\}$ is said *to be equivalent to the μ-fundamental sequence* $\{b_{\dot{n}}\}$:

$$\{a_{\dot{n}}\}\doteq\{b_{\dot{n}}\},$$

whenever $\{a_{\dot{n}}\dot{+}b_{\dot{n}}\}$ is a μ-null-sequence.

α.9.1. THEOREM. The equivalence $(\doteq)$ of fundamental sequences is reflexive, symmetric and transitive.

It is also invariant with respect to the $(=^{\mu})$ equality, (it is $(=^{J})$ equality), and with respect to $(\doteq)$-equality.

α.9.2. THEOREM. Any two μ-null-sequences are $(\doteq)$-equivalent, and $(\doteq)$-equivalent to the constant μ-null-sequence $\{O, O, \ldots, O, \ldots\}$.

α.9.3. DEFINITION. For fundamental μ-sequences we define the *operation of the algebraic addition and multiplication* as follows:

$$\{a_{\dot{n}}\}+\{b_{\dot{n}}\}=_{\mathrm{df}}\{a_{\dot{n}}\dot{+}b_{\dot{n}}\},\qquad \{a_{\dot{n}}\}\cdot\{b_{\dot{n}}\}=_{\mathrm{df}}\{a_{\dot{n}}\cdot b_{\dot{n}}\}.$$

α.9.4. THEOREM. The above operations on fundamental sequences, are invariant with respect to the ($\dot{=}$)-equality of μ-fundamental sequences. They are commutative, associative and distributive. They obey the rules of algebra of Stone's ring, where the zero and the unit of the ring are defined as $\{0, 0, ..., 0, ...\}$, $\{1, 1, ..., 1, ...\}$ and denoted $\{0\}$, $\{1\}$ respectively.

Thus the collection of all μ-fundamental sequences is organized into a Stone's ring with ($\dot{=}$) as governing equality.

α.9.5. THEOREM. The corresponding tribe [α.1.2.] is given by the ordering

$$\{a_{\dot{n}}\} \dot{\leq} \{b_{\dot{n}}\} \cdot =_{df} \{a_{\dot{n}}\} \cdot \{b_{\dot{n}}\} \dot{=} \{a_{\dot{n}}\}$$

or equivalently by

$$\{a_{\dot{n}}\} \dot{\leq} \{b_{\dot{n}}\} \cdot \dot{=}_{df} \{a_{\dot{n}}\} + \{b_{\dot{n}}\} \dot{=} \{b_{\dot{n}}\}.$$

This tribe is finitely additive.

Its somata are μ-fundamental sequences and ($\dot{=}$) is its governing equality.

If $\mathfrak{T}$ is the given tribe, we shall denote by $\bar{\mathfrak{T}}$ the new tribe, made of μ-fundamental sequences.

The addition of somata in $\bar{\mathfrak{T}}$ is

$$\{a_{\dot{n}}\} \dot{+} \{b_{\dot{n}}\} \dot{=} \{a_{\dot{n}} + b_{\dot{n}}\}.$$

We also have

$$\text{co}\,\{a_{\dot{n}}\} \dot{=} \{\text{co}\, a_{\dot{n}}\},$$

$$\{a_{\dot{n}}\} - \{b_{\dot{n}}\} \dot{=} \{a_{\dot{n}} - b_{\dot{n}}\},$$

and

$$\{a_{\dot{n}}\} \dot{+} \{b_{\dot{n}}\} \dot{=} \{a_{\dot{n}} \dot{+} b_{\dot{n}}\}.$$

The somata $\{1\}$, $\{0\}$ are the same both in the ring as well as in the tribe $\bar{\mathfrak{T}}$.

α.9.9. The somata $\{a, a, ..., a, ...\}$ of $\bar{\mathfrak{T}}$ — are order — and operation-isomorphic to the somata a of $\mathfrak{T}$.

Thus the tribe $\overline{\mathfrak{T}}$ is a finitely genuine supertribe of $\mathfrak{T}$ through the above isomorphism, [A.7].

The tribe $\overline{\mathfrak{T}}$ will be termed *Cantor-Mac Neille's extension of* $\mathfrak{T}$.

α.10. The tribe $\overline{\mathfrak{T}}$ having been introduced, we are now going to consider measure-circumstances.

α.10.2. THEOREM. If $\{a_n\}$ is a μ-fundamental sequence then $\lim_{n\to\infty} \mu(a_n)$ exists.

PROOF. Let $\{a_n\}$ be a fundamental sequence, and let $\varepsilon > 0$. There exists n_0, such that for every $n \geq n_0$ and every $m \geq n_0$ we have [α.8.1.]:

$$\mu(a_n \dot{+} a_m) \leq \varepsilon.$$

Taking account of the inequality [α.5.3.]:

$$|\mu(a_n) - \mu(a_m)| \leq \mu(a_n \dot{+} a_m),$$

we get

$$|\mu(a_n) - \mu(a_m)| \leq \varepsilon \text{ for } n,\ m \geq n_0.$$

Consequently

$$\lim_{n\to\infty} \mu(a_n) \text{ exists.}$$ Q.E.D.

α.10.3 THEOREM. Iif $\{a_n\}$, $\{b_n\}$ are fundamental sequences, and if

$$\{a_n\} \doteq \{b_n\},$$

then

$$\lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} \mu(b_n).$$

PROOF. Let $\{a_n\}$, $\{b_n\}$ be two equivalent fundamental sequences, then, [α.9.], $\{a_n \dot{+} b_n\}$ is a null-sequence. Hence, [α.8.2.],

$$\mu(a_n \dot{+} b_m) \to 0 \text{ for } n \to \infty.$$

It follows, [α.5.3.],

(1) $$|\mu(a_m)-\mu(b_n)|\to 0 \text{ for } n\to\infty.$$

The limits

$$\lim_{n\to\infty}\mu(a_n), \qquad \lim_{n\to\infty}\mu(b_m)$$

exist, [α.10.2.].

Consequently, by (1),

$$\lim_{n\to\infty}\mu(a_n)=\lim_{n\to\infty}\mu(b_n).$$ Q.E.D.

α.11. DEFINITION. If $\{a_{\dot{n}}\}$ is a μ-fundamental sequence, then the number

$$\bar{\mu}\{a_{\dot{n}}\}=_{df}\lim_{n\to\infty}\mu(a_{\dot{n}})$$

is *termed measure of* $\{a_{\dot{n}}\}$.

α.11.1. THEOREM. If $\{a_{\dot{n}}\}$ is a fundamental sequences, then

$$\bar{\mu}\{a_{\dot{n}}\}\geq 0.$$

The measure $\bar{\mu}$ is invariant with respect to the equivalence $(\doteq)$ of fundamental sequences, [α.10.3.].

The measure $\bar{\mu}$ is not trivial, because $\bar{\mu}\{1, 1, ..., 1, ...\}=\mu(1)>0$.

α.11.2. THEOREM. The measure $\bar{\mu}$ is effective on the tribe $\bar{\mathfrak{T}}$, [α.1.1.].

PROOF. Suppose $\bar{\mu}\{a_{\dot{n}}\}=0$, where $\{a_{\dot{n}}\}$ is a μ-fundamental sequence.

We have, [α.11.],

$$\lim_{n\to\infty}\mu(a_n)=0.$$

Hence $\{a_{\dot{n}}\}$ is a μ-null sequence, [α.8.2.]; hence, [α.9.2.], it is equivalent to the constant sequence $\{0\}$, which is the zero of the tribe $\bar{\mathfrak{T}}$, [α.9.4.].

The theorem is proved.

α.**11.3.** THEOREM. The measure $\bar{\mu}$ is an extension of the measure μ on $\mathcal{T}$ to $\bar{\mathcal{T}}$ through the correspondence

(2) $$a \rightsquigarrow \{a, a, ..., a, ...\}$$

valid for all $a \in \mathrm{O}\,\mathcal{T}$.

PROOF. The correspondence (2) is $1 \to 1$, [α.9.9.], and we have

$$\bar{\mu}\{a, a, ..., a, ...\} = \mu(a), \qquad [\alpha.11].$$

α.**11.4.** THEOREM. For μ-fundamental sequences $\{a_n\}$, $\{b_n\}$ we have

$$\bar{\mu}(\{a_n\} + \{b_n\}) \leq \bar{\mu}\{a_n\} + \bar{\mu}\{b_n\}.$$

PROOF. We have, [α.5.2.],

(3) $$\mu(a_n + b_n) \leq \mu(a_n) + \mu(b_n), \qquad (n = 1, 2, ...).$$

We also have

$$\mu(\{a_n\} + \{b_n\}) = \lim_{n \to \infty} \mu(a_n + b_n), \qquad [\alpha.11.],$$

and

$$\bar{\mu}\{a_n\} = \lim \mu(a_n),$$

$$\bar{\mu}\{b_n\} = \lim \mu(b_n).$$

Hence, from (α), by going to limit, we get:

$$\bar{\mu}(\{a_n\} + \{b_n\}) \leq \bar{\mu}\{a_n\} + \bar{\mu}\{b_n\}. \qquad \text{Q.E.D.}$$

α.**11.5.** THEOREM. The measure $\bar{\mu}$ on $\bar{\mathcal{T}}$ is finitely additive.

PROOF. Let $\{a_n\}$, $\{b_n\}$ be fundamental sequences, and suppose that

$$\{a_n\} \cdot \{b_n\} \doteq \{O\}.$$

We have, [α.9.3.],

$$\{a_{\dot n}\cdot b_{\dot n}\}\doteq\{0\};$$

hence $\{a_{\dot n}\cdot b_{\dot n}\}$ is a null-sequence, [α.8.2.], [α.9.2.]; this gives

$$\lim_{n\to\infty}(a_n\cdot b_n)=0. \tag{4}$$

Now, we have for any n:

$$a_n+b_n=(a_n-a_n\cdot b_n)+a_n\cdot b_n+(b_n-b_n\cdot a_n),$$

where all three terms, on the right, are disjoint.

Since μ is additive, we get

$$\mu(a_n+b_n)=\mu(a_n-a_n\cdot b_n)+\mu(a_n\cdot b_n)+(b_n-a_n\cdot b_n)=$$
$$=\mu(a_n)+\mu(b_n)-\mu(a_n\cdot b_n).$$
$$=\mu(a_n)-\mu(a_n\cdot b_n)+\mu(a_n\cdot b_n)+\mu(b_n)-\mu(a_n\cdot b_n)=$$

Proceeding to the limit, we get, by (4),

$$\lim_{n\to\infty}\mu(a_n\cdot b_n=0);$$

hence

$$\bar\mu\{a_{\dot n}+b_{\dot n}\}=\bar\mu\{a_n\}+\bar\mu\{b_n\}.$$ Q.E.D.

α.11.6. Theorm. If 1. $\{a_{\dot n}\}$, $\{b_{\dot n}\}$ are μ-fundamental sequences,

2. $\{a_{\dot n}\}\leq\{b_{\dot n}\}$,

then

$$\bar\mu\{a_{\dot n}\}\leq\bar\mu\{b_{\dot n}\}.$$

Proof. Put

$$c_n=_{df}\ b_n-a_n, \qquad (n=1,2,\ldots).$$

We have

$$a_n+c_n=a_n+b_n,\ \text{and}\ a_n\cdot c_n=0.$$

Hence, [α.11.5.],

(5) $$\{c_{\dot{n}}\} = \{a_{\dot{n}}\} + \{a_{\dot{n}} + b_{\dot{n}}\}.$$

By hyp. 2 we have

$$\{a_{\dot{n}}\} + \{b_{\dot{n}}\} \doteq \{b_{\dot{n}}\}, \qquad [\alpha.9.5.];$$

hence by (5)

(6) $$\{a_{\dot{n}}\} + \{c_{\dot{n}}\} \doteq \{b_{\dot{n}}\}.$$

Now

$$a_n \cdot c_n = 0;$$

hence

$$\{a_{\dot{n}}\} \cdot \{c_{\dot{n}}\} = \{0\}.$$

Applying [α.11.5.], we get

$$\bar{\mu}\{a_{\dot{n}}\} + \bar{\mu}\{c_{\dot{n}}\} = \bar{\mu}\{b_n\},$$

which implies

$$\bar{\mu}\{a_{\dot{n}}\} \leq \bar{\mu}\{b_{\dot{n}}\}.$$ Q.E.D.

α.11.7. THEOREM. *If* $\{a_{\dot{n}}\} < \{b_{\dot{n}}\}$, which means that $\{a_{\dot{n}}\} \dot{\leq} \{b_{\dot{n}}\}$, but $\{a_{\dot{n}}\}$, $\{b_{\dot{n}}\}$ are ($\doteq$)-different, then

$$\bar{\mu}\{a_{\dot{n}}\} < \bar{\mu}\{b_{\dot{n}}\}.$$

PROOF. By the preceeding theorem, we have

(7) $$\bar{\mu}\{a_{\dot{n}}\} \leq \bar{\mu}\{b_{\dot{n}}\}.$$

Put

$$c_n =_{df} b_{\dot{n}} - a_{\dot{n}}.$$

We have

$$a_n + c_n = a_n + b_n;$$

hence, [α.11.5.],

(8) $$\{a_{\dot{n}}\}+\{c_{\dot{n}}\}\doteq\{a_{\dot{n}}+b_{\dot{n}}\}.$$

By hypothesis we have, [α.9.5.],

$$\{a_{\dot{n}}\}+\{b_{\dot{n}}\}\doteq\{b_{\dot{n}}\};$$

hence by (8) we get

(9) $$\{a_{\dot{n}}\}+\{c_{\dot{n}}\}\doteq\{b_{\dot{n}}\}.$$

We also have

$$a_n \cdot c_n = 0;$$

hence

(10) $$\{a_n\}\cdot\{c_n\}=0.$$

From (9) and (10) we get, by virtue of additivity of the measure $\bar{\mu}$:

(11) $$\bar{\mu}\{b_{\dot{n}}\}=\bar{\mu}\{a_{\dot{n}}\}+\bar{\mu}\{c_{\dot{n}}\}.$$

I say, that we cannot have $\bar{\mu}\{c_{\dot{n}}\}=0$.
Indeed, suppote that $\bar{\mu}\{c_{\dot{n}}\}=0$.
By virtue of effectiveness of the measure $\bar{\mu}$ we would get

$$\{c_{\dot{n}}\}=\{0\}.$$

Taking account of (9), we would get

$$\{a_{\dot{n}}\}\dot{+}\{c_{\dot{n}}\}\doteq\{b_{\dot{n}}\}\doteq\{a_{\dot{n}}\}\dot{+}\{0\}\doteq\{a_{\dot{n}}\},$$

so we would have

$$\{a_{\dot{n}}\}\doteq\{b_{\dot{n}}\},$$

which is excluded by hypothesis.
Thus we have proved that

$$\bar{\mu}\{c_{\dot{n}}\}>0.$$

Taking account of (11) we get

$$\mu\{b_{\dot{n}}\} > \mu\{a_{\dot{n}}\}. \qquad \text{Q.E.D.}$$

α.**11.8.** We have constructed a tribe $\bar{\mathcal{T}}$ whose somata are μ-fundamental sequences and whose governing equality is the equivelence of these sequences. The given tribe $\mathcal{T}$ is a finitely genuine, finitely additive subtribe of $\bar{\mathcal{T}}$, [A.7], through the correspondence

$$\{a, a, ..., a, ...\} \rightsquigarrow a.$$

We have also extended the measure μ on $\mathcal{T}$ into a new one, $\bar{\mu}\{a_{\dot{n}}\}$ of somata of $\bar{\mathcal{T}}$.

We have $\bar{\mu}(a) = \mu\{a, a, ..., a, ...\}$ for all $a \in \mathcal{T}$.

The measure $\bar{\mu}$ has been proved to be effective and finitely additive.

α.**11.9.** Remark. We may remark that, if we use the above extension of tribe and measure, starting with $\mathcal{T}_\mu$ i.e. with $\mathcal{T}_J$ (the tribe $\mathcal{T}$ modulo J, [α.7.6.]), we shall get the extension $\bar{\mathcal{T}}_J$ of the tribe $\mathcal{T}_J$ and measure, which, however coincides with $\bar{\mathcal{T}}$.

α.**12.** Now we are going to prove that the tribe $\bar{\mathcal{T}}$ and the measure $\bar{\mu}$ are both denumerably additive.

The proof will be supplied by means of several lemmas.

α.**12.1.** The applied method of fundamental sequences of somata is general, so it can also be applied to the tribe $\bar{\mathcal{T}}$ and $\bar{\mu}$. The only difference is that the measure μ is effective, while $\bar{\mu}$ may be not.

α.**12.2.** Definition. Thus we put for two fundamental sequences, A, B,

$$\| A, B \| =_{df} \mu(A \dot{+} B),$$

the « *distance* » *between two somata of* $\bar{\mathcal{T}}$.

α.**12.3.** Theorem. This notion is invariant with respect to the equivalence ($\doteq$) of somata of $\mathscr{C}$, [α.9.]; i.e. if $A \doteq A'$, $B \doteq B'$, then

$$\| A, B \| = \| A', B' \| .$$

α.12.4. THEOREM. We have

$$| \bar{\mu}(A) = \bar{\mu}(B) | \leq \| A, B \| .$$

α.12.5. THEOREM. The followings rules are valid:

1) $\| A, A \| = 0$,

2) $\| A, B \| = \| B, A \|$,

3) $\| A, B \| \leq \| A, C \| + \| C, B \|$,

α.12.6. THEOREM. The following are equivalent:

I. $\| A, B \| = 0$.

II. $A \doteq B$.

PROOF. Let II; i.e. $A \doteq B$.
We have

$$A \dotplus B \doteq A \dotplus A,$$

because the algebraic addition is ($\doteq$)-invariant.
As

$$A \dotplus A \doteq \{O\},$$

we get

$$A \dotplus B \doteq \{O\},$$

and then

$$\| A, B \| = \bar{\mu}(A \dotplus \bar{B}) = \mu\{O\} = 0,$$

so I follows.
Now let I, i.e. $\| A, B \| = 0$.
By [α.12.2.] we have

$$\bar{\mu}(A \dotplus B) = 0.$$

Since the measure $\bar{\mu}$ is effective, it follows

$$A \dot{+} B \doteq \{O\}.$$

This gives:

$$A \dot{+} B \dot{+} B \doteq \{O\} \dot{+} B;$$

hence

$$A \dot{+} \{O\} \doteq \{O\} \dot{+} B;$$

hence

$$A \doteq B, \quad \text{i.e. II.}$$

The theorem is established.

α.13. The notion of distance $\|A, B\|$ of two fundamental sequences organizes the tribe $\bar{\mathfrak{T}}$ into a metrice space, owing to the property [α.12.6.].

α.13.1. Definition. In this thopology we can define the *notion of limit of a sequence*

$$(1) \qquad P_1, P_2, \ldots, P_n, \ldots$$

of somata of $\bar{\mathfrak{T}}$ in the usual way:

We say that P a *limit of the sequence* (1), whenever for every $\varepsilon > 0$, there exists and index n_0 such that, for every $n \geq n_0$ we have

$$\|P_n, P\| \leq \varepsilon.$$

α.13.2 Theorem. If a sequence (1) possesses a limit, this limit is $(\doteq)$-unique.

Proof. Suppose that P', P'' are limits of the sequence $P_1, P_2, \ldots, P_n$.

Choose $\varepsilon > 0$ and find n_0 such that for all $n \geq n_0$ we have

$$(2) \qquad \|P_n, P'\| < \varepsilon$$

Also find an index m_o such that, for all $n \geq m_o$,

(3) $$\| P_n, P'' \| \leq \varepsilon.$$

It follows that, for all $n \geq \max(n_o, m_o)$, we have valid both the inequalities (2) and (3).

We get, [α.12.5.]:

$$\| P', P'' \| \leq \| P', P_n \| + \| P_n, P'' \| \leq 2\varepsilon.$$

This being true for all $\varepsilon > 0$, it follows

$$\| P', P'' \| = 0.$$

Hence, by [α.12.6.] we obtain

$$P' \doteq P''.$$ Q.E.D.

α.13.3. DEFINITION. The limit P of the sequence $P_1, P_2, \ldots, P_n, \ldots$ *will be denoted*

$$\operatorname{Lim}_{n \to \infty} P_n .$$

So we have

$$P = \operatorname{Lim}_{n \to \infty} P_n .$$

α.13.4. THEOREM. The finite operations on somata of $\bar{\mathfrak{T}}$ are continuous in the topology on $\bar{\mathfrak{T}}$.

PROOF. Let

$$P \doteq \operatorname{Lim} P_n ,$$

$$Q \doteq \operatorname{Lim} Q_n .$$

Take $\varepsilon > 0$ and find n_o such that, for all $n \geq n_o$, we have

$$\| P, P_n \| \leq \frac{\varepsilon}{2}, \qquad \| Q, Q_n \| \leq \frac{\varepsilon}{2}.$$

It follows that, [α.5.]:

$$\| P+Q,\ P_n+Q_n \| \leq \| P,\ P_n \| + \| Q.\ Q_n \| \leq \varepsilon.$$

This completes the proof of the continuity of the somatic addition in $\overline{\mathfrak{T}}$.

α.13.4a. Let

$$P \doteq \operatorname*{Lim}_{n\to\infty} P_n .$$

Take $\varepsilon > 0$ and find n_0 such that for all $n \geq n_0$:

(1) $$\| P_n ,\ P \| \leq \varepsilon.$$

We have in general

$$\| A,\ B \| = \| \mathrm{co}\, A,\ \mathrm{co}\, B \| ,$$

Hence from (4) it follows

$$\| \mathrm{co}\, P_n ,\ \mathrm{co}\, P \| \leq \varepsilon,$$

which completes the proof of continuity of the operation of complementation.

α.13.4b. To prove the continuity of all other finite operations, we notice that

$$A \cdot B \doteq \mathrm{co}\,(\mathrm{co}\, A + \mathrm{co}\, B),$$

$$A - B \doteq A \cdot \mathrm{co}\, B,$$

$$A \dotplus B \doteq (A-B)+(B-A),$$

and apply [α.13.4.] and [α.13.4a.].

α.13.5. THEOREM. The measure $\bar{\mu}(\dot{P})$ on $\overline{\mathfrak{T}}$ is a continuous function of $\dot{P}$ in the $\overline{\mathfrak{T}}$-topology.

PROOF. Let

$$P \doteq \operatorname*{Lim}_{n\to\infty} P_n .$$

We have for all n:

(1.1) $$|\bar{\mu}(P)-\bar{\mu}(P_n)|\leq\bar{\mu}(P\dot{+}P_n),\qquad [\alpha.5.3.].$$

Let $\varepsilon>0$. Find n_o such that for all $n\geq n_o$

$$\|P, P_n\|\leq\varepsilon.$$

Now we have

$$\|P, P_n\|=\bar{\mu}(P\dot{+}P_n).$$

Applying (1) we get

$$|\bar{\mu}(P)-\bar{\mu}(P_n)|\leq\|P, P_n\|\leq\varepsilon$$

for all $n\geq n_o$. Thus the continuity is established.

α.14. DEFINITION. *Let* $P_1, P_2, \ldots, P_n, \ldots$ be a sequence of somata of $\bar{\mathfrak{T}}$. We say that the *sequence satisfies the Cauchy-condition*, whenever for every $\varepsilon>0$ there exists an index n_o such that, for every $n'\geq n_o$ and $n''\geq n_o$, we have

$$\|P_{n'}, P_{n''}\|\leq\varepsilon.$$

α.14.1. THEOREM. If $\operatorname{Lim} P_n$ exists, then the sequence $P_1, P_2, \ldots, P_n, \ldots$ satisfies the Cauchy-condition.

PROOF. Take $\varepsilon>0$. Let

$$P\doteq\operatorname{Lim}_{n\to\infty} P_n.$$

Find n_o such that for every $n\geq n_o$ we have

$$\|P_o, P\|\leq\frac{\varepsilon}{2}.$$

Take any indices $n', n''\geq n_o$.
We have

$$\|P_{n'}, P\|\leq\frac{\varepsilon}{2},$$

$$\|P_{n''}, P\|\leq\frac{\varepsilon}{2}.$$

We get

$$\| P_{n'}, P_{n''} \| \leq \| P_{n'}, P \| + \| P_{n''}, P \| \leq \varepsilon,$$

which completes the proof.

α.14.2. DEFINITION. Infinite sequence P_n of somata of $\bar{\mathfrak{T}}$ satisfying the Cauchy-condition may be termed *fundamental sequences in* $\bar{\mathfrak{T}}$.

α.14.3. THEOREM. A constant sequence P, P, ..., P, ..., where $P_n \in \bar{\mathfrak{T}}$, possesses the limit P; hence it is a fundamental sequence in $\bar{\mathfrak{T}}$.

α.14.4. LEMMA. Let

$$A =_{df} \{a_1, a_2, ..., a_n, ...\}$$

be a fundamental sequence of somata of $\mathfrak{T}$.

We shall consider the following constant fundamental sequences:

$$A^1 =_{df} \{a_1, a_1, ..., a_1, ...\},$$

$$A^2 =_{df} \{a_2, a_2, ..., a_2, ...\},$$

. . . . .

$$A^m =_{df} \{a_m, a_m, ..., a_m, ...\}.$$

We shall prove that

$$\operatorname*{Lim}_{m \to \infty} A^m = A.$$

PROOF. Let $\varepsilon > 0$; find and index n_o, such that for any $n \geq n_o$ and any $k \geq n_o$, we have

(6) $$\mu(a_n \dot{+} a_k) \leq \varepsilon.$$

Keep n fixed and vary k. We have the sequence

$$a_n \dot{+} a_1, a_n \dot{+} a_1, ..., a_n \dot{+} a_k, ...$$

This is a fundamental sequence, because so is

$$\{a_n, a_n, ..., a_n, ...\}$$

and

$$\{a_1, a_2, ..., a_k, ...\}$$

too.

From (6) we get

$$\lim_{k\to\infty} \mu\{a_n \dot{+} a_k\} \leq \varepsilon,$$

[4], for all $n \geq n_0$.

Now let us vary $n=1, 2, ...$. We get

$$\overline{\lim_{n\to\infty}} \lim_{k\to\infty} \mu\{a_n \dot{+} a_k\} \leq \varepsilon.$$

This being true for all $\varepsilon > 0$, it follows

$$\overline{\lim_{n\to\infty}} \lim_{k\to\infty} \mu\{a_n \dot{+} a_k\} = 0;$$

hence

$$\lim_{n\to\infty} \lim_{k\to\infty} \mu\{a_n \dot{+} a_k\} \equiv 0,$$

because the measure μ is non negative.

It follows, by [α.11.], that

$$\lim_{n\to\infty} \bar{\mu}\{a_n \dot{+} a_1, a_n \dot{+} a_2, ...\} = 0,$$

$$\lim_{n\to\infty} \bar{\mu}[\{a_n, a_n, ...\} \dot{+} \{a_1, a_2, ...\}] = 0,$$

i.e.

$$\lim_{n\to\infty} \bar{\mu}\{A^n + A\} = 0.$$

This says that given $\sigma > 0$, we can find an index m_0, such that for every $n \geq n_0$;

$$\bar{\mu}\{A^n \dot{+} A\} \leq \sigma,$$

i.e.

$$\| A^n, A \| \leq \sigma$$

for all $n \geq n_0$. Hence

$$\operatorname*{Lim}_{n\to\infty} A^n \doteq A. \qquad \text{Q.E.D.}$$

α.**14.5.** THEOREM. If the sequence $A_1, A_2, \ldots, A_n, \ldots$ of somata of $(\bar{\mathfrak{T}})$ satisfies the Cauchy-condition, then the limit

$$\operatorname*{Lim}_{n\to\infty} A_n$$

exists, and is a soma of $(\bar{\bar{\mathfrak{T}}})$.

PROOF. Let

$$A_1 = \{a_1^1, a_2^1, \ldots, a_n^1, \ldots\} = \{a_n^1\},$$

$$A_2 = \{a_1^2, a_2^2, \ldots, a_n^2, \ldots\} = \{a_n^2\},$$

. . . . .

$$A_k = \{a_1^k, a_2^k, \ldots, a_n^k, \ldots\} = \{a_n^k\},$$

. . . . .

By virtue of the lemma [α.14.4.],

$$A_k = \operatorname*{Lim}_{n\to\infty} \{a_n^k, a_n^k, \ldots, a_n^k, \ldots\}, \ (k=1, 2, \ldots), \text{ i.e.}$$

the sequence of constant fundamental sequences

$$\{a_1^k, a_1^k, \ldots, a_1^k, \ldots\}, \{a_2^k, a_2^k, \ldots, a_2^k, \ldots\}, \ldots \{a_n^k, a_n^k, \ldots, a_n^k, \ldots\}, \ldots$$

tends to a_k.

Take $\varepsilon > 0$. By definition of Lim there exists, for every k, an index $n(k)$, such that

(1) $$\| A_k, \{a^k_{n(k)}, a^k_{n(k)}, \ldots, a^k_{n(k)}, \ldots\} \| \leq \frac{\varepsilon}{k}.$$

Put

(2) $$b(k) =_{df} a^k_{n(k)}, \ (k=1, 2, \ldots)$$

for all k. We get

(3) $$\| A_k , \{b(k), b(k), ..., b(k), ...\} \| \leq \frac{\varepsilon}{k},$$

and if we define

$$B_k =_{df} /^{(k)}\{b(k), b(k), ..., b(k), ...\} \text{ for all } k,$$

we get

(3.1) $$\| A_k , B_k \| \leq \frac{\varepsilon}{k} \text{ for all k.}$$

Now we have supposed that the sequence $A_1, A_2, ..., A_k, ...$ satisfies the Cauchy-condition. Therefore there exists k_o such that if $k' \geq k_o$ and $k'' \geq k_o$, we get

(4) $$\| A_{k'} , A_{k''} \| \leq \varepsilon.$$

From (3) we get

(5) $$\| A_{k'} , B_{k'} \| \leq \frac{\varepsilon}{k'}$$

and

(6) $$\| A_{k''} , B_{k''} \| \leq \frac{\varepsilon}{k''} \text{ for all } k', '' \geq k_o .$$

We have

$$\| B_{k'} , B_{k''} \| \leq \| B_{k'} , A_{k'} \| + \| A_{k'} , A_{k''} \| + \| A_{k''} , B_{k''} \|;$$

hence, by (4), (5), (6),

$$\| B_{k'} , B_{k''} \| < \frac{\varepsilon}{k'} + \varepsilon + \frac{\varepsilon}{k''};$$

hence

(7) $$\| B_{k'} , B_{k''} \| \leq 3\varepsilon \text{ for all } k', k'' \geq k_o .$$

Since $B_{k'}$ and $B_{k''}$ are constant fundamental sequences, it follows that $B_{k'} \dotplus B_{k''}$ is also a constant fundamental sequence.

We have

$$B_{k'}=\{b(k'),\ b(k'),\ \dots,\ b(k'),\ \dots\},$$

$$B_{k''}=\{b(k''),\ b(k''),\ \dots,\ b(k''),\ \dots\};$$

hence

$$B_{k'}\dot{+}B_{k''}=\{b(k')\dot{+}b(k''),\ b(k')\dot{+}b(k''),\ \dots\},$$

and then

$$\bar{\mu}(B_{k'}\dot{+}B_{k''})=\bar{\mu}\{b(k')+b(k''),\ b(k')\dot{+}b(k''),\ \dots\}=$$
$$=\mu\{b(k')\dot{+}b(k'')\}.$$

Hence, by (7),

$$\mu\{b(k')\dot{+}b(k'')\}\leq 3\varepsilon;$$

hence

$$|\, b(k'),\ b(k'')\,|\leq 3\varepsilon, \text{ for all } k',\ k''\geq k_o\,,$$

which implies that

$$B=_{df}\{b(1),\ b(2),\ \dots,\ b(k),\ \dots\}$$

is a fundamental sequence.

Now we can see that $\operatorname*{Lim}_{n\to\infty} A_n=B$, so that the limit $\operatorname*{Lim}_{n\to\infty} A_n$ exists.

Indeed, we have, by the Lemma [α.14.4.]:

$$B=\operatorname*{Lim}_{n\to\infty} B_n\,;$$

hence

$$\|\, B,\ B_n\,\|\leq\varepsilon$$

for sufficiently great index n.

We also have for sufficientl great index, (5),

$$\|\, A_n\,,\ B_n\,\|\leq\varepsilon.$$

Hence

$$\|\, A_n\,,\ B\,\|\leq\|\, A_n\,,\ B_n\,\|+\|\, B_n\,,\ B\,\|\,.$$

Consequently

$$\| A_n , B \| \leqq 2\varepsilon$$

for sufficiently great index n.

This proves that

$$\lim_{n\to\infty} A_n = B.$$

The theorem is established.

α.14.6. The last theorem shows that the $\bar{\mathfrak{T}}$-topology is complete.

α.14.7. One can see that if starting with the tribe $\bar{\mathfrak{T}}$, instead of $\mathfrak{T}$, we can apply the above method of completing, considering $\bar{\mathfrak{T}}$-fundamental sequences, the tribe $\bar{\mathfrak{T}}$ will be not essentially extended; this extension will be only illusionary: it will give a tribe which is isomorphic and isometric with $\bar{\mathfrak{T}}$.

α.15. Till now have proved that the tribe $\bar{\mathfrak{T}}$ is finitely additive and that the measure $\bar{\mu}$ on it is also finitely additive.

Now we are going to prove that $\bar{\mathfrak{T}}$ is denumerably additive i.e. that if A_1, A_2, A_n, ... are somata of $\bar{\mathfrak{T}}$, then the lattice union $\sum_{n=1}^{\infty} A_n$ exists (in $\bar{\mathfrak{T}}$).

α.15.1. THEOREM. We have already proved that if

(1) $$A_1, A_2, \ldots, A_n, \ldots$$

are somata of $\bar{\mathfrak{T}}$, i.e. fundamental sequences in $\mathfrak{T}$, then the following are equivalent:

I. $\{A_1, A_2, \ldots, A_n, \ldots\}$ is a μ-fundamental sequence, in $\bar{\mathfrak{T}}$.

II. $\lim_{n\to\infty} A_n$ exists, and is a soma of $\bar{\mathfrak{T}}$.

III. The sequence (1) satisfies the Cauchy's condition:

For exery $\varepsilon > 0$, there exists n_0 such that, if $n' \geq n_0$ and $n'' \geq n_0$,

then

$$\| A_{n'}, A_{n''} \| = \bar{\mu}(A_{n'} \dotplus A_{n''}) \leq \varepsilon.$$

We shall state some properties of « Lim »:

α.**15.2.** THEOREM. If $\{A_n'\}$, $\{A_n''\}$ are fundamental sequences in $\bar{\mathfrak{T}}$, and if for every n we have

$$A_n' \doteq A_n'', \qquad \text{i.e. } \bar{\mu}(A_n' \dotplus A_n'') = 0,$$

then

$$\operatorname*{Lim}_{n\to\infty} A_n' \doteq \operatorname*{Lim}_{n\to\infty} A_n''.$$

α.**15.3.** THEOREM. If $\operatorname*{Lim}_{n\to\infty} \doteq B$ and $B \doteq B'$, then $\operatorname*{Lim}_{n\to\infty} A_n \doteq B'$.

α.**15.4.** THEOREM. If

$$\operatorname*{Lim}_{n\to\infty} A_n \doteq B$$

and

$$\operatorname*{Lim}_{n\to\infty} A_n \doteq C,$$

then $B \doteq C$, [α.13.2.].

α.**15.5.** THEOREM. If A_n is a constant sequence $\{A, A, ..., A, ...\}$ in $\bar{\mathfrak{T}}$, then

$$\operatorname*{Lim}_{n\to\infty} A_n \doteq A.$$

α.**15.6.** THEOREM. If

$$\operatorname*{Lim}_{n\to\infty} A_n \doteq A,$$

then

$$\operatorname*{Lim}_{n\to\infty} (\text{co } A_n) \doteq \text{co } A.$$

α.**15.7.** THEOREM. If

$$\operatorname*{Lim}_{n\to\infty} A_n \doteq A,$$

$$\operatorname*{Lim}_{n\to\infty} B_n \doteq B,$$

then

$$\operatorname*{Lim}_{n\to\infty}(A_n+B_n)\doteq A+B,$$

$$\operatorname*{Lim}_{n\to\infty}(A_n\cdot B_n)\doteq A\cdot B,$$

$$\operatorname*{Lim}_{n\to\infty}(A_n-B_n)\doteq A-B,$$

$$\operatorname*{Lim}_{n\to\infty}(A_n\dot{+}B_n)\doteq A\dot{+}B.$$

α.16.8. THEOREM.

$$\text{Lim}\,\{\bar{0},\bar{0},\ldots,\bar{0},\ldots\}\doteq\bar{0}\ \text{i.e.}\ \{0\}$$

$$\text{Lim}\,\{\bar{1},\bar{1},\ldots,\bar{1},\ldots\}\doteq\bar{1},\ \text{i.e.}\ \{1\}.$$

α.16.9. THEOREM. If

$$A_n\dot{\leq}B_n,\qquad (n=1,\ 2,\ \ldots)$$

and

$$\operatorname*{Lim}_{n\to\infty}A_n\doteq A,\qquad \operatorname*{Lim}_{n\to\infty}B_n\doteq B,$$

then $A\dot{\leq}B$.

PROOF.

$$A_n\dot{\leq}B_n$$

is equivalent to

$$A_n+B_n\doteq B_n\,.$$

Hence,

$$\operatorname*{Lim}_{n\to\infty}A_n+\operatorname*{Lim}_{n\to\infty}B_n\doteq\operatorname*{Lim}_{n\to\infty}B_n\,.$$

i.e.

$$A+B\doteq B,$$

which implies

$$A\dot{\leq}B.$$

Q.E.D.

α.**16.10.** LEMMA. If

1. $P =_{df} \{p_1, p_2, ..., p_n, ...\}$,

   $Q =_{df} \{q_1, q_2, ..., q_n, ...\}$

are somata of $\bar{\bar{\mathscr{C}}}$,

2. $\{p_1, p_1, ..., p_1, ...\} \dot{\leq} Q$,

   $\{p_2, p_2, ..., p_2, ...\} \dot{\leq} Q$,

   . . . . .

   $\{p_n, p_n, ..., p_n, ...\} \dot{\leq} Q$,

   . . . . . ,

then

$$P \dot{\leq} Q.$$

Put

$$\{p_k, p_k, ..., p_k, ...\}, \qquad (k=1, 2, ...).$$

We have, by hypothesis,

$$P \dot{\leq} Q \text{ for all } k.$$

Applying [α.16.9.], we get

$$\operatorname*{Lim}_{k\to\infty} P_k \dot{\leq} \operatorname{Lim} \{Q, Q, .., Q, ...\};$$

hence

$$\operatorname*{Lim}_{k\to\infty} P_k \dot{\leq} Q.$$

Applying the lemma [α.14.4.], we obtain

$$P \dot{\leq} Q. \qquad \text{Q.E.D.}$$

α.**16.11.** LEMMA. If

1. $p_n \in \mathscr{C}$, $\qquad (n=1, 2, ...)$,
2. $p_1 \leq p_2 \leq ... \leq p_n, ...$,

then

$$\{p_1, p_2, \ldots, p_n, \ldots\} \in \bar{\mathfrak{T}},$$

i.e. it is a $\mathfrak{T}$-fundamental sequence.

PROOF. We have

$$\mu(p_1) \leq \mu(p_2) \leq \ldots \leq \mu(p_n) \leq \ldots \tag{1}$$

That sequence is bounded, because $\mu(p_n) \leq \mu(1)$ for all $n=1, 2, \ldots$ and the terms $\mu(p_n)$ are all non-negative.

Take $\varepsilon > 0$. There exists n_o such that if $n^2 \geq n_o$, $n'' \geq n'$, we have

$$\mu(p_{n''}) - \mu(p_{n'}) \leq \varepsilon.$$

It follows

$$\mu(p_{n''} - p_{n'}) \leq \varepsilon.$$

As $p_{n'} - p_{n''} = 0$, we get

$$\mu(P_{n''} - p_{n'}) + \mu(p_{n'} - {}_{n''}) < \varepsilon.$$

The somata $p_{n''} - p_{n'}$, $p_{n'} - {}_{n''}$ being disjoint, it follows

$$\mu[(p_{n''} - p_{n'}) + (p_{n'} - p_{n''})] \leq \varepsilon,$$

i.e.

$$\mu(p_{n'} \dotplus p_{n''}) \leq \varepsilon$$

for all

$$n', n'' \geq n_o,$$

Hence $\{p_{\dot{n}}\}$ is a $\mathfrak{T}$-fundamental sequence; hence it is a soma of $\bar{\mathfrak{T}}$. The lemma is established.

α.16.12. LEMMA. If

1. $p_1, p_2, \ldots, p_n, \ldots$
2. $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots,$
3. $P_k =_{df} \{p_k, p_k, \ldots, p_k, \ldots,\}$ $(k=1, 2, \ldots),$

then we have, for the $\mathcal{C}$-fundamental sequence, [α,16.11.],

$$P =_{df} \{p_1, p_2, ..., p_n, ...\},$$

the inequalties $P_k \dot{\leq} P$ for every $k = 1, 2, ...$.

PROOF. The thesis of the theorem is equivalent to the statement:

$$P_k \cdot P \dot{=} P_k \text{ for all } k = 1, 2, ...$$

Hence it is also equivalent to

(1) $$\lim_{n\to\infty} \mu((p_k \cdot p_n \dot{+} p_k) = 0 \text{ for all } k.$$

Now we have for all n and k:

$$p_k \cdot p_n \dot{+} p_k = (p_k p_n - p_k) + (p_k - p_k \cdot p_n).$$

Hence

$$p_k \cdot p_n \dot{+} p_k = p_k - p_k p_n ,$$

because

$$p_k \cdot p_n - p_k = 0,$$

hence

$$p_k \cdot p_n \dot{+} p_k = p_k - p_n .$$

Consequently the thesis is equivalent to:

(2) $$\lim_{n\to\infty} \mu(p_k - p_n) = 0 \text{ for all } k.$$

We can prove that (2) is true.
Indeed let us fix k and vary n.
We have for all $n > k$:

$$p_k - n_n = 0,$$

by hypothesis 2.
Hence

$$\mu(p_k - p_n = 0 \text{ for } n > k,$$

which gives

$$\lim_{n\to\infty} \mu(p_k - p_n) = 0,$$

so (2) is true, and then the thesis is true.

α.16.13. LEMMA. If

1. $a_1, a_2, \ldots, a_n, \ldots$ are somata of $\mathfrak{T}$;
2. $b_1 =_{df} a_1$, $b_2 =_{df} a_1 + a_2$, ..., $b_n =_{df} a_1 + a_2 + \ldots + a_n$, ...;
3. B is the sequence $b_1, b_2, \ldots, b_n, \ldots$;
4. $A_k =_{df} \{a_k, a_k, \ldots, a_k, \ldots\}$,

then

1) B is a $\mathfrak{T}$-fundamental sequence, i.e. a soma of $\bar{\mathfrak{T}}$;

2) the somatic union $\sum_{k=1}^{\infty} A_k$ exists in $\bar{\mathfrak{T}}$;

3) $\sum_{k=1}^{\infty} A_k \doteq B$.

PROOF. Put $B_k =_{df} \{b_k, b_k, \ldots\}$, By virtute of Lemma [$\alpha$.16.11] the sequence $\{b_1, b_2, \ldots, b_n, \ldots\}$ is a $\mathfrak{T}$-fundamental sequence, i.e. a soma of $\bar{\mathfrak{T}}$. Hence, [α.16.12.],

(3) $$B_k \dot{\leq} B, \qquad (k = 1, 2, \ldots).$$

Having that, suppose that for a given soma Q of $\bar{\mathfrak{T}}$ we have

(4) $$B_k = \{b_k, b_k, \ldots\} \dot{\leq} Q \text{ for every } k = 1, 2, \ldots$$

Since B is a soma of $\bar{\mathfrak{T}}$, and we have $B_k \dot{\leq} Q$ for $k = 1, 2, \ldots$ and also we have

$$\operatorname{Lim}_{k\to\infty} B_k \doteq B.$$

Hence, by [α.16.9.],

$$\operatorname{Lim}_{k\to\infty} B_k \dot{\leq} Q,$$

Thus we have the statements:

(5) $$B_k \leq B, \text{ fora all } k;$$

and if $B_k \leq Q$ for all k, then $B \leq Q$.

It follows, by virtue of defnition of lattice-union of somata, that the denumerable somatic sum

$$B_1 + B_2 + \ldots + B_k + \ldots$$

exists in $\bar{\mathfrak{T}}$, and that

$$B \doteq B_1 + B_2 + \ldots + B_k + \ldots .$$

We have

$$B_1 = \{b_1, b_1, \ldots\} = \{a_1, a_1, \ldots\} = A_1$$

$$B_2 = \{b_2, b_2, \ldots\} = \{a_1 + a_2, a_1 + a_2, \ldots\} = A_1 + A_2,$$

$$\cdots\cdots$$

$$B_k = \{b_k, b_k, \ldots\} = \{a_1 + \ldots + a_k, a_1 + \ldots + a_k, \ldots\} = A_1 + A_2 + \ldots + A_k$$

$$\cdots\cdots$$

Hence

$$B_k = A_1 + A_2 + \ldots + A_k, \qquad (k = 1, 2, \ldots).$$

It follows, by [A], that

$$B \doteq A_1 + A_2 + \ldots + A_n + \ldots .$$

The theorem is established.

α.17. REMARK. Till now we have proved that the infinite union in $\bar{\mathfrak{T}}$

$$A_1 + A_2 \ldots + A_n + \ldots$$

exists for somata A_k of $\mathfrak{T}$, which have the form

$$A_k = \{a_k, a_k, \ldots\},$$

where $a_k \in \bar{\mathfrak{T}}$.

Now we shall prove that the union exists for any somata $\mathcal{T}$.

α.**17.1.** Starting with $\mathcal{T}$ and μ, we have constructed μ-fundamental sequences, getting another tribe $\bar{\mathcal{T}}$, whose somata are $\mathcal{T}$-μ-fundamental sequences.

Now we can apply to $\bar{\mathcal{T}}$ a similar construction, getting a tribe $\bar{\bar{\mathcal{T}}}$, whose somata are $\bar{\mu}$-fundamental sequences of somata of $\bar{\mathcal{T}}$,

$$\{A_1, A_2, \ldots, A_n, \ldots\}.$$

Denoting by capitals A, B the somata of $\bar{\mathcal{T}}$ and by fat capitals **A**, **B** - the somata of $\bar{\bar{\mathcal{T}}}$, consider an infinite sequence

$$(1) \qquad A', A'', \ldots, A^n, \ldots$$

of somata of $\bar{\bar{\mathcal{T}}}$. Let us write (1) in the form

$$\{a_1^{(1)}, a_2^{(1)}, \ldots\}, \{a_1^{(2)}, a_2^{(2)}, \ldots\}, \ldots$$

We can prove that the somatic union

$$(2) \qquad \{A^1, A^1, \ldots\}+\{A^2, A^2, \ldots\}+\ldots+\{A^n, A^n, \ldots\}+\ldots$$

exists in $\bar{\bar{\mathcal{T}}}$

Let us write (2) in the form

$$\mathbf{A}^1+\mathbf{A}^2+\ldots$$

Now between $\bar{\mathcal{T}}$ and $\bar{\bar{\mathcal{T}}}$ there is the one-to one isomorphic correspondence, defined by a new notion Lim of the limit. We have

$$A^1 \rightsquigarrow \mathbf{A}^1, \ A^2 \rightsquigarrow \mathbf{A}^2, \ \ldots$$

It follows, by (2), that the somatic union

$$A^1+A^2+\ldots$$

exists in $\bar{\bar{\mathcal{T}}}$.

α.**17.2.** THEOREM. We have proved that the tribe $\bar{\mathcal{T}}$ is denumerably additive, i.e. if $A_1, A_2, \ldots$ is an infinite sequence of somata of $\bar{\mathcal{T}}$, then the somatic (lattice) union exists in $\bar{\mathcal{T}}$.

α.**18.** Let us go over to measure circumstances. We have the theorem:

THEOREM. If

1. $A_1, A_2, \ldots, A_n, \ldots \in \mathfrak{a}\overline{\mathfrak{T}}$,
2. they are disyoint, i.e.

$$_1 \cdot A_k \doteq 0 \text{ for } i \neq k,$$

3. $A \doteq A_1 + A_2 + \ldots + A_n + \ldots$

then

$$\bar{\mu}A = \bar{\mu}(A_1) + \bar{\mu}(A_2) + \ldots + \bar{\mu}(A_n) + \ldots$$

PROOF. We take over the notations of the preceeding proof.

$$\sum_{k=1}^{\infty} B_k \doteq \operatorname*{Lim}_{n \to \infty} B_n .$$

Put

$$B_n =_{df} A_1 + A_2 + \ldots + A_n .$$

We get

$$\sum_{k=1}^{\infty} B_k = \operatorname*{Lim}_{n \to \infty} \bar{\mu}(B_n),$$

because $\bar{\mu}$ is continuous.

Since

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} A_k ,$$

we get

$$\bar{\mu} \sum_{k=1}^{\infty} A_k = \operatorname*{Lim}_{n \to \infty} \bar{\mu}(A_1 + \ldots + A_n) = \operatorname*{Lim}_{n \to \infty} [\bar{\mu}A_1) + \ldots + \bar{\mu}A_n)],$$

because the somata A_n are disjoint and $\bar{\mu}$ is finitely additive.

Thus

$$\bar{\mu} \sum_{k=1}^{\infty} A_n = \sum_{k=1}^{\infty} \bar{\mu}(A_n). \qquad \text{Q.E.D.}$$

α.19. We have proved that the tribe $\bar{\mathfrak{T}}$ is d e n u e m e r a b l y a d d i t i v e , but we can have more, by proving that it is completely additive.

Indeed. Wecken has proved the following general theorem [AI.1.1]: If

1. $\mathfrak{G}$ is a denumerably additive, not trivial tribe;
2. $\mathfrak{G}$ admits a denumerably additive not trivial measure $\sigma(\dot{a}) \geq 0$;
3. The measure $\sigma(\dot{a})$ is effective i.e. if $\sigma(a)=0$, then $a=0$,

then

1) $\mathfrak{G}$ is completely additive i.e. if M is any not empty collection of somata of $\mathfrak{G}$, then

$$\sum_{a \in M} a \in \mathfrak{G};$$

this even is true when the collection M is non denumerable.

2) if M is any non empty collection of somata of $\mathfrak{G}$, then there exists an infinite sequence p_1, p_2, ..., p_n, ... of somata of M, such that

$$\sum_{a \in M} A = \sum_{n=1}^{\infty} p_n .$$

Applying that theorem to our case $\bar{\mathfrak{T}}$, we see that $\bar{\mathfrak{G}}$ is completely additive.

α.19.1. Let us make the following remark:

The tribe $\mathfrak{T}_J$, [A], cannot be considered as an extension of $\mathfrak{T}$, because some somata of $\mathfrak{T}$ are considered as $(=^J)$-equal; hence the situation in $\mathfrak{T}_J$ looks like reducing the number of somata.

Now we see that the tribes $\bar{\mathfrak{T}}$ and $\bar{\mathfrak{T}}_J$, made of fundamental sequences, are composed of the same somata of $\bar{\mathfrak{T}}$, though with differently defined equalities.

But the tribe $\bar{\mathfrak{T}}$ contains elements which do not exist in $\mathfrak{T}_J$; thus $\bar{\mathfrak{T}}$ can be considered as a true extension of $\mathfrak{T}_J$.

α.19.2. It will be interesting to apply the obtained results to a particular case, where the somata of $\mathfrak{T}$ are some subsets of a given variety $\mathfrak{M}$.

Especially let $\mathfrak{M}$ be the set of all points of the half open interval

$$(0,\ 1\rangle =_{\mathrm{df}} \{x \mid 0 < x \leq 1\},$$

hence of a set of numbers.

Let $\mathfrak{T}$ be the collection of all finite unions

$$(\alpha_1,\ \beta_1\rangle + (\alpha_2,\ \beta_2\rangle + \ldots + (\alpha_n,\ \beta_n\rangle, \qquad (n = 1,\ 2,\ \ldots),$$

of half open subintervals of $(0,\ 1\rangle$, i.e.

$$(\alpha_k,\ \beta_k\rangle = \{x \mid \alpha_k < x \leq \beta_k\}.$$

Notice that

$$(\alpha_k,\ \beta_k\rangle \neq \emptyset, \quad \text{(empty set)},$$

whenever $\alpha_k < \beta_k$.

Let the ordering in $\mathfrak{T}$ be the inclusion of sets: $p \leq q$.

We see that $\mathfrak{T}$ is a finitely additive tribe, whose somata are some subsets of $(0,\ 1\rangle$.

We define on $\mathfrak{T}$ the following finitely additive measure $\mu(p)$:

If

$$p = (\alpha_1,\ \beta_1\rangle + (\alpha_2,\ \beta_2\rangle + \ldots + (\alpha_n,\ \beta_n\rangle,$$

with disjoint terms, we define

$$\mu(p) =_{\mathrm{df}} \sum_{k=1}^{n} [f(\beta_k) - f(\alpha_k)],$$

where $f(x)$ is an increasing and bounded real number-valued function of the variable point x in $(0,\ 1\rangle$.

Thus if

$$x_1 < x_2 \in (0,\ 1\rangle,$$

then

$$f(x_1) < f(x_2).$$

We see that the measure $\mu \geq 0$ is effective and finitely additive.

α.19.3. We shall consider the following example of the function f(x):

Let

$$f(x) =_{df} x \text{ for } x \in (0, \tfrac{1}{2}\rangle,$$
$$f(x) =_{df} \tfrac{1}{2}x + \tfrac{1}{2} \text{ for } x \in (\tfrac{1}{2}, 1\rangle.$$

We shall prove that there does not exist any correspondence Φ between the somata of $\bar{\mathfrak{T}}$ and subsets of the interval (0, 1⟩ which satisfies the following conditions:

1) $\Phi\{(\alpha, \beta\rangle, (\alpha, \beta\rangle, \ldots, (\alpha, \beta\rangle, \ldots\} = (\alpha, \beta\rangle$

for all α, β in (0, 1⟩,

2) If A, B are somata $\bar{\mathfrak{T}}$ i.e. fundamental sequences in $\mathfrak{T}$, with $A \cdot B = \bar{O}$, then $\Phi(A) \cdot \Phi(B) = \emptyset$.

3) If $A^1 \dot{\leq} A^2 \dot{\leq} \ldots \dot{\leq} A^n \dot{\leq} \ldots$ are somata of $\bar{\mathfrak{T}}$, with

$$\sum_{n=1}^{\infty} A^n \dot{=} A,$$

then

$$\Phi(A^1) \dot{\leq} \Phi(A^2) \leq \ldots$$

with

$$\sum_{n=1}^{\infty} \Phi(A^n) \dot{=} \Phi(A).$$

4) If $A^1 \dot{\geq} A^2 \dot{\geq} \ldots \dot{\geq} A^n \dot{\geq} \ldots$ with

$$\sum_{n=1}^{\infty} A^n \dot{=} B,$$

then

$$\Phi(A^1) \geq \Phi(A^2) \geq \ldots \geq \Phi(A^n) \geq \ldots,$$

with

$$\sum_{n=1}^{\infty} \Phi(A^n) = \Phi(B).$$

Let us remark that sets making up the region of the correspondence Φ, should be understood as sets taken modulo the ideal of null sets.

These entities are sets, but they are provided with a governing equality, determined by the ideal of null-sets.

The non-existence of the correspondence Φ implies that the MacNeilles extension may differ from the generalized Lebesgue extension [AI] within a supertribe.

Especially we shall prove that, in our example, to the fundamental sequence

$$\{(\tfrac{1}{2}-\varepsilon_n, \tfrac{1}{2}+\varepsilon_n\rangle\}$$

there cannot correspond any subset of $(0, 1\rangle$, but only an abstract entity, which is not set all of the kind considered.

To prove that, suppose that the said correspondence Φ exists, and consider the following somata of $\overline{\mathfrak{T}}$:

$$P =_{df} \{(0, \tfrac{1}{2}-\varepsilon_n\rangle\}, \qquad Q =_{df} \{(\tfrac{1}{2}-\varepsilon_n, \tfrac{1}{2}+\varepsilon_n\rangle\},$$

$$R =_{df} \{(\tfrac{1}{2}+\varepsilon_n, 1\rangle\},$$

where

$$\varepsilon_n > 0, \qquad \varepsilon_n \to 0.$$

These somata are disjoint, and their corresponding values of the $\bar{\mu}$-measure are respectively:

$$\lim_{n\to\infty}[f(\tfrac{1}{2}-\varepsilon_n)-f(0)], \qquad \lim_{n\to\infty}[f(\tfrac{1}{2}+\varepsilon_n)-f(\tfrac{1}{2}-\varepsilon_n)], \tag{1}$$

$$\lim_{n\to\infty}[f(1)-f(\tfrac{1}{2}+\varepsilon_n)].$$

Now we have

$$f(\tfrac{1}{2}-\varepsilon_n) = f(\tfrac{1}{2}) - \eta_n, \text{ where } \eta_n \to 0$$

$$f(\tfrac{1}{2}+\varepsilon_n) = \tfrac{3}{4}+\eta_n', \text{ with } \eta_n' \to 0.$$

Hence the measures (1) are respectively the limits of the sequences:

$$\tfrac{1}{2}-\eta_n, \qquad (\tfrac{3}{4}+\eta_n')-(\tfrac{1}{2}-\eta_n), \qquad 1-(\tfrac{3}{4}+\eta'_n).$$

Hence these measures are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$.

Thus we have

$$\bar{\mu}(P)=\tfrac{1}{2}, \qquad \bar{\mu}(Q)=\tfrac{1}{4}, \qquad \bar{\mu}(R)=\tfrac{1}{4}.$$

As the function f(x) is continuous in $\langle 0, \tfrac{1}{2})$ and in $(\tfrac{1}{2}, 1\rangle$, it follows that

$$\Phi(P)=(0, \tfrac{1}{2}),$$

$$\Phi(R)=(\tfrac{1}{2}, 1).$$

Since the three fundamental sequences P, Q, R are disjoint, we get

$$\Phi(Q)=(\tfrac{1}{2})$$

i.e. the set composed of the single number $\tfrac{1}{2}$.

It follows that

$$\bar{\mu}\Phi(\tfrac{1}{2})=(\tfrac{1}{2}).$$

By continuity of f(x), the measure of every point in $(0, \tfrac{1}{2}\rangle$ and $(\tfrac{1}{2}, 1\rangle$ is $=0$. The only point, having a positive messure is the point $\tfrac{1}{2}$.

But then it follows that

$$f(\tfrac{1}{2})=\tfrac{3}{4},$$

which is not true, because by definition

$$f(\tfrac{1}{2})=\tfrac{1}{2}.$$

The contradiction proves that the supposed correspondence Φ does not exist.

α.**20**. THEOREM. Whatever the non-trivial tribe $\mathcal{T}$ may be, and whatever any (finitely additive, finite, non trivial, non negative measure on it may be, the *Cantor-Mac Neille's extension* $\bar{\mathcal{T}}$ always exists. It is completely additive, and the extended measure is effective and denumerably additive. The extension does not always coincide with Lebesgue's covering extension. If the somata of $\mathcal{T}$ are sets, the Cantor-Mac Neille's extension may contain somata which are no sets at all of the kind considered.

Chapter β.

A study in the cartesian product of abstract, measured Boole'an lattices.

β.0. Though the notion of cartesian product (sometimes called « cross-product ») is an important notion and of frequent use, I did not succeed to find in the literature a satisfactory foundation of the theory of this notion.

In the present paper I am presenting a precise and clear setting of the theory of that matter, owing to a suitable change of the definition of inclusion of elements of the cross-product, and by means of using a special notion of « grate ».

The content of the present paper has not been published, but only presented in schetch at a meeting of the Amer. Math. Soc. some years ago. The paper gives explicite proofs of basic theorems, but it limits itself to the very foundations only. I believe the paper will be useful.

For terminology we refer to the preceeding chapter α. Usually the domain of an ordering (A) is denoted by A.

β.1. Let (T′), (T″) be two non trivial tribes [6]. Denote by $0', 0''$; $1', 1''$ their respective *zeros* and *units*.

Their governing equalities may be different and so may be with the operations. Nevertheless we shall use for both the same symbols $(=)$, $+$, $\cdot$, $-$, co.

β.1.1. Definition. By *rectangle* we denote every ordered couple $[a', a'']$, where $a' \in T'$, $a'' \in T''$.

[6]) This means that $0' \neq 1'$, $0'' \neq 1''$ with respect to the equalities governing in (T′), (T″) respectively.

β.1.1.1. DEFINITION. Two rectangles $[a_1', a_1'']$, $[a_2', a_2'']$ will be said to be *β-disjoint*, whenever either

$$a_1' \cdot a_2' = 0' \text{ or } a_1'' \cdot a_2'' = 0''.$$

β.1.2. DEFINITION. The rectangles $[a', 0'']$, $[0', a'']$ will be termed: *rectangles null.*

Especially $[0', 0'']$ is a rectangle null.

β.1.2.1. A rectangle null is β-disjoint with every rectangle; hence also with itself.

β.1.2.2. There exists a rectangle, which is not null, wiz. $[1', 1'']$.

β.1.3. DEFINITION. If $[a', a'']$ is a rectangle, the soma a' will be termed *abscissa of the rectangle,* and the soma a'' will be termed *ordinate of the rectangle.*

β.1.4. DEFINITION. Every non empty finite set of rectangles, such that any two of them are β-disjoint, is called *figure.*

Thus every rectangle is a figure [7]).

β.1.4.1. DEFINITION. A figure is called *null-figure,* whenever it is a finite, not empty collection of null-rectangles.

A null-figure is a figure, because the null-rectangles are β-disjoint.

β.1.4.2. DEFINITION. Two figures A, B are said to be *β-disjoint,* whenever every rectangle of A is β-disjont wth every rectangle of B, (according to Def. [β1.1.1.]).

We see that, for figures composed of a single rectangle, the definitions [β.1.1.1.] and [β.1.4.2.] are giving the same, so Def [β.1.4.2.] constitutes a good generalization of Def. [β.1.1.1.].

A null-figure is β-disjoint with any figure.

β.1.5. DEFINITION. If A, B are figures, we say that *A is β-included*

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7) It should be said: set, composed of a single rectangle only. But, dealing with finite, non empty collections, we may — for sake of simplicity — not make any discrimination between a set composed of a single object and the object itself.

in B,

$$A \dot{\leq} B, \qquad (B \dot{\geq} A),$$

whenever every rectangle, which is β-disjoint with B, is also β-disjoint with A.

β.1.5.1. If A is a null-figure, we have

$$A \dot{\leq} B$$

for every figure B.

β.1.5.2. We also have for any figure A:

$$A \dot{\leq} [1', 1''].$$

β.1.5.3. We have for any figure A:

$$A \dot{\leq} A.$$

If $A \dot{\leq} B$, $B \dot{\leq} C$, then $A \dot{\leq} C$.

β.1.5.4. THEOREM. If [a′, a″], [b′, b″] are rectangles, then the following are equivalent:

I. $[a', a''] \dot{\leq} [b', b'']$,

II. $a' \leq b'$ and $a'' \leq b''$.

PROOF. Suppose I. This statement, $[a', a''] \dot{\leq} [b', b'']$, means by [β.1.5.Def.] that, in general, if [p′, p″] is a rectangle β-disjoint with [b′, b″], [β.1.1.], then it is also β-disjoint with [a′, a″].

Concerning the rectangles [p′, p″] which are β-disjoint with [a′, a″], there are two categories:

1) $p' \cdot a' = O'$.

2) $p'' \cdot a'' = O''$.

Denote by E the collection of all rectangles [p′, p″], where $p' \cdot a' = O'$, and denote by F the collection of all rectangles [p′, p″] for wich $p'' \cdot a'' = O''$.

These collections may be even empty. The union $E \cup F$ is just the set of all rectangles $[p', p'']$, which are disjoint with $[a', a'']$.

The sets E, F may be not disjoint.

The condition $p' \cdot a' = 0'$, which caracterizes E, is equivalent to $p' \subseteq \mathrm{co}\, a'$ and the condition, $p'' \cdot a'' = 0''$, which caracterizes F, is equivalent $p'' \subseteq \mathrm{co}\, a''$.

Concerning rectangles $[p', p'']$ which are disjoint with $[b', b]''$, there are two kinds of them: one, whose collection we denote by E_1, is caracterized by the condition $p' \subseteq \mathrm{co}\, b'$; the other F_1 is caracterized by the condition $p'' \subseteq \mathrm{co}\, b''$.

Having this, the condition I can be restated by: If a rectangle $[p', p'']$ belongs to E_1, then it belongs to E and if a rectangle belongs to F_1, then it belongs to F.

This can be restated as:

If for a rectangle $[p', p'']$ we have $p' \subseteq \mathrm{co}\, b'$, then $p' \subseteq \mathrm{co}\, a'$, and if for a rectangle $[p', p'']$ we have $p'' \subseteq \mathrm{co}\, ''$, then $p'' \subseteq \mathrm{co}\, a''$.

It follows that $\mathrm{co}\, b' \subseteq \mathrm{co}\, a'$, $\mathrm{co} b'' \subseteq \mathrm{co}\, a''$, hence

$$a' \leq b', \qquad a'' \leq b'' \text{ i.e. II.}$$

Now suppose II, i.e.

$$a' \leq b', \qquad a'' \leq b''.$$

It follows that $\mathrm{co}\, b' \leq \mathrm{co}\, a'$, $\mathrm{co}\, b'' \leq \mathrm{co}\, a''$; hence
if $p' \leq \mathrm{co}\, b'$, then $p' \leq \mathrm{co}\, a'$, and
if $p'' \leq \mathrm{co}\, b''$, then $p'' \leq \mathrm{co}\, a''$.

Hence if a $[p', p'']$ belongs to E_1, it belongs to E and if $[p', p''] \in F_1$ then it belongs to F.

It follows that if a rectangle belongs to $E_1 \cup F_1$, then it also belongs to $E \cup F$, hence

$$[a', a''] \dot{\leq} [b', b''] \text{ i.e. I.}$$

The theorem is established.

β.1.6. Definition. If A, B are figures, then A is said to *be β-equivalent to B,* whenever

$$A \dot{\leq} B \text{ and } B \dot{\leq} A.$$

We shall write $A \doteq B$.

β.**1.6.1.** If A, B, C are figures, then:

1) $A \doteq A$;

2) if $A \doteq B$, then $B \doteq A$;

3) if $A \doteq B$, $B \doteq C$, then $A \doteq C$.

β.**1.6.2.** Then notion of a figure being included in another one is equivalence ($\doteq$)-invariant. This means that

$$\text{if } A \doteq A',\ B \doteq B',\ A \dot{\leq} B, \text{ then } A' \dot{\leq} B'.$$

β.**1.6.3.** If A, B are both null figures, [Def.β.1.4.1.], then $A \doteq B$.

β.**1.6.4.** If A is a figure, and we add to it a finite number of any null-figures, then the new figure B will be β-equivalent to A.

β.**1.6.5.** If A is a figure having some elements, which are null-rectangles, and if we remove all or some of them, the new figure B, if available, will be β-equivalent to A.

β.**1.6.5.1.** THEOREM. It follows that, if A is a figure not null, it is β-equivalent to a figure B, which is composed of mutually β-disjoint rectangles, no one of them being a null rectangle.

β.**1.6.6.** REMARK. The notion of β-inclusion of figures organizes the collection of all figures into an ordering [A.1.1.] with zero and unit, and with β-equivalence [β.1.6.] as governing equality [A.4.]. Any null-figure is the *zero*, and any figure, β-equivalent to [1′, 1″], is the *unit* of the ordering.

Indeed we have [β.1.5.1.] and [β.1.5.2.].

β.**1.6.7.** Our principal aim will be to prove that the above ordering is a lattice [A.1.3.].

To do that in an easy way, we shall introduce an auxiliary notion.

β.**2.** DEFINITION., A figure A is termed *total grate*, whenever it is composed of a finite number of rectangles

$$(1) \qquad [a_i' \ a_j''],$$

where

1) $a_i' \neq 0'$, $i=1, ..., n$, $n \geq 1$,

$a_j'' \neq 0''$, $j=1, ..., m$, $m \geq 1$,

2) $a_1' + a_2' + ... + a_n' = 1'$,

$a_1'' + a_2'' + ... + a_m'' = 1''$,

3) $a_\alpha' \cdot a_\beta' = 0$ for $\alpha \neq \beta$, $\alpha, \beta = 1, ..., n$, $n \geq 1$,

$a_\alpha'' \cdot a_\beta'' = 0''$ for $\alpha \neq \beta$, $\alpha, \beta = 1, ..., m$, $m \geq 1$.

The conditions 2), 3) say that we have to do with partitions of $1'$ and $1''$ into disjoint somata [8]) respectively.

REMARK. If $n=1$ or $m=1$, the corresponding sums are reduced to single somata.

β.2.1. If A is a total grate, then the rectangles (1) are mutually β-disjoint.

β.2.2. The figure $[1', 1'']$ is a total grate.

β.2.3. The total grate A is β-equivalent to $[1', 1'']$.

β.2.4. A null figure never is a total grate.

β.2.4.1. DEFINITION. Let A be a non null figure and G a total grate.

We say *that A and G fit together,* whenever A is β-equivalent to a not empty subset A° of G.

Remind that a figure is a set of rectangles; so it is a total grate. We have $A \doteq A^\circ$, $A^\circ \subseteq G$.

Under these circumstances A° is termed *grate of A generated by G.*

8) Concerning a precise theory of partition in a tribe, see the paper by O. M. Nikodým:
« On extension of a given finitely additive field-valued, non negative measure, on a finitely additive Boolean tribe, to another more ample ». Rendiconti del Seminario Matematico della Università di Padova. Vol. 26, 1958, pp. 265-267.

β.2.4.2. If G is a total grate and F a not empty subset of G, then G and F fit together, and F is the grate of F, generated by G.

β.2.4.2.1. If

1) the figure F and the total grate G fit together,

2) $F \doteq F'$,

then F′ and G also fit together.

β.2.4.3. THEOREM. If A is not a null-figure, then there exists a total grate G which fits A.

PROOF. By [β.6.5.1.] F is β-equivalent to a finite collection of mutually disjoint non-null-rectangles.

Denote them: $A_1, A_2, \ldots, A_n$, $(n \geqq 1)$.

Let $A_1 = [a_1', a_1'']$. We have $a_1' \neq 0'$, $a_1'' \neq 0''$. Consider the rectangles

(1) $[a_1', a_1''], [co' a_1', a_1''], [a_1', co'' a_1''], [co' a_1', co'' a_1'']$.

They are β-disjoint, and we have

$$a_1' + co' a_1' = 1', \qquad a_1'' + co'' a_1'' = 1''.$$

It may happen that

$$co' a_1' = 0, \qquad co'' a_1'' = 0.$$

If so, we omit in the sequence (1) the corresponding somata. We will get

$$[a_1', a_1''], \qquad [a_1', co'' a_1''],$$

or

$$[a_1', a_1''], \qquad [co' a_1', a_1''],$$

or else $[a_1', {}_1'']$, instead of the sequence (1).

In al these cases the remaining rectangles (1) make up a total grate, say G_1. We see that G_1 is fitting A_1. The figure G_1 is composed of four, two or one rectangle according to the case.

Now suppose that we have already found a total grate G_i fitting the figure composed of the rectangles

$$A_1, A_2, ..., A_i, \qquad i \geq 1).$$

Put

$$A_{i+1} = [a'_{i+1}, a''_{i+1}],$$

and let G_i be composed of the non-null rectangles

$$[g_\alpha', g_\beta''], \qquad \alpha = 1, ..., p, \ p \geqq 1, \qquad \beta = 1, ..., q, \ q \geq 1,$$

all disjoint, and where

$$g_1' + ... + g_p' = 1',$$

$$g_1'' + ... + g_q'' = 1''.$$

The somata of (T'): $g_\alpha' a'_{i+1}$, g_α' co' a'_{i+1} are disjoint, and so are the somata $g_\beta'' a''_{i+1}$, g_β'' co'' a''_{i+1} of (T''). We have

$$\sum_\alpha g_\alpha' a'_{i+1} + \sum_\alpha g_\alpha' \text{ co}' a'_{i+1} = 1',$$

$$\sum_\beta g_\beta'' a''_{i+1} +_\beta'' \text{ co}'' a''_{i+1} = 1''.$$

We see that, if we omit those products which are null, we obtain a total grate G_{i+1} which fits the figure composed of the rectangles $A_1, ..., A_i, A_{i+1}$. The theorem follows by induction.

β.2.4.4. Definition. If G_1, G_2 are total grates and G_2 is a subpartition of G_1, then G_2 is termed *total subgrate of* G_1.

β.2.4.5. Theorem. If the non null figure fits the total grate G, and G_1 is a total subgrate of G, then A also fits G_1.

β.2.4.6. Theorem. If $A_1, A_2, ..., A_n$, $(n \geq 2)$ are non-null-figures, then there exists a total grate G fitting all these figures.

Proof. It suffice to prove this in the case $n=2$.

Let G_1 be a total grate fitting A_1, and G_2 a total grate fitting A_2. Let G_1 be composed of rectangles $g_1', g_2', ..., g_n'$ and G_2 of rectangles $g_1'', g_2'', ..., g_m''$, where $n \geq 1$, $m \geq 1$.

The figure, composed of all g_α', g_β'', which are not null, is a total grate G°, which is a total subgrate of G_1, and at the same time a total subgrate of G_2. From [2.4.5.] it follows that G° is fitting A_1 and A_2. The theorem can now be proved by induction.

β.**2.4.7.** THEOREM. If A, B are non-null-figures, then the following are equivalent:

I. A is β-disjoint with B.

II. There exists a total grate G fitting both A and B, such that, if we denote by A° the grate of A generated by G, and denote by B° the grate of B generated by G, then the collection A° and B° are disjoint, i.e. all rectangles of A° are different from all rectangles of B°.

III. For every total grate G fitting both A and B, the collection A°, B° have an element in common.

PROOF. Let I, and let G be a total grate fitting both A and B. Suppose that the collections A° and B° are not disjoint, so they have at least one element in common, say $[a', a'']$.

Since $A \doteq A^\circ$, $B \doteq B^\circ$, the A°, B° are β-disjoint figures. Hence all rectangles of A° are β-disjoint with all rectangles of B°.

Hence $[a', a'']$ is β-disjoint with itself; hence either $a' \neq a'$ or $a'' \neq a''$, which is absurd.

Thus we have proved that $I \rightarrow II$. A similar proof will be for $I \rightarrow III$. The converse implications of statements are evident. Thus we have:

$$II \rightarrow I, \qquad III \rightarrow I.$$

It follows that

$$I \rightarrow II \rightarrow III \rightarrow I.$$

The theorem is proved.

β.**2.4.8.** THEOREM. If A, B are non-null-fgures, then the following are equivalent:

I. $A \dot{\leq} B$.

II. There exists a total grate G fitting both A and B such that, if we denote by A^o, B^o the grates of A, B, determined by G, then every element of the collection A^o is contained in the collection B^o, i.e. $A^o \subseteq B^o$.

III. If G is any total grate fitting A and B, then

$$A^o \subseteq B^o. \tag{1}$$

PROOF. Let I. Since $A \doteq A^o$, $B \doteq B^o$, we have by [β.1.6.2.],

$$A^o \dot{\leq} B^o.$$

Suppose (1) be not true. Then it is not true, that every element of A^o also belongs to B^o. Hence there exists an element $[a', a'']$ of A^o which does not belong to the collection B^o.

$[a', a'']$ is β-disjoint with B^o. Hence the rectangle $[a', a'']$ is also β-disjoint with A^o, which contradicts the fact that $[a', a'']$ belongs to A^o.

Thus we have proved that I → II. A similar proof will be for the implication I → III.

The converse implications II → I, III → I are easy to prove. It follows that II → I → III → I → II.

Hence I, II, III are equivalent statements.

β.2.4.9. THEOREM. If A, B are non null figures, then the following are equivalent:

I. $A \doteq B$.

II. There exists a total grate G fitting both A and B, such that, the grate A^o of A generated by G, and the grate B^o of B generated by G, are identical, $A^o \equiv B^o$, i.e. composed of the same rectangles.

III. For every total grate G which fits both A and B, the grate A^o of A generated by G, and the grate B^o of B generated by G, are identical, $A^o \equiv B^o$, i.e. composed of the same rectangles.

This follows from [β.2.4.8.].

β.2.5. DEFINITION. The total grate G_1 is said to *be finer than the total grate* G whenever every mesh of G_1 is included in some mesh of G.

β.2.5.1. THEOREM. If the total grate G_1 is finer than the total grate G, and if the figure A fits G, then A also fits G_1.

If we consider the grates A_G, A_{G_1} of A generated by G, G_1 respectively, then every mesh of A_{G_1} is included in some mesh of A_G. We say that A_{G_1} is finer than A_G.

β.2.5.2. THEOREM. If G, H are total grates, G is composed of meshes A_α, and H is composed of meshes B_λ, then the collection of all rectangles

(1) $$A_\alpha, B_\lambda,$$

after having been omitted those rectangles (1), which are null-rectangles, is a total grate which is finer than G and finer than H.

β.3. Let A, B be two non null figures. Consider a grate G fitting both A and B and consider the corresponding grates G_A, G_B of A, B respectively, generated by G.

Then the union $G_A \cup G_B$ of the sets G_A, G_B of rectangles also fits G.

Let G′ be another total grate fitting A and B, and consider the corresponding G_A', G_B'.

Under these circumstances we have

(1) $$G_A \cup G_B \doteq G_A' \cup G_B'.$$

β.3.1. DEFINITION. Every figure C which β-equals (1) will be termed *union (sum, join) of* A *and* B, and denoted by A+B.

So we have

(2) $$A+B \doteq G_A \cup B_B.$$

PROOF. Take [β.9.5.2.].

First we prove the statements in the case where G′ is finer than G, and afterwords we consider the general case.

β.3.2. THEOREM. If

$$A \doteq A', \qquad B \doteq B' \qquad C \doteq C', \qquad C \doteq A+B,$$

then

$$C' \dot{=} A' + B'.$$

Thus the operation of union of two figures is invariant with respect to the equality ($\dot{=}$).

β.3.3. DEFINITION. Let us denote by *0* any null-figure, and by *1* any figure β-equivalent to [*1'*, *1''*].

β.3.3.1. DEFINITION. We complete the definition [β.3.1.] by taking also care of of null-figures:

$$A + 0 \dot{=} A, \qquad A + 1 \dot{=} 1, \qquad 0 + A \dot{=} A, \qquad 1 + A \dot{=} 1.$$

β.3.3.2. THEOREM. The operation of addition for all figures is β-invariant, i.e.:

If A, B, C are any figures and if

$$A \dot{=} A', \qquad B \dot{=} B', \qquad C \dot{=} C', \qquad C \dot{=} A + B,$$

then

$$C' \dot{=} A' + B'.$$

β.3.3.3. THEOREM. We have for any figures

$$A + B \dot{=} B + A, \qquad (A + B) + C \dot{=} A \dot{+} (B \dot{+} C), \qquad A + A \dot{=} A.$$

β.3.3.4. THEOREM. The following are equivalent for any figures

I. $C \dot{=} A + B$,

II. $A \dot{\leq} C$, $B \dot{\leq} C$ and if $A \dot{\leq} C'$, $B \dot{\leq} C'$, then $C \dot{\leq} C'$.

This shows that the ordering ($\dot{\leq}$) of all figures admits the unions.

PROOF. By considering a total grate for al figures considered.

β.3.4. DEFINITION. We are going to define the *multiplication* (*intersection, meet*) of figures.

Let A, B be figures and G a total grate fitting them, hence inducing grates A_G, B_G of A, B.

We define the *product* (*meet, intersection*), $A \cdot B$, as any figure which is β-equivalent to the intersection $A_G \cap B_G$, in the case where this set is not empty.

If

$$A_G \cap B_G = \emptyset,$$

we define

$$A \cdot B \doteq_{df} 0,$$

i.e. the null-figure.

The product $A \cdot B$ does not depend on the choice of the grate G fitting A and B.

Concerning null-sets, we define:

$$A \cdot 0 \doteq_{df} 0, \qquad A \cdot 1 \doteq_{df} A, \qquad 0 \cdot A =_{df} 0, \qquad 1 \cdot A \doteq_{df} A.$$

β.3.4.1. Theorem. The operation of multiplication of figures is β-equality-invariant.

β.3.4.2. Theorem. We have for any figures A, B, C:

$$A \cdot B \doteq B \cdot A, \qquad (A \cdot B) \cdot C \doteq A \cdot (B \cdot C), \qquad A \cdot A \doteq A.$$

β.3.4.3. Theorem. The following are equivalent for any figures.

I. $C \doteq A \cdot B$.

II. $C \dot{\leq} A$, $C \dot{\leq} B$, and if

$C' \dot{\leq} A$, $C' \dot{\leq} B$, then $C' \dot{\leq} C$.

We see that the ordering ($\dot{\leq}$) of figures admits meets.

β.3.4.4. Theorem. Under the above definitions of $\dot{\leq}$, $\doteq$, $+$, $\cdot$, *0*, *1*, we have a lattice with zero and unit. The lattice is distributive i.e. for any figures A, B, C we have

$$(A + B) \cdot C \doteq A \cdot C + B \cdot C.$$

β.4. Definition. We shall define the *complement of a figure.*

Let A be a figure differing from *0* and *1*. Take a total grate G of A and determine the grate A_G of A with respect to G. Consider the set H of all rectangles in G, which do not belong to A_G.

We have

$$A_G \cup A = H, \qquad A_G \cap H = \emptyset.$$

Now we define the *complement* (co A) *of* A as any figure, which β-equivalent to H. The defintion of co A does not depend on the choice of the total grate fitting the figure A.

We complete this definition by the following:

$$\text{co}\, 1 \doteq 0, \qquad \text{co}\, 0 \doteq 1.$$

β.4.1. THEOREM. The notion of co A is β-equality invariant, i.e. if

$$B \doteq \text{co A}, \qquad B \doteq B', \qquad A \doteq A',$$

then

$$B \doteq \text{co A}'.$$

β.4.2. THEOREM. We have for all figures the relations:

$$A + \text{co}\, A \doteq 1,$$

$$A \cdot \text{co}\, A \doteq 0.$$

β.4.3. THEOREM. The following relations take place:

$$\text{co}\,(\text{co}\, A) = A,$$

$$\left.\begin{aligned} \text{co}\,(A+B) &= \text{co}\, A \cdot \text{co}\, B, \\ \text{co}(A \cdot A) &= \text{co}\, A + \text{co}\, B. \end{aligned}\right\} \text{ De Morgan's laws.}$$

β.4.4. THEOREM. The above discussion shows that the β-ordering ($\dot{\leq}$) of figures is a Boole'an finitely additive lattice. The figures are its somata, *0* its zero and *1* its unit. The governing equality is the β-equality ($\doteq$).

β5. The notion of the cartesian product of two tribes and some of its properties having been explained, we suppose that the tribes T', T'' are provided with finitely additive measures μ', μ'', i.e. with functions $\mu'(\dot{A}')$, $\mu''(\dot{A}'')$, which are finite, non trivial, non negative and satisfying the conditions:

1) $\mu'(A'+B')=\mu'(A')+\mu'(B')$,

whenever

$$A'\cdot B'=O'; \qquad A',\ B'\in T';$$

2) $\mu''(A''+B'')=\mu''(A'')+\mu''(B'')$,

whenever

$$A''\cdot B''=O'', \qquad A'',\ B''\in T''.$$

3) The measures are invariant with respect to the equalities governing in T', T'' respectively.

β.5.1. Definition. We shall define the measure on $T=_{df}T'\times T''$ as usually in the following way:

If $A\in T$ is a null-figure, i.e. $A\doteq O$, [β.3.3.], we put

$$\mu(A)=_{df}0.$$

If $A\in T$ is not a null figure, we define

$$\mu(A)=_{df}\sum_{i,j}\mu'(a_i')\cdot\mu''(a_j''),$$

where $A\doteq\sum[a_i', a_j'']$ with disjoint rectangles $[a_i', a_j'']$ finite in number.

β.5.2. Theorem. To justify that definition one can prove that if

$$A\doteq\sum_{i,j}[a_i',\ a_j'']\doteq\sum_{r,s}[b_r',\ b_s''],$$

with disjoint $[a_i', a_j'']$, as well as with disjoint $[b_r', b_s'']$, then

$$\sum_{i,j}\mu'(a_i')\mu''(a_j'')=\sum_{r,s}\mu'(b_r')\cdot\mu''(b_s'').$$

The proof is based on considering grates of A.

β.5.3. If the figure A is a rectangle [a′, a″], then

$$\mu(A)=\mu'(a')\cdot\mu''(a'').$$

β.5.4. If $A \doteq B$ and $\in T$, then $\mu(A)=\mu(B)$.

β.5.5. If $A, B \in T$, $A\cdot B=O$, then

$$\mu(A+B)=\mu(A)+\mu(B),$$

i.e. the measure μ on T is finitely additive.

β.5.6. The measure μ is effective if, and only if both the measures μ′, μ″ are effective.

β.5.7. The measure μ is not trivial, whenever both the measures μ′, μ″ are not trivial.

β.6. The tribe T can be extended, together with the measure μ, by the Cantor-Mac Neille's device, explained in Chapter [α], and based on consideration of μ-fundamental sequences of somata of T,

$$A_1, A_2, \ldots, A_n, \ldots$$

with

$$\lim_{n,m} (A_n \dotplus A_m), \text{ for } n, m \rightarrow \infty,$$

where $A_n \dotplus A_m$ denote the algebraic addition:

$$A_n \dotplus A_m =_{df} (A_n - A_m) + (A_m - A_n).$$

The tribe $\bar{T}$, thus obtained in this way, is completely additive with denumerably additive, effective measure $\bar{\mu}$.

β.7. If the somata of T′ and T″ are sets or some entities, the construction of somata of $T' \times T''$ does need any sophisticated foundation, as it is needed in the case where T′ and T″ are abstract.

In the case of sets, we also consider « rectangles » [a′, a″], but we define the inclusion

$$[a', a''] \dot{\leq} [b', b'']$$

as $a' \leq b'$, $a'' \leq b''$, and this will yield a quite simple theory, based on consideration of points in the cartesian product of subset of the domains $\mathfrak{a}T'$, $\mathfrak{a}T''$ of T′ and T″.

The somata of T′ will be sets of couples of points.

In our case of sets, we can, of course as well use the general theory, as the usual just mantioned method. Both resulting tribes are isomorphic and isometric.

CHAPTER γ

Section 1.

A special metric topology on the lattice $\mathfrak{L}$ of all closed subspaces of the separable and complete Hilbert-Hermite space **H**.

γ.1. We shall consider a Hilbert-Hermite abstract space **H**, (H·H-space), which is not trivial, i.e. not confined to the null-vector only. $(\vec{\xi}, \vec{\eta})$ denotes the *scalar product* of two vectors $\vec{\xi}, \vec{\eta}$ of **H**, where

$$\lambda(\vec{\xi}, \vec{\eta}) = (\vec{\xi}, \lambda\vec{\eta}).$$

The *norm of the vector* $\vec{\xi}$ will be denoted by $\vec{\xi}$:

$$|\vec{\xi}| = \sqrt{(\vec{\xi}, \vec{\xi})}.$$

This norm defines the *distance*

$$|\vec{\xi}, \vec{\eta}| =_{\mathrm{df}} |\vec{\xi} - \vec{\eta}|$$

betveen the vectors $\vec{\xi}, \vec{\eta}$, and in turn, it organizes **H** into *metric topology* (**H**-*topology*). We shall consider subspaces *a* of **H**, which are closed sets in the **H**-topology. We call them simply *spaces*.

The collection of all spaces, ordered by the ordinary relation of inclusion of sets of vectors, constitutes a complete lattice, denoted by $\mathfrak{L}$, [A.1.4.1.]. This lattice will be the main object of study in this paper. We suppose that **H** is *separable* and *complete* in its topology.

For terminology we refer to our book:

« Mathematical Apparatus for Quantum-Theories » (sec. [A]).

γ.1.1. We shall define and study a special notion $\| a, b \|$ of distance between a and b called *γ-distance between a and b*.

The topology generated by it, will be called *γ-topology*.

γ.1.1.a. As **H** is separable, there exists a denumerable sequence of vectors

$$\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n, \ldots$$

which is everywhere dense in the **H**-topology, and where $\vec{\xi}_n \neq \vec{0}$ for $n = 1, 2, \ldots$. We shall keep fixed this sequence.

We put

$$\vec{x}_k =_{df} \frac{1}{|\vec{\xi}_k|} \cdot \vec{\xi}_k, \qquad (k = 1, 2, \ldots).$$

We have

$$(1) \qquad |\vec{x}_k| = 1, \qquad (k = 1, 2, \ldots),$$

If $\vec{x}$ is a vector of **H** and p a space, we denote by $\text{Proj}_p \vec{x}$ *the projection of $\vec{x}$ on p.*

γ.1.2. Definition. Choose a sequence

$$(1.1) \qquad \lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$$

of positive numbers, such that the series

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k + \ldots$$

converges.

Now, for every two spaces *a, b* we define the function $\| a, b \|$ by means of the equation

$$\| a, b \| =_{df} \sum_{k=1}^{\infty} \lambda_k \, | \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} | . \tag{2}$$

We call $\| a, b \|$ *γ-distance between the spaces* a, b.
This sum (2) converges. Indeed we have

$$| \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} | \leq | \mathrm{Proj}_a \overrightarrow{x_k} | + | \mathrm{Proj}_b \overrightarrow{x_k} | \leq$$
$$\leq | \overrightarrow{x_k} | + | \overrightarrow{x_k} | = 2 \, | \overrightarrow{x_k} | = 2,$$

and the series of positive numbers

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k + \ldots$$

converges too.

γ.1.3. THEOREM. We shall prove that this notion of distance organizes the collection $\mathfrak{L}$ of spaces into a metrice space.

PROOF. We have

$$\| a, b \| \geq 0 \tag{3}$$

for all spaces *a, b*.
We have

$$\| a, b \| = \| b, a \| , \tag{4}$$

which follows from the relation

$$| \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} | = | \mathrm{Proj}_b \overrightarrow{x_k} - \mathrm{Proj}_a \overrightarrow{x_k} | .$$

We have for any three spaces *a, b, c,*

$$\| a, b \| \leq \| a, c \| + \| c, b \| , \tag{5}$$

Indeed this follows from the inequality

$$| \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} | \leq | \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_c \overrightarrow{x_k} | + | \mathrm{Proj}_c \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} | .$$

γ.1.4. THEOREM. Thus we have proved that if

$$a=b, \text{ then } \|a, b\|=0. \tag{6}$$

We must prove that the proposition (6) c a n b e i n v e r t e d.

PROOF. Suppose that

$$\|a, b\|=0.$$

By definition of the γ-distance we have

$$\sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} \,| = 0.$$

Since all terms in this sum are non negative, we get

$$\lambda_k \,|\, \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} \,| = 0 \text{ for all } k=1, 2, \ldots$$

Since $\lambda_k > 0$, it follows

$$|\, \mathrm{Proj}_a \overrightarrow{x_k} - \mathrm{Proj}_b \overrightarrow{x_k} \,| = 0. \tag{7}$$

Since the topology **H**, created by the **H**-distance of vectors, is a metric space, the equality (7) implies for each k separately

$$\mathrm{Proj}_a \overrightarrow{x_k} = \mathrm{Proj}_b \overrightarrow{x_k} ;$$

hence

$$\mathrm{Proj}_a \frac{\overrightarrow{\xi_k}}{|\overrightarrow{\xi_k}|} = \mathrm{Proj}_b \frac{\overrightarrow{\xi_k}}{|\overrightarrow{\xi_k}|} ;$$

Hence

$$\mathrm{Proj}_a \overrightarrow{\xi_k} = \mathrm{Proj}_b \overrightarrow{\xi_k}, \text{ for } k=1, 2, \ldots . \tag{8}$$

This being established, we shall prove that for every vector $\overrightarrow{\xi} \in \mathbf{H}$ we have

$$\mathrm{Proj}_a \overrightarrow{\xi} = \mathrm{Proj}_b \overrightarrow{\xi}.$$

To do that take an arbitrary vector $\overrightarrow{\xi}$.

Since the collection

$$\vec{\xi}_1, \vec{\xi}_2, \dots, \vec{\xi}_k, \dots$$

is dense in **H** with respect to the **H**-topology, therefore there exists a sequence

$$\vec{\xi}_{v(k)}, \quad k=1, 2, \dots,$$

with

$$v(1)<v(2)<\dots<v(k)<\dots,$$

where

$$\vec{\xi}=\lim_{k\to\infty}\vec{\xi}_{v(k)} \tag{9}$$

in the **H**-topology.

From (9) it follows, by virtue of continuity of the operation of projecting a vector on a space:

$$\left\{\begin{array}{l}\mathrm{Proj}_a\vec{\xi}_{v(k)}\to\mathrm{Proj}_a\vec{\xi},\\ \mathrm{Proj}_b\vec{\xi}_{v(k)}\to\mathrm{Proj}_b\vec{\xi}.\end{array}\right. \tag{10}$$

From (8) we have

$$\mathrm{Proj}_a\vec{\xi}_{v(k)}=\mathrm{Proj}_b\vec{\xi}_{v(k)}, \qquad k=1, 2, \dots;$$

hence, by (10),

$$\mathrm{Proj}_a\vec{\xi}=\mathrm{Proj}_b\vec{\xi}. \tag{11}$$

Since (11) holds true for any vector $\vec{\xi}$, it holds also for a vector $\vec{\eta}\in a$.

We have from (11):

$$\mathrm{Proj}_a\vec{\eta}=\mathrm{Proj}_b\vec{\eta};$$

hence

$$\vec{\eta}=\mathrm{Proj}_b\vec{\eta},$$

which implies

$$\vec{\eta} \in b.$$

Thus we have proved that, if $\vec{\eta} \in a$, than $\vec{\eta} \in b$, which implies that

(12) $$a \leq b.$$

In a similar way, taking a vector $\vec{\xi} \in b$, we get

(13) $$b \leq a.$$

From (12) and (13) we deduce

(14) $$a = b.$$ Q.E.D.

Thus we have proved that the following are equivalent, (6), (14):

I. $\| a, b \| = 0$,

II. $a = b$.

This and (3), (4), (5), (6) show that:

γ.**1.5**. THEOREM. The topology, generated in ($\mathfrak{L}$) by the notion of γ-distance $\| a, b \|$, is a metric space.

We call it *γ-metric space* or *γ-topology*.

γ.**1.6**. DEFINITION. We shall introduce a special auxiliary kind of measure of spaces which we shall call *Φ-measure*:

We choose an infinite sequence

$$\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_k, \ldots$$

of vectors in **H**, constituting an everywhere dense set. We suppose that $\vec{\xi}_k \neq \vec{O}$ for all k.

We choose an infinite sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$$

of positive numbers with

$$\sum_{k=1}^{\infty} \lambda_k < +\infty .$$

We put

$$\vec{x}_k =_{df} \frac{1}{|\vec{\xi}_k|} \cdot \vec{\xi}_k .$$

Now if c is a space, we define

$$\Phi(c) =_{df} \sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_c \vec{x}_k \,|^2. \tag{1}$$

γ.**1.7.** THEOREM. If $c_1, c_2, \ldots, c_n, \ldots$ is an infinite sequence of mutually orthogonal spaces, then

$$\Phi(\sum_{n=1}^{\infty} c_n) = \sum_{n=1}^{\infty} \Phi(c_n).$$

PROOF. We have, by (1):

$$\sum_{n=1}^{\infty} \Phi(c_n) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_{c_n} \vec{x}_k \,|^2 =$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_{c_n} \vec{x}_k \,|^2 =$$

$$= \sum_{k=1}^{\infty} \lambda_k \sum_{n=1}^{\infty} |\, \mathrm{Proj}_{c_n} \vec{x}_k \,|^2 =$$

$$= \sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_{\sum_{n=1}^{\infty} c_n} \vec{x}_k \,|^2 =$$

$$= \Phi(\sum_{n=1}^{\infty} c_n).$$

γ.**1.7.1.** THEOREM. If $\Phi(c)=0$, then $c=O$.

PROOF. Let $\Phi(c)=0$ i.e.

$$\sum_{n=1}^{\infty} \lambda_n \,|\, \mathrm{Proj}_c \vec{x}_n \,|^2 = 0.$$

The terms of the sum being non-negative, we get

$$\lambda_n \,|\, \mathrm{Proj}_c \overrightarrow{x_n} \,|^2 = 0 \text{ for all } n = 1, 2, \ldots;$$

hence

$$|\, \mathrm{Proj}_c \overrightarrow{x_n} \,|^2 = 0 \text{ for all } n;$$

hence

$$|\, \mathrm{Proj}_c \overrightarrow{\xi_n} \,|^2 = 0;$$

hence

(2) $$\mathrm{Proj}_c \overrightarrow{\xi_n} = \overrightarrow{0} \text{ for all } n.$$

Now, let $\overrightarrow{y} \in c$. We have

$$\overrightarrow{\xi_n} = \mathrm{Proj}_c \overrightarrow{\xi_n} + \mathrm{Proj}_{co\,c} \overrightarrow{\xi_n} = \mathrm{Proj}_{co\,c} \overrightarrow{\xi_n};$$

hence

$$\overrightarrow{\xi_n} \in co\; c \text{ for all } n,$$

i.e., we get for the scalar products:

$$(\overrightarrow{y}, \overrightarrow{\xi_n}) = 0 \text{ for all } n.$$

Since the set of all $\overrightarrow{\xi_n}$ is everywhere dense in **H** in the **H**-topology, it follows that $\overrightarrow{y} = \overrightarrow{O}$.

Thus we have proved that in general, if $\overrightarrow{y} \in c$, then $\overrightarrow{y} = \overrightarrow{O}$. It follows that $c = O$, so the theorem is established.

We also have $\Phi(O) = 0$, which is obvious.

γ.1.7.1.1. Theorem. Thus we have proved that $\Phi(\dot{a})$ is a kind of denumerably additive and effective measure of spaces.

γ.1.7.2. Theorem. If

1. $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots,$
2. $a = \sum_{n=1}^{\infty} a_n,$

then

$$\lim_{n \to \infty} \|\, a_n, a \,\| = 0,$$

where $\|\ \|$ denotes the γ-distance between the spaces a, a_n in the γ-topology [γ.1.5.].

PROOF. We have $a_n \leq a$, hence $a_n a = a_n$. This gives

$$a_n - a = a_n \text{ co } a = (a_n a) \text{ co } a = a_n(a \text{ co } a) = a_n \cdot O = O.$$

Consequently

$$a_n \dot{+} a = (a_n - a) + (a - a_n) = a - a_n .$$

We also have

$$a = a_n + (a - a_n)$$

with $a_n + a - a_n$, [D.2.4], [D.2.4a].

Applying [D.2.6.5], we get

$$\text{Proj}_a \overrightarrow{X} = \text{Proj}_{a_n} \overrightarrow{X} + \text{Proj}_{a - a_n} \overrightarrow{X},$$

i.e.

$$\text{Proj}_a \overrightarrow{X} - \text{Proj}_{a_n} \overrightarrow{X} = \text{Proj}_{a - a_n} \overrightarrow{X} = \text{Proj}_{a \dot{+} a_n} \overrightarrow{X}.$$

We have

$$\| a_n , a \| = \sum_{k=1}^{\infty} \lambda_k \, | \, \text{Proj}_a \overrightarrow{x_k} - \text{Proj}_{a_n} \overrightarrow{x_k} \, | = \tag{3}$$

$$= \sum_{k=1}^{\infty} \lambda_k \, | \, \text{Proj}_{a \dot{+} a_n} \overrightarrow{x_k} \, | ,$$

Since $a_n \leq a$, the spaces a_n, a are compatible [D.5], [D.8], [D.5.1], for every $n = 1, 2, \ldots$.

Let us white (3) in te form

$$\| a_n , a \| = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \cdot \sqrt{\lambda_k} \, | \, \text{Proj}_{a - a_n} \overrightarrow{x_k} \, | .$$

Applying the Cauchy-Schwartz inequality, we get:

$$\| a_n , a \|^2 \leq \sum_{k=1}^{\infty} \lambda_k \cdot \sum_{k=1}^{\infty} \lambda_k \, | \, \text{Proj}_{a \dot{+} a_n} \overrightarrow{x_k} \, |^2 .$$

Hence

$$\| a_n , a \|^2 \leq \sum_{k=1}^{\infty} \lambda_k \cdot \Phi(a - a_n) \tag{4}$$

Now we are going to prove that the spaces

γ.1.7.3.

$$a_1 , a_2 - a_1 , a_3 - a_2 , ..., a_{k+1} = a_k , ... \tag{4.1}$$

(where, in general, $p - q =_{df} p \cdot \mathrm{co}\, q$) are mutually orthogonal.

PROOF. Notice that $\mathfrak{L}$ is a complementary lattice, [A].

We have

$$a_1 \cdot (a_2 - a_1) = a_1 \cdot (a_2 \mathrm{co}\, a_1) = (a_1 \cdot \mathrm{co}\, a_1) \cdot a_2 = 0 \cdot a_2 = 0,$$

and

$$(a_{n+1} - a_n) \cdot (a_{n+k+1} - a_{n+k}) = (_{n+1} \cdot \mathrm{co}\, a_n \cdot a_{n+k+1} \cdot \mathrm{co}\, a_{n+k} =$$
$$= (a_{n+1} \cdot a_{n+k+1}) \cdot (\mathrm{co}\, a_n \cdot \mathrm{co}\, a_{n+k}) = a_{n+1} \cdot \mathrm{co}\, a_{n+k},$$

because

$$a_{n+1} \cdot \mathrm{co}\, a_{n+k+1} = a_{n+1}$$

and

$$\mathrm{co}\, a_n \cdot \mathrm{co}\, a_{n+k} = \mathrm{co}\, a_{n+k} .$$

It follows that:

$$(a_{n+1} - a_n) \cdot (a_{n+k+1} - a_{n+k}) \leq a_{n+1} \cdot \mathrm{co}\, a_{n+k} \leq a_{n+k} \cdot \mathrm{co}\, a_{n+k} = 0.$$

Thus the spaces (4.1) are mutually orthogonal. Q.E.D.
In addition to that we have

$$a - a_n = (a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + ...,$$

which implies, by [γ.7.]:

$$\Phi(a - a_n) = \Phi(a_{n+1} - a_n) + \Phi(a_{n+2} - a_{n+1}) + ...$$

Given $\varepsilon > 0$, there exists n_o such that for all $n \geq n_o$ we get:

$$\Phi(a - a_n) \leq \varepsilon.$$

Consequently, by (4) we obtain:

$$\| a_n , a \|^2 \leq \sum_{k=1}^{\infty} \lambda_k \cdot \varepsilon.$$

This proves that

$$\| a_n , a \|^2 \leq \sqrt{\sum_{k=1}^{\infty} \lambda_k} \cdot \sqrt{\varepsilon},$$

which completes the proof of theorem [γ.1.7.2.].

γ.1.7.4. THEOREM. Having that, consider the collection $\mathfrak{L}'$ of all finitely dimensional, closed subspaces of **H**. This collection if ordered geometrically, i.e. by the relation of inclusion of sets of vectors in **H**, constitutes a lattice $\mathfrak{L}'$ which is a sublattices of $\mathfrak{L}$. We shall prove that the domain of the lattice $\mathfrak{L}'$ is-everywhere dense in $\mathfrak{L}$.

PROOF. First we notice that if a is a finite dimensional space, then we have

$$a, a, a, \ldots, a, \ldots \rightarrow a$$

in the γ-topology.

Let *a* be an arbitrary space with infinite dimensions.
Choose in *a* a saturated orthonormal set of vectors

$$\vec{\varphi_1}, \vec{\varphi_2}, \ldots, \vec{\varphi_n}, \ldots .$$

Denote by a_s the space spanned by $\vec{\varphi_1}, \vec{\varphi_2}, \ldots, \vec{\varphi_s}$. We have

$$a_1 \leq a_2 \leq \ldots \leq a_s \leq \ldots,$$

where al these spaces are finitely dimensional. We have

$$a = \sum_{s=1}^{\infty} a_s ,$$

so we are in the conditions of the theorem [γ.1.7.2.].

It follows that

$$\| a_n - a \| \to 0 \text{ for } n \to \infty .$$

Hence a_n tends to a in the γ-topology.
The theorem is established.

γ.**1.7.9.** The γ-topology has been defined by means of an infinite sequence of vectors [γ.1.6.]

(1) $$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

in **H**, which makes up an everywhere dense set in **H**. Let us take that sequence and keep it fixed.

Consider the set $\mathcal{H}$ of all finite dimensional spaces made up by finite subcollections of (1). This collection is denumerable.

Indeed $\mathcal{H}$ is composed of the spaces

(2) $$[\vec{\xi}_\alpha], [\vec{\xi}_\alpha, \vec{\xi}_\beta], [\vec{\xi}_\alpha, \vec{\xi}_\beta, \vec{\xi}_\gamma], \ldots,$$

where $\alpha, \beta, \gamma, \ldots = 1, 2, 3, \ldots$. They are spanned by one, two, three etc. vectors taken from the sequence $\vec{\xi}_1, \vec{\xi}_2, \ldots$.

γ.**1.7.9.1.** Theorem. We shall prove that taken any p-dimensional space c, we can it γ-approximate with spaces taken from $\mathcal{H}$.

Let $\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_p$ be an orthogonal system of vectors, saturated in c.

Since the collection of vectors $\{\vec{\xi}_n\}$, $n = 1, 2, \ldots$ is everywhere dense in **H**, we can find its partial infinite sequences:

(3) $$\begin{cases} \vec{\xi}_{1,1}, \vec{\xi}_{21}, \ldots, \vec{\xi}_{n1}, \ldots \to \vec{\psi}_1 \\ \vec{\xi}_{12}, \vec{\xi}_{22}, \ldots, \vec{\xi}_{n2}, \ldots \to \vec{\psi}_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \vec{\xi}_{1p}, \vec{\xi}_{2p}, \ldots, \vec{\xi}_{np}, \ldots \to \vec{\psi}_p . \end{cases}$$

Let c_n be the space spanned by the vectors

$$\vec{\xi}_{n1}, \vec{\xi}_{n2}, \ldots, \vec{\xi}_{np} .$$

γ.**1.7.9.2.** Now we can, by using the known E. Schmidt's orthogonalizing process, accompanied by some approximation, transform the system of vectors

(3.1.) $$\vec{\xi}_{n1}, \vec{\xi}_{n2}, \dots, \vec{\xi}_{np},$$

into an orthonormal system in a_n

(4) $$\vec{z}_{n1}, \vec{z}_{n2}, \dots, {}_{np},$$

but still conserving the convergences indicated in (2).

So we get the relations

(5) $$\begin{cases} \vec{z}_{11}, \vec{z}_{21}, \dots, \vec{z}_{n1} \to \vec{\psi}_1 \\ \vec{z}_{12}, \vec{z}_{22}, \dots, \vec{z}_{n2} \to \vec{\psi}_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \vec{z}_{1p}, \vec{z}_{2p}, \dots, \vec{z}_{np} \to \vec{\psi}_p . \end{cases}$$

The vectors (4) are linear combinations of the vectors (3.1).

γ.**1.7.9.3.** Let us we call the mentioned E. Schmidt-device.

Put

$$\vec{y}_{n1} =_{df} \frac{\vec{\xi}_{n1}}{|\vec{\xi}_{n1}|}; \text{ we have } |\vec{y}_{n1}| = 1$$

and

$$\vec{y}_{n,s+1} =_{df} \sum_{i=1}^{s} \frac{1}{|\vec{y}_{ni}|^2} \cdot (\vec{y}_{ni}, \vec{\xi}_{n,s+1}) \vec{y}_{ni} + \vec{\xi}_{n,s+1} .$$

The vectors

$$\vec{y}_{n1}, \vec{y}_{n2}, \dots, \vec{y}_{n,s+1}$$

are orthogonal with one another, and $\neq \vec{O}$.

Put

$$\vec{z}_{ni} =_{df} \frac{\vec{y}_{ni}}{|\vec{y}_{ni}|} .$$

The vectors $z_{n1}, z_{n2}, \ldots, z_{np}$ constitute an orthonormal system of vectors.

We have in the **H**-topology:

$$\lim_{n\to\infty} \vec{z}_{ni} = \vec{\psi}_i, \qquad (i=1, 2, \ldots, p).$$

γ.**1.7.9.4.** Having that all, take an arbitrary vector $\vec{x}$.

We have

$$\mathrm{Proj}_{c_n}\vec{x} - \mathrm{Proj}_c\vec{x} = \sum_{i=1}^{p} (\vec{z}_{ni}, \vec{x})\cdot z_{ni} - \sum_{i=1}^{p} (\psi_i, \vec{x})\vec{\psi}_i =$$

$$= \sum_{i=1}^{p} (\vec{z}_{ni}, \vec{x})\vec{z}_{ni} - \sum_{i=1}^{p} (\vec{z}_{ni}, \vec{x})\vec{\psi}_i + \sum_{i=1}^{p} (\vec{z}_{ni}, \vec{x})\vec{\psi}_i - \sum_{i=1}^{p} (\vec{\psi}_i, \vec{x})\vec{\psi}_i =$$

$$= \sum_{i=1}^{p} (\vec{z}_{ni}, \vec{x})\cdot(\vec{z}_{ni} - \vec{\psi}_i + \sum_{i=1}^{p} (\vec{z}_{ni} - \vec{\psi}_i, \vec{x})\vec{\psi}_i .$$

Hence

$$|\mathrm{Proj}_{c_n}\vec{x} - \mathrm{Proj}_c\vec{x}| \leq \sum_{i=1}^{p} |\vec{z}_{ni}|\cdot|\vec{x}|\cdot|\vec{z}_{ni} - \vec{\psi}_i| + \sum_{i=1}^{p} |\vec{z}_{ni} - \vec{\psi}_i|\cdot|\vec{x}|\cdot|\vec{\psi}_i| \leq$$

$$\leq \sum_{i=1}^{p} |\vec{x}|\cdot|\vec{z}_{ni} - \vec{\psi}_i| + \sum_{i=1}^{p} |\vec{z}_{ni} - \vec{\psi}_i|\cdot|\vec{x}| = 2|\vec{x}|\cdot\sum_{i=1}^{p} |\vec{z}_{ni} - \vec{\psi}_i| .$$

Let $\varepsilon > 0$. There exists N' such that if $n \geq N'$, then

$$|z_{ni} - \vec{\psi}_i| \leq \frac{\varepsilon}{2p} .$$

It follows that

$$|\mathrm{Proj}_{c_n}\vec{x} - \mathrm{Proj}_c\vec{x}| \leq \varepsilon\cdot|\vec{x}| \text{ for all } n \geq N'. \tag{6}$$

That formula shows a kind of uniform convergence, because it is valid for every $\vec{x}$, and N' does not depend on $\vec{x}$.

We can write

$$\| c_n, c \| = \sum_{k=1}^{\infty} \lambda_k |\mathrm{Proj}_{c_n}\vec{x}_k - \mathrm{Proj}_c\vec{x}_k| \leq \sum_{k=1}^{\infty} \lambda_k |\vec{x}_k|\cdot\varepsilon \leq$$

$$\leq \varepsilon\cdot\sum_{k=1}^{\infty} \lambda_k \text{ for all } n \geq N'.$$

Thus we have proved that $\| c_n, c \| \rightarrow 0$ for $n \rightarrow \infty$, i.e. c_n-tends to c in the γ-topology.

The space c was any one with p dimension and c_n was the space spanned by the vectors

$$\vec{\xi}_{n1}, \vec{\xi}_{n2}, \ldots, \vec{\xi}_{np},$$

which belong to the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$.

It follows that $\mathcal{H}$ is everywhere dense in $\mathcal{L}$, with respect to the γ-topology.

Since c_n to spanned by a finite number of vectors $\vec{\xi}_n$, it follows that:

The γ-topology in the lattice $\mathcal{L}$ of all closed subspaces of **H** is γ-separable.

γ.1.7.10. We have proved the THEOREM: If

1. **H** is a non trivial Hilbert-Hermite separable and complete space with norm denoted by $|\ |$.

2. $\mathcal{L}$ is the lattice of all closed subspaces of **H**, ordered geometrically i.e. by the relation of inclusion of sets of vectors.

3. The γ-distance between spaces a, b is defined by

$$\| a, b \| = \sum_{k=1}^{\infty} \lambda_k \, | \operatorname{Proj}_a x_k - \operatorname{Proj}_b x_k | ,$$

where

$$\vec{x}_k =_{df} \frac{1}{| \xi_k |} \vec{\xi}_k , \qquad k = 1, 2, \ldots$$

and $\vec{\xi}_1, \vec{\xi}_2, \ldots$ is a set of vectors everywhere dense in **H**, and λ_k a converging sequence of positive numbers, then this distance organizes the lattice $\mathcal{L}$ into a metric space, which is separable.

γ.1.7.10.1. Notice that there are many γ-topologies on $\mathcal{L}$, depending of the choice of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ and the set $\vec{\xi}_n$ of vectors.

Section 2.

Measure-topologies on a geometrical tribe of spaces in the H.H.-space.

γ.**8.** Now we shall study topologies on a geometrical tribe of spaces in **H**. First we shall prove an identity:

THEOREM. If $(\mathfrak{T})$ is a geometrical tribe of spaces in **H**, then for any spaces a, b and any vector $\vec{y}$ in $(\mathfrak{T})$ we have.

$$(1) \qquad |\mathrm{Proj}_a \vec{y} - \mathrm{Proj}_b \vec{y}|^2 = |\mathrm{Proj}_{a-b} \vec{y}|^2 + |\mathrm{Proj}_{b-a} \vec{y}|^2.$$

PROOF. Remind that the spaces of $(\mathfrak{T})$ are compatible with one another.

First we prove the

γ**8.a.** LEMMA. If p, q are compatible spaces and $\vec{\zeta}$ is a vector, then $(\mathrm{Proj}_p \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) = (\mathrm{Proj}_{pq} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta})$, (scalar product).

PROOF. We have

$$(\mathrm{Proj}_p \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) = (\mathrm{Proj}_{pq+(p-q)} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) =$$
$$= (\mathrm{Proj}_{pq} \vec{\zeta} + \mathrm{Proj}_{p-q} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) = (\mathrm{Proj}_{pq} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) +$$
$$+ (\mathrm{Proj}_{p-q} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}).$$

Since $p-q \perp q$, we have

$$(\mathrm{Proj}_{p-q} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) = 0.$$

Consequently

$$(\mathrm{Proj}_p \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}) = (\mathrm{Proj}_{pq} \vec{\zeta}, \mathrm{Proj}_q \vec{\zeta}). \qquad \text{Q.E.D.}$$

γ.**8.b.** The lemma having been established, consider the expression

$$|\mathrm{Proj}_a \vec{y} - \mathrm{Proj}_b \vec{y}|^2.$$

We have

$$(2)\qquad |\,\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y}\,|^2=(\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y},\ \mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y})=$$
$$=(\mathrm{Proj}_a\vec{y},\ \mathrm{Proj}_a\vec{y})-(\mathrm{Proj}_b\vec{y},\ \mathrm{Proj}_a\vec{y})-(\mathrm{Proj}_a\vec{y},\ \mathrm{Proj}_b\vec{y})+$$
$$+(\mathrm{Proj}_b\vec{y},\ \mathrm{Proj}_b\vec{y})=[(\mathrm{Proj}_a\vec{y},\ \mathrm{Proj}_a\vec{y})-(\mathrm{Proj}_{ba}\vec{y},\ \mathrm{Proj}_a\vec{y})]+$$
$$+[(\mathrm{Proj}_b\vec{y},\ \mathrm{Proj}_b\vec{y})-(\mathrm{Proj}_{ab}\vec{y},\ \mathrm{Proj}_b\vec{y})],$$

because, by virtue of the lemma:

$$(\mathrm{Proj}_b\vec{y},\ \mathrm{Proj}_a\vec{y})=(\mathrm{Proj}_{ba}\vec{y},\ \mathrm{Proj}_a\vec{y})$$

and

$$(\mathrm{Proy}_a\vec{y},\ \mathrm{Proj}_b\vec{y})=(\mathrm{Proj}_{ab}\vec{y},\ \mathrm{Proj}_b\vec{y}).$$

Hence we obtain the equality:

$$|\,\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y}\,|^2=(\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_{ba}\vec{y},\ \mathrm{Proj}_a\vec{y})+$$
$$+(\mathrm{Proj}_b\vec{y}-\mathrm{Proj}_{ab}\vec{y},\ \mathrm{Proj}_b\vec{y})=$$
$$=(\mathrm{Proj}_{a-ba}\vec{y},\ \mathrm{Proj}_a\vec{y})+(\mathrm{Proj}_{b-ab}\vec{y},\ \mathrm{Proj}_b\vec{y})=$$
$$=(\mathrm{Proj}_{a-b}\vec{y},\ \mathrm{Proj}_a\vec{y})+(\mathrm{Proj}_{b-a}\vec{y},\ \mathrm{Proj}_b\vec{y}).$$

Applying once more the lemma, we get

$$|\,\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y}\,|^2=(\mathrm{Proj}_{a-b}\vec{y},\ \mathrm{Proj}_{(a-b)a}\vec{y})+(\mathrm{Proj}_{b-a}\vec{y},\ \mathrm{Proj}_{(b-a)b}\vec{y})=$$
$$=(\mathrm{Proj}_{a-b}\vec{y},\ \mathrm{Proj}_{a-b}\vec{y})+(\mathrm{Proj}_{b-a}\vec{y},\ \mathrm{Proj}_{b-a}\vec{y})=$$
$$=|\,\mathrm{Proj}_{a-b}\vec{y}\,|^2+|\,\mathrm{Proj}_{b-a}\vec{y}\,|^2.$$

Q.E.D.

γ.**8.1**. This being established, we can write:

$$(3)\qquad |\,\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y}\,|^2=|\,\mathrm{Proj}_{(a-b)+(b-a)}\vec{y}\,|^2,$$

and then

$$(4)\qquad |\,\mathrm{Proj}_a\vec{y}-\mathrm{Proj}_b\vec{y}\,|^2=|\,\mathrm{Proj}_{a\dotplus b}\vec{y}\,|^2,$$

because the spaces a—b, b, b—a are orthogonal to one another.

γ.**8.2.** THEOREM. Let

1. $(\mathfrak{T})$ be a geometrical tribe of spaces,

2. $\mu(\dot{a})$ an effective measure on $(\mathfrak{T})$,

3. $a_1, a_2, \ldots, a_n, \ldots$

an infinite sequence of somata of $(\mathfrak{T})$ such that

$$\mu(a_n \dotplus a) \rightarrow 0,$$

4. $\| p, q \|$ is, in general, the γ-distance between two somata of $(\mathfrak{T})$, [γ.q.10.], then

$$\| a_n, a \| \rightarrow 0 \text{ for } n \rightarrow \infty.$$

PROOF. We shall apply an argument similar to that used in [γ.1.7.2.].

Consider the expression:

$$\| a_n, a \| =_{df} \sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_{a_n} \overrightarrow{x_k} - \mathrm{Proj}_a \overrightarrow{x_k} \,| =$$

$$= \sum_{k=1}^{\infty} \sqrt{\lambda_k} \cdot \sqrt{\lambda_k} \,|\, \mathrm{Proj}_{a_n} \overrightarrow{x_k} - \mathrm{Proj}_a \overrightarrow{x_k} \,| .$$

We get

$$\| a_n, a \|^2 \leq \sum_{k=1}^{\infty} \lambda_k \cdot \sum_{k=1}^{\infty} \lambda_k \,|\, \mathrm{Proj}_{a_n} \overrightarrow{z_k} - \mathrm{Proj}_a \overrightarrow{x_k} \,|^2.$$

Applying the formula (3) in [γ.8.1.], we get

$$\| a_n, a \|^2 \leq \sum_{k=1}^{\infty} \lambda_k \cdot \sum_{k=1}^{\infty} \lambda_k \cdot |\, \mathrm{Proj}_{a_n \dotplus a} \overrightarrow{x_k} \,|^2,$$

(5) $$\| a_n, a \|^2 \leq \left(\sum_{k=1}^{\infty} \lambda_k \right)^2 \cdot \Phi(a_n \dotplus a).$$

Now we have proved in [γ.1.7.] that $\Phi(p)$ is a denumerably additive measure, and by [γ.1.7.1.] we have that if $\Phi(p)=0$, then $p=0$. So $\Phi(p)$ is a denumerably additive and effective measure on $(\mathcal{T})$.

Since $\mu(p)$ and $\Phi(p)$ are both effective, denumerably additive measures, it follows that they induce equivalent topologies on $(\mathcal{T})$ i.e.

γ.**8.2.1.** THEOREM. The following are equivalent for somata of $(\mathcal{T})$:

I. $\lim_{n\to\infty} \Phi(a_n \dot{+} a)=0,$

II. $\lim_{n\to\infty} \mu(a_n \dot{+} a)=0.$

We have supposed that $\mu(a_n \dot{+} a) \to 0$.
It follows that $\Phi(a_n \dot{+} a) \to 0$. Hence by (5)

$$\| a_n , a \| \to 0. \qquad \text{Q.E.D.}$$

γ.**8.3.** THEOREM. Under hypothesis I, II, if

$$\| a_n , a \| \to 0, \text{ then } \mu(a_n \dot{+} a) \to 0 \text{ for } n \to \infty.$$

PROOF. Let $\| a_n , a \| \to 0$. Take $\varepsilon>0$ and find N such that if $n>n$, then

$$\| a_n , a \| \leq \varepsilon.$$

Hence, by definition of the γ-distance of somata:

$$\sum_{k=1}^{\infty} \lambda_k \cdot | \mathrm{Proj}_{a_n} \vec{x}_k - \mathrm{Proj}_a \vec{x}_k | \leq \varepsilon. \tag{6}$$

Since $|\vec{x}_k|=1$, we have

$$| \mathrm{Proj}_{a_n} \vec{x}_k - \mathrm{Proj}_a \vec{x}_k | \leq | \mathrm{Proj}_{a_n} \vec{x}_k | + | \mathrm{Proj}_a \vec{x}_k | \leq$$
$$\leq |\vec{x}_k| + |\vec{x}_k| = 2.$$

Consequently

$$0 \leq \tfrac{1}{3} | \mathrm{Proj}_{a_n} \vec{x}_k - \mathrm{Proj}_a \vec{x}_k | \leq \tfrac{2}{3} < 1.$$

Hence

$$\tfrac{1}{9} | \mathrm{Proj}_{a_n} \vec{x}_k - \mathrm{Proj}_a \vec{x}_k |^2 \leq \tfrac{1}{3} | \mathrm{Proj}_{a_n} \vec{x}_k - \mathrm{Proj}_a \vec{x}_k | .$$

It follows that

$$\frac{1}{9}\cdot\sum_{k=1}^{\infty}\lambda_k \,|\, \mathrm{Proj}_{a_n}\overrightarrow{x_k}-\mathrm{Proj}_a\overrightarrow{x_k}\,|^2\leq\frac{1}{3}\sum_{k=1}^{\infty}\lambda_k \,|\, \mathrm{Proj}_{a_n}\overrightarrow{x_k}-\mathrm{Proj}_a\overrightarrow{x_k}\,|;$$

hence

$$\frac{1}{9}\sum_{k=1}^{\infty}\lambda_k \,|\, \mathrm{Proj}_{a_n}\overrightarrow{x_k}-\mathrm{Proj}_a\overrightarrow{x_k}\,|^2\leq\frac{1}{3}\varepsilon.$$

Applying the identity (4) in [γ.8.1.], we get

$$\sum_{k=1}^{\infty}\lambda_k \,|\, \mathrm{Proj}_{a_n\dot{+}a}\overrightarrow{x_k}\,|^2\leq 3\varepsilon. \tag{7}$$

It follows, [(1), γ1.6.],

$$\Phi(a_n\dot{+}a)\leq 3\varepsilon \text{ for all } n\geq N.$$

It follows that

$$\Phi(a_n\dot{+}a)\to 0 \text{ for } n\to 0,$$

and then, by [γ.8.2.1.],

$$\mu(a_n\dot{+}a)\to 0.$$

γ.8.3.1. Theorem. The theorems [γ.8.2.], [γ.8.2.1.] show that on ($\mathfrak{T}$) the μ-topology and the γ-topology restricted to ($\mathfrak{T}$) are equivalent.

γ.8.3.1.1. We have proved that the γ-topology on the lattice $\mathfrak{L}$ is γ-separable.

Hence i.e.: there exists a denumerable collection $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ of spaces (in $\mathfrak{L}$), approximating exery soma of $\mathfrak{T}$. But to prove that the μ-topology is separable *we must prove* the existence of such a collection not only in $\mathfrak{L}$ but also in ($\mathfrak{T}$).

γ.8.3.2. The vectors $\overrightarrow{\xi_1}, \overrightarrow{\xi_2}, \ldots, \overrightarrow{\xi_n}$ give approximations of somata of ($\mathfrak{T}$) but these spaces, generated by ξ_k, may not belong to ($\mathfrak{T}$). We must find another sequence, taken from ($\mathfrak{T}$).

γ.8.4. Let us consider the sequence $\overrightarrow{\xi_1}, \overrightarrow{\xi_2}, \ldots, \overrightarrow{\xi_n}, \ldots$ and the set [γ.7.9.1.] of all spaces spanned by finite numbers of these vectors.

Take a natural number n. Denote by $\mathcal{H}_n$ the set of all somata $b \in \mathcal{H}$, for which there exists a space $a \in \mathcal{T}$ with

$$(1) \qquad \| a, b \| \leq \frac{1}{n}.$$

For every $b \in \mathcal{H}_n$ choose a soma $a \in T$ such that (1) takes place.

Denote by $\mathcal{T}_n$ the set of all a obtained in this way for different $b \in \mathcal{H}_n$.

The set $\mathcal{H}_n$ is at most denumerable and the same holds for $\mathcal{T}_n$, hence also for their set union $\mathfrak{R}$.

Evidently $\mathfrak{R} \subseteq \mathcal{T}$.

We shall prove that $\mathfrak{R}$ is everywhere dense in $(\mathcal{T})$. To do that, let d be any space of $(\mathcal{T})$.

The set $\mathcal{H}$ being everywhere dense in $\mathcal{L}$, composed of all spaces, there evists $b'_n \in \mathcal{H}$ such that

$$\| d, b'_n \| \leq \frac{1}{n}.$$

Hence b'_n belongs to $\mathcal{H}_n$.

Let $a'_n \in \mathcal{T}_n$ be the space corresponding to b'_n.

Hence we have

$$\| a'_n, b'_n \| \leq \frac{1}{n}.$$

It follows that

$$\| d, a'_n \| \leq \frac{\varepsilon}{n}, \text{ where } a'_n \in R.$$

This being true for all $n = 1, 2, \ldots$, we have

$$\| d, a'_n \| \to 0, \text{ where } a'_n \in \mathfrak{R}.$$

Hence $\mathfrak{R}$ is everywhere dense in $\mathcal{T}$.

The two topologies considered being equivalent, it follows that $\mathfrak{R}$ is a subset of $\mathcal{T}$ and everywhere dense in the $\mathcal{T}$-topology, induced by the measure μ.

Thus we have proved the

γ.**8.5.** THEOREM. If

1. **H** is a separable and complete H.H.-space.

2. $(\mathfrak{T})$ a geometrical tribe of spaces.

3. μ a denumerably additive effective measure on $(\mathfrak{T})$, defining the distance of two spaces

$$|\, a,\ b\, | =_{\mathrm{df}} \mu(a \dotplus b),$$

then the topology, generated by $|\, a,\ b\, |_\mu$ is separable.

REMARK. We do not know whether this topology is complete, but we shall prove that on a measured tribe of spaces it is.

γ.**9.** For the sake of comprehensiveness, we shall give a proof that the (μ)-topology in $(\mathfrak{T})$ is complete, i.e. if $\{a_n\}$ satisfies the Cauchy condition, then a_n possesses a μ-limit in $(\mathfrak{T})$[9].

γ.**9.1.** PROOF. We shall prove that a necessary and sufficient condition for the existing of a soma, where

$$\lim_{n \to \infty} |\, a_n \dotplus a\, |_\mu = 0,$$

is a following one: For every $\sigma > 0$ there exists an index n_o, such that if $n' \geq n_o$, $n'' \geq n_o$, then

$$|\, a_{n'} \dotplus a_{n''}\, |_\mu \leq \sigma,$$

i.e.

$$\mu(a_{n'} \dotplus a_{n''}) \leq \sigma.$$

The necessity of the condition is obvious. To prove its sufficiency,

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[9]) The theorem with proof is printed in the paper by the author: « Sur une généralisation des intégrales de M. J. Radon », Fund. Math., Vol. 14. However one must know that this theorem has been found simultaneously and independently by Aronszajn, though not published.

The following proof is taken from the above paper by the author.

consider a sequence of positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$, such that $\sum_{n=1}^{\infty} \sigma_n$ converges.

Find the natural numbers $v_1, v_2, \ldots, v_n, \ldots$, such that

$$(1) \qquad | a_n, a_m |_\mu < \sigma_k,$$

when n, m are $\geq v_k$, $(k=1, 2, \ldots)$. Consider the soma

$$(2) \qquad a =_{df} (a_{v_1} + a_{v_2} + \ldots)(a_{v_2} + a_{v_3} + \ldots)\ldots$$

We have

$$a \in T.$$

We also have

$$a_{v_k} + a_{v_{k+1}} + \ldots = a_{v_k} + (a_{v_{k+1}} - a_{v_k}) + (a_{v_{k+2}} - a_{v_{k+1}}) + \ldots$$

Hence

$$(a_{v_k} - a_{v_{k+1}} + \ldots) - a_{v_{k+1}} \leq (a_{v_{k+1}} - a_{v_k}) + (a_{v_{k+2}} - a_{v_{k+1}}) + \ldots$$

Since the left hand side equals

$$(a_{v_k} - a_{v_k}) + (a_{v_{k+1}} - a_{v_k}) \cdot \text{co}\, a_{v_k} + (a_{v_{k+2}} - a_{v_{k+1}}) \cdot \text{co}\, a_{v_k} + \ldots,$$

we get

$$\mu[(a_{v_k} + a_{v_{k+1}} + \ldots) - a_{v_k}] \leq \mu(a_{v_{k+1}} - a_{v_k}) + \mu(a_{v_{k+2}} - a_{v_{k+1}}) + \ldots \leq$$
$$\leq \sigma_k + \sigma_{k+1} + \ldots$$

Consequently

$$(3) \qquad \mu[(a_{v_k} + a_{v_{k+1}} + \ldots) - a_{v_k}] \to 0 \text{ for } k \to \infty.$$

As

$$a_{v_k} - (a_{v_k} + a_{v_{k+1}} + \ldots) = 0,$$

we have

$$\mu[a_{v_k} - (a_{v_k} + a_{v_{k+1}} + \ldots)] = 0,$$

which gives with (3):

(4) $$| a_{v_k}+a_{v_{k+1}}+\ldots,\ a_{v_k} | \rightarrow 0 \text{ for } k\rightarrow \infty.$$

On the other hand, as

$$a \leq a_{v_k}+a_{v_{k+1}}+\ldots,$$

we get

$$\mu[a-(a_{v_k}+a_{v_{k+1}}+\ldots)]=0,$$

$$\mu[(a_{v_k}+a_{v_{k+1}}+\ldots)-a]=\mu(a_{v_k}+a_{v_{k+1}}+\ldots)-\mu(a),$$

which implies:

(5) $$| a_{v_k}+a_{v_{k+1}}+\ldots,\ a |_\mu \rightarrow 0 \text{ for } k\rightarrow \infty,$$

because by (2)

$$\mu(a)=\lim_{k\rightarrow\infty} \mu(a_{v_k}+a_{v_{k+1}}+\ldots).$$

It follows by (4) and (5), that

$$| a_{v_k},\ a | \rightarrow 0 \text{ for } k\rightarrow \infty.$$

Now let $\sigma>0$. Choose k such that

$$\sigma_k<\frac{\sigma}{2},\ | a_{v_k},\ a |_\mu<\frac{\sigma}{2},$$

Let $n\geq v_k$. We have, by (1),

$$| a_{v_k},\ a_n |_\mu<\sigma_k<\frac{\sigma}{2}.$$

Hence:

$$| a,\ a_n |_\mu \leq | a,\ a_{v_k} |_\mu+| a_{v_k},\ a_n | <\sigma.$$

This proves that

$$| a_n,\ a |_\mu \rightarrow 0 \text{ for } n\rightarrow \infty,$$

hence

$$a_n \rightarrow_\mu a.$$

The theorem is established.

γ.9.2. REMARK. We have proved that the special γ-topology considered in the lattice $\mathfrak{L}$ of all spaces is separable (under the condition that **H** is separable), but we do not know whether this γ-topology is complete not, and also we do not know whether all γ-topologies in a given H.H.-space are isomorphic or not.

Section 3.

Linearisation of a geometrical tribe of spaces in the H.H.-space.

We have proved, [DI, 11], the following theorem:

γ.10. THEOREM. Let

$$T_1, T_2, \ldots, T_\omega, \ldots, T_\alpha, \ldots$$

be a finite or transfinite well ordered sequence of geometrical tribes in **H**, such that if

$$\alpha' \leq \alpha'', \text{ then } T_{\alpha'} \subseteq T_{\alpha''}$$

in the sense that the ordering in $(T_{\alpha'})$ is a strict subordering of $T_{\alpha''}$.

Under these circumstances

$$T =_{df} \bigcup_\alpha T_\alpha$$

is also a geometrical tribe set in **H** in the sense, that the ordering in (T_α) is a strict subordering of (T).

γ.11. THEOREM. Let M be any non empty set of mutually compatible spaces.

Then there exists a smallest geometrical tribe (T) of spaces such that every space $\in M$ is a soma of (T). This tribe (T) is unique.

PROOF. Let

(1) $$a_1, a_2, \ldots, a_\omega, \ldots, a_\alpha, \ldots$$

be a well ordering with domain M and where all spaces (1) are different.

Denote by (T_1) the smallest geometrical tribe composed of somata $0, 1, a_1$. The elements of (T_1) are:

$$0,\ 1,\ a_1, \text{co}\, a_1$$

with ordering

$$0 \leq a_1 \leq 1, \qquad 0 \leq \text{co}\, a_1 \leq 1.$$

It may contain only the two elements $0, 1$.

Suppose that we have already defined all tribes (T_β) with $\beta < \alpha$, where α is fixed for a moment, and suppose that all somata of T_β are compatible with the spaces (1).

Let $\alpha - 1$ exist; then we define T_α as the smallest tribe containing the tribe $(T_{\alpha-1})$ and a_α. Such tribe exists and is unique; its somata are

$$p \cdot a_\alpha + q\, \text{co}\, a_\alpha,\ p,\ q \in T_{\alpha-1}, \qquad [\text{DI}.8.2].$$

This is possible because the somata of $(T_{\alpha-1})$ are compatible with a_α.

Now suppose that α is a limit-ordinal, then we define

$$T_\alpha =_{\text{df}} \bigcup_{\beta<\alpha} T_\beta .$$

The somata of T_α are just the somata of T_β, where $\beta < \alpha$; hence they are compatible with a_α.

γ.**11.1.** The above construction gives a finite on transfinite sequence of embedded tribes (T_α).

The tribe

$$(T) =_{\text{df}} \bigcup_{\alpha} (T_\alpha)$$

contains all spaces (1). One proves easily that (T) is a smallest tribe containing (1), and that such a tribe is unique.

γ.**11.2.** We may call (T) the tribe spanned by the set M of spaces.

γ.**12.** Definition. By a *linearly ordered set of spaces* we shall understand any subordering ($\mathfrak{L}'$) of ($\mathfrak{L}$) such that

1) if a, $b \in \mathfrak{L}'$, then either $a \leq b$ or $b \leq a$,

2) $0 \in \mathfrak{L}'$, $1 \in \mathfrak{L}'$.

Thus we can say that ($\mathfrak{L}'$) is finitely genuine strict subordering of ($\mathfrak{L}$). The governing equality in ($\mathfrak{L}'$) is just the restriction to the set $\mathfrak{L}$, of the equality « = » (=), governing in ($\mathfrak{L}$). The domain $\mathfrak{L}'$ of a linearly ordered set of spaces has at least the two somata *0, 1.*

γ.**12.1.** If ($\mathfrak{L}'$) is a linearly ordered set of spaces, then all is somata are compatible with one another.

γ.**12.2.** Definition. If *a*, $b \in \mathfrak{L}'$, we call the space $b - a$ *segment of* ($\mathfrak{L}'$). The spaces *0* and *1* are segments of ($\mathfrak{L}'$).

The domain of the linearly ordered set of spaces possesses at least two somata, viz. *0, 1.*

We have proved, in [γ.1.7.3.], the following

Theorem. Let ($\mathfrak{L}'$) be a linear ordering of spaces and

$$a_1 \leq a_2 \leq \ldots \leq a_n, \qquad n \geq 3$$

a sequence of somata of ($\mathfrak{L}'$).

Then

1) $a_n - a_1 = (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$.

2) The terms $a_k - a_{k-1}$ are all disjoint, and even orthogonal with one another.

γ.**12.4.** Theorem. If

1) $a - a'$, $b - b'$ are both segmants in ($\mathfrak{L}'$),

2) $a - a' = b - b' \neq 0$,

then

$$a=b,\ a'=b'.$$

PROOF. By hyp. 2) we have

$$a'<a,\ b'<b.$$

There are four possibilities:

1°. $b'\leq a'<a\leq b$,

2°. $a'\leq b'<b\leq a$,

3°. $a'\leq b'\leq a\leq b$,

4°. $b'\leq a'\leq b\leq a$.

The possibilities 2°, 4° differ from 1°, 3° only by notation, so it suffices to consider only 1° and 3°.

Take the case 1°:

$$b'\leq a'<a\leq b$$

and denote by

$$\Delta_1,\ \Delta_2,\ \Delta_3$$

the intervals

$$a'-b',\ a-a',\ b-a.$$

We have

$$a-a'=b-b',$$

hence

$$\Delta_2=\Delta_1+\Delta_2+\Delta_3 .$$

Relying on theorem 2.3, we get

$$\Delta_1=0 \text{ i.e. } a'=b'$$

and

$$\Delta_3=0 \text{ i.e. } a=b;$$

so in that case the theorem is established.

$$a' \leq b' \leq a \leq b.$$

Denote the segments

$$b' - a',\ a - b',\ b - a$$

by

$$\Delta'_1,\ \Delta'_2,\ \Delta'_3.$$

We have, by hypothesis,

$$a - a' = b - b'.$$

Hence

$$\Delta'_1 + \Delta'_2 = \Delta'_2 + \Delta'_3.$$

It follows $\Delta'_1 = \Delta'_3$. But Δ'_1, Δ'_3 are disjoint.
Consequently

$$\Delta'_1 = \Delta'_3 = O.$$

Hence $a' = b'$ and $a = b$.
The theorem is established.

γ.**12.5**. THEOREM. The product of two segments is also a segment.

PROOF. Let p, q be two segments. If one is contained in the other, the theorem holds true.

In the remaining case we have the following situation:

$$(1) \qquad \begin{cases} p = a - b,\ q = c - d, \\ b \leq d \leq a \leq c, \end{cases}$$

or another one, where a, b are replaced by b, a.
Take (1), and put

$$\Delta_1 = d - b,\ \Delta_2 = a - d,\ \Delta_3 = c - a.$$

We have

$$p=\Delta_1+\Delta_2\,,$$

$$q=\Delta_2+\Delta_3\,.$$

Hence

$$p\cdot q=(\Delta_1+\Delta_2)(\Delta_2+\Delta_3)=\Delta_2\,,$$

so the theorem follows.

γ.**12.6**. THEOREM. If p is the sum of a finite number of segmenta, then p can be represented as the sum of a finite number of disjoint (hence orthogonal) segments.

PROOF by induction.

γ.**12.7**. THEOREM. If p is the sum of a finite number of segments, then co p is also a finite sum of segments.

Let

$$p=p_1+p_2+\ldots+p_n\,,\qquad n\geq 1$$

where p_i are segments.

By de Morgan law we have

$$\text{co}\ p=\text{co}\ p_1\cdot\text{co}\ p_2\cdot\ \ldots\ \text{co}\ p_n\,.$$

Let

$$p_i=b_i-a_i\,.$$

We have

$$\text{co}\ p_i=(a_i-0)+(b_i\cdot 1-b_i),$$

which we shall write in the form s_i+t_i.

Hence

$$\text{co}\ p=(s_1+t_1)\cdot(s_2+t_2),\ \ldots,\ (s_n+t_n).$$

The distributive law can be applied, because the spaces s_i, t_i are all compatible with one another. Hence co p will be represented as a finite sum of finite products of segments.

Hence, by [γ.12.5.] the theorem follows.

γ.**12.8.** We shall use Lemma D [γ.5.1.], stating that if S is a non empty collection of spaces satisfying the conditions:

1° if a, b∈S, then a+b∈S,

2° if a∈S, then co a∈S,

3° the spaces of S are mutually compatible,

then (S) ardered as in $\mathfrak{L}$, is a geometrical tribe.

Now the elments of T are compatible with one another [DI.4] and the condition:

if a, b∈T, then a+b∈T is obvious.

Then to prove the theorem it suffices to show that if

a∈T, then co a∈T.

This however is true.

γ.**13.** In [D.1]we have proved the following theorem:

Let S be a geometrical tribe, which may be finitely additive only. We shall consider denumerably infinite somatic operations, taken from $\mathfrak{L}$. We know that the spaces obtained in this way are all compatible with one another and also compatible with the somata of S.

The collection of all spaces

$$p=\sum_{i=1}^{\infty}\prod_{k=1}^{\infty} a_{ik}\,, \text{ where } a_{ik}\in S,$$

make up a geometrical tribe, which is an extension of S.

The same tribe will be obtained by taking collection of all spaces

$$q=\prod_{i=1}^{\infty}\sum_{k=1}^{\infty} b_{ik}\,.$$

γ.**13.1.** We call the extended tribe (S^b), the *borelian extension of* S in **H**.

This is the smallest geometrical denumerably additive tribe, containing (S).

γ.**13.2.** If **H** is supposed to be a separable and complete H.H.-space, then (S^b) is completely additive.

γ.**14.** DEFINITION. Let (T) be a denumerably additive tribe. We say that (T) *can be linearized* whenever we can find a linear subordering $(\mathfrak{L}')$ of (T), such that (T) is the smallest borelian extension of the collection of all segments of $(\mathfrak{L}')$.

γ.**15.** REMARK. Starting with a linear ordering $\mathfrak{L}'$ of spaces, we have constructed a tribe, which is denumerably and even completely additive, containing $(\mathfrak{L}')$. Now we shall start with a given tribe (T) and find a kind of « *linearization* » *of* (T).

γ.**15.1.** Let (T) be a given denumerably additive geometrical tribe. We know that (T) admits a denumerably additive effective measure $\mu(a)$.

That measure generates a topology (metric space), where the distance $|a, b|_\mu$ between two somata a, b of (T) is defined by

$$|a, b|_\mu =_{df} \mu(a \dot{+} b) = \mu(a-b) + \mu(b-a).$$

We know that this topology, called μ-topology, is complete.

We have also proved that the topology is separable, which means that there exists a denumerable collection of somata $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ of (T), which is everywhere dense in (T) with respect to the μ-topology.

These facts enable us to « linearize » (T) in an important manner.

γ.**15.2.** Let $a_1, a_2, \dots, a_n, \dots$ be an infinite sequence of somata of (T).

We are going to construct an infinite sequence of linear orderings of spaces, whith more and more ample domains:

$$\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \subseteq \dots \subseteq \mathfrak{L}_n \subseteq \dots,$$

and such that their union contains al the spaces a_i, $(i=1, 2, \dots)$.

We define $(\mathfrak{L}_1)$ as the ordering whose domain is composed of three spaces:

$$0 \leq a_1 \leq 1.$$

1) Suppose that $n \geq 1$, and that we have already defined the linear orderings

$$\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \subseteq \ldots \subseteq \mathfrak{L}_k ,$$

where each of them is a sequence of a finite number of somata.

2) consider the tribes $T(\mathfrak{L}_1)$, $T(\mathfrak{L}_2)$, ..., $T(\mathfrak{L}_k)$, containing $\mathfrak{L}_1$, $\mathfrak{L}_2$, ..., $\mathfrak{L}_k$ respectively.

Every one of them is a finitely genuine, strict supertribe of the preceding ones.

We suppose that $a_1, a_2, \ldots, a_k$ are somata of $T(\mathfrak{L}_k)$.

Let the elements of (L_k) be

$$(1) \qquad 0 \leq b_1^{(k)} \leq b_2^{(k)} \leq \ldots \leq b_{s(k)}^{(k)} \leq 1$$

We shall define $(\mathfrak{L}_{k+1})$.

To do that, first notice, that in general, if $p_1 \leq p_2$ and q is arbitrary, we have:

$$(2) \qquad p_1 \leq p_1 + (p_2 - p_1) \cdot q \leq p_2 .$$

Indeed, we have

$$(p_2 - p_1) \cdot q = p_2 \cdot \text{co}\, p_1 \cdot q \leq p_2 .$$

This implies that

$$(3) \qquad \begin{aligned} 0 \leq b_1^{(k)} a_{k+1} \leq b_1^{(k)} \leq b_1^{(k)} + (b_2^{(k)} - b_1^{(k)}) a_{k+1} \leq \\ \leq b_2^{(k)} \leq b_2^{(k)} + (b_3^{(k)} - b_2^{(k)}) a_{k+1} \leq b_3^{(k)} \leq \ldots \leq \\ \leq b_{s(k)-1}^{(k)} + (b_{s(k)}^{(k)} - b_{s(k)-1}^{(k)}) a_{k+1} \leq b_{s(k)}^{(k)} \leq 1. \end{aligned}$$

This is a linear ordering. We denote it as

$$0 \leq b_1^{(k+1)} \leq b_2^{(k+1)} \leq \ldots \leq b_{s(k+1)}^{(k+1)} \leq 1.$$

and call it $(\mathfrak{L}_{k+1})$.

We have $\mathfrak{L}_k \subseteq \mathfrak{L}_{k+1}$ with preservation of order.

Indeed, in (3) we have the somata

$$b_1^{(k)} \leq b_2^{(k)} \leq \ldots \leq b_{s(k)}^{(k)},$$

which belong to $\mathfrak{L}_k$ with preservation of order.

I say that $a_{k+1} \in T(\mathfrak{L}_{k+1})$.

Indeed, we have [A.2.6.4.],

$$b_1^{(k)} + (b_2^{(k)} - b_1^{(k)}) + (b_3^{(k)} - b_2^{(k)}) + \ldots + (1 - b_{s(k)}^{(k)}) = 1;$$

hence

$$(4) \qquad b_1^{(k)} a_{k+1} + (b_2^{(k)} - b_1^{(k)}) a_{k+1} + \ldots + (1 - b_{s(k)}^{(k)}) a_{k+1} = a_{k+1}.$$

Now, we have

$$(5) \qquad (b_{i+1}^{(k)} - b_i^{(k)}) a_{k+1} = [b_i^{(k)} + (b_{i+1}^{(k)} - b_i^{(k)}) a_{k+1}] - b_i^{(k)}.$$

Since both terms on the right are somata of $T(\mathfrak{L}_{k+1})$, it follows, that so is also

$$(b_{i+1}^{(k)} - b_i^{(k)}) a_{k+1},$$

and then, by (4), $a_{k+1} \in T(\mathfrak{L}_{k+1})$.

Thus we have got, by induction, an infinite sequence of nested linear orderings

$$(6) \qquad \mathfrak{L}_1 \subseteq \mathfrak{L}_2 \subseteq \ldots \subseteq \mathfrak{L}_n \subseteq \ldots,$$

such that $\mathfrak{L}_n$ contain all somata $a_1, a_2, \ldots, a_n$.

γ.15.3. Denote by $(\mathfrak{L}')$ the linear ordering generated by all $(\mathfrak{L}_i)$, $i = 1, 2, \ldots$.

The ordering $(\mathfrak{L}')$ is defined as follows:

Let b, c be somata of

$$\bigcup_{i=1}^{\infty} \mathfrak{a}(\mathfrak{L}_i).$$

There exist indices s, t, such that $b \in \mathfrak{a}\mathfrak{L}_s$, $c \in \mathfrak{a}\mathfrak{L}_t$.

By (6), b, $c \in \mathfrak{L}_r$, where $r = \max(s, t)$.

Now we define the correspondence

$$b(\mathfrak{L}')c$$

as $b \leq c$ in $(\mathfrak{L}_r)$. This definition does not depend on the choice of *s* and t. We can say, that the ordering in $(\mathfrak{L}')$ is taken from the orderings in $(\mathfrak{L}_1), (\mathfrak{L}_2), \ldots, (\mathfrak{L}_i), \ldots$.

The just defined ordering is a linear ordering.

γ.15.4. Consider the tribes $T(\mathfrak{L}_i)$, generated by $(\mathfrak{L}_i)$.

The tribe $T(\mathfrak{L}_i)$ is composed of all finite unions of segments of $\mathfrak{L}_i$.

Now all the segments belonging to $(\mathfrak{L}_i)$, are also segments of $(\mathfrak{L}_{i+k})$ for any $k = 1, 2, \ldots$. Thus the segments belonging to $(\mathfrak{L}_i)$ are also belonging to $(\mathfrak{L}')$, $i = 1, 2, \ldots$, and conversely every segment, of $(\mathfrak{L}')$ belongs to some $(\mathfrak{L}_i)$.

Now we know that the union of all

$$T(\mathfrak{L}_1), T(\mathfrak{L}_2), \ldots, T(\mathfrak{L}_i), \ldots$$

is a finitely additive tribe (S) with domain

$$\mathfrak{a}\, S = \bigcup_{i=1}^{\infty} \mathfrak{a}\, T(\mathfrak{L}_i).$$

The thribe (S) is a finitely genuine subtribe of the given tribe (T); we can write

$$S = \bigcup_{i=1}^{\infty} T(\mathfrak{L}_i).$$

The tribe (S) contains all the spaces

$$a_1, a_2, \ldots, a_n, \ldots$$

We have proved the following THEOREM:

γ.15.5. If

1. (T) is a finitely additive tribe,
2. $a_1, a_2, \ldots, a_n, \ldots$ any sequence of somata of (T) then there

exists a smallest linear ordering ($\mathfrak{L}'$) such that

1) if $b \in \mathfrak{L}'$, then $b \in T$,
2) all a_i, ($i=1, 2, \ldots$), belong to $\mathfrak{L}'$,
3) the ordering ($\mathfrak{L}'$) is a subordering of the ordering (T).

γ.15.6. Having that, let us remind that the μ-topology on (T) is separable. Let

$$(7) \qquad a_1, a_2, \ldots, a_n, \ldots$$

be a sequence of somata of (T), everywhere dense in the μ-topology, and let us apply the just proved theorem.

Take the linear ordering ($\mathfrak{L}'$) and T($\mathfrak{L}'$).

Let $p \in (T)$. There exists a subsequence of (7)

$$(8) \qquad a_{k(1)}, a_{k(2)}, \ldots, a_{k(n)}, \ldots$$

such that

$$\lim_{n\to\infty} a_{k(n)} = p.$$

Now there exists a subsequence of (8)

$$a_{kl(1)}, a_{kl(2)}, \ldots, a_{kl(n)}, \ldots,$$

such that

$$\lim_{n\to\infty} a_{kl(n)} = p.$$

Hence

$$(a_{kl(1)} + a_{kl(2)} + a_{kl(3)} + \ldots)(a_{kl(2)} + {}_{kl(3)} + \ldots)\ldots = p.$$

This says that p belongs to the borelian extension (S^b) of (S).

$$(9) \qquad (T) \subseteq (S^b),$$

hence

$$(T^b) \subseteq (S^b).$$

Now, as we had (S)$\subseteq$(T), it follows that

$$(S^b) \subseteq (T^b).$$

Finally we get $(S^b) = (T^b) = (T)$.

γ.15.7. If (T) is a denumerably additive geometrical tribe of spaces, and μ an effective measure on (T), then there exists a linear subordering $\mathfrak{L}'$ of (T), such that

1) $\mathsf{a}(\mathfrak{L}' \subseteq \mathsf{a}(T)$,

2) (T) is the borelian extension of the tribe (S) composed of all finite sums of segments of $(\mathfrak{L}')$.

We may say that (T) can be « linearized » in the μ-topology.

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