RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 42 (1969), p. 267-297

<http://www.numdam.org/item?id=RSMUP_1969__42__267_0>

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ON THE INTERIOR REGULARITY OF WEAK SOLUTIONS OF NON-STATIONARY NAVIER-STOKES EQUATIONS ON A RIEMANNIAN MANIFOLD

di MILAN Đ. ĐURIĆ *)

Introduction.

To the mathematical investigation concerning the existence, uniqueness and regularity of solutions of non-stationary Navier-Stokes equations in the case of an n-dimensional Eucliedan space E_n, where n is, in the main, either 2 or 3, has been devoted a large number of works by various authors. We mention some of main results. So, J. Leray [10] established the existence of a classical solution which is local in time by means of non-stationary potentials. E. Hopf [3] proved the existence of a weak solution which is global in time. A. Kiselev and O. Ladyzhenskaia [5] showed the local existence and uniqueness of a weak solution of a various type. The paper of T. Kato and H. Fujita [4] represents an attempt to deduce an existence and uniqueness theorem in its classical form by means of Hilbert space theory. Next, we mention papers by J. Lions [11], G. Prodi [13], P. Sobolevskii [15], O. Ladyzhenskaia [8] and so on. For details and a more complete literature we refer to the book [9] and to publications of Steclov Mathematical Institute of Soviet Academy of Sciences.

There is a need for consideration of the above mentioned questions in the case of non-Euclidean space. The present paper just represents such an approach. Namely, in this paper we deal with the existence and regularity of a weak solution of non-stationary Navier-Stokes equations on a Riemannian manifold. We first establish the existence of a weak solution and then show that it is a regular solution in the interior of

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the domain where the external force satisfies the Hölder condition with respect to (x, t).

Some another questions will not be treated for the present. They will be the object of a subsequent publication.

Chapter I.

Preliminaries and definition of generalized solution

1. Some basic notions and notations.

We devote this section to some notations and some basic notions concerning Riemannian manifold and to well-known results by G. de Rham and W. V. D. Hodge [2] and K. Kodaira [7]. Also, we introduce other notations needed for the later work.

Let R be an n-dimensional orientable C^∞ Riemannian manifold. We denote by $(x)=(x_1, \ldots, x_n)$ a system of local coordinates with positive orientation. The local coordinates of a point q on R will be denoted by $x^j(q)$, $j=1, 2, \ldots, n$. $\mathfrak{C}$ will denote a family of coordinate systems whose domains $U_{(k)}$ cover R. With ∇ we shall denote covariant derivatives with respect to the Riemannian connection whose components in a system of local coordinates are Cristoffel symbols $\{{}_j{}^i{}_k\}$. The square of the geodesic distance between two closed points x and ξ according to the metric $ds^2=g_{ij}dx^i dx^j$ will be denoted by $\Gamma=\Gamma(x, \xi)$. Tensor fields of rank ρ on R we shall denote by $\varphi^\rho=\varphi^{i_1, \ldots, i_\rho}(x)$. The fields φ^ρ are said to be continuous and to have continuous derivatives if their components $\varphi^{i_1, \ldots, i_\rho}$, in a local system of coordinates (x), are continuous and have continuous derivatives.

If φ is an antisymmetric ρ-field we define two operators, the exterior derivative operator d and its dual operator δ, see [2]. The operator d, which a ρ-field sets in correspondence with a $(\rho+1)$-field, is defined as follows

$$(d\varphi)_{i_1, \ldots, i_{\rho+1}}=\delta^{\nu\mu_1, \ldots, \mu_\rho}_{i_1, \ldots, i_{\rho+1}} \nabla_\nu \varphi_{\mu_1, \ldots, \mu_\rho}, \tag{1.1}$$

where

$$\delta^{j_1, \ldots, j_\rho}_{i_1, \ldots, i_\rho}=\| \delta^{j_s}_{i_t} \|, \quad \rho>1 \text{ and } s, t=1, \ldots, \rho,$$

is the Kronecker symbol, which for $\rho=1$ is $\delta_i^j=\begin{cases}1, & j=i,\\ 0, & j\neq i\end{cases}$. Dual operator δ, which a ρ-field sets in correspondence with a $(\rho-1)$-field, is

$$(1.2)\qquad (\delta\varphi)_{i_1,\dots,i_{\rho-1}}=-\delta^{\mu_1,\dots,\mu_\rho}_{\nu i_1,\dots,i_{\rho-1}}\nabla^\nu\varphi_{\mu_1,\dots,\mu_\rho}.$$

The above operators can also have the following forms

$$(1.3)\qquad (d\varphi)_{i_1,\dots,i_{\rho+1}}=(-1)^{\nu-1}\partial_{i_\nu}\varphi_{i_1,\dots,\widehat{i_\nu},\dots,i_{\rho+1}},$$

respectively

$$(1.4)\qquad (\delta\varphi)^{i_1,\dots,i_{\rho-1}}=-\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\varphi^{ii_1,\dots,i_{\rho-1}}),$$

where $\wedge$ means which index is to be omitted, $g=\det(g_{ij})$, and in repeated indeces, as usually, one makes summation.

According to K. Kodaira [7] one defines an operator Δ, which to a ρ-field φ sets in correspondence a ρ-field $\Delta\varphi$, as follows

$$(1.5)\qquad \Delta=-(d\delta+\delta d).$$

Furthermore, we always have that $dd=0$ and $\delta\delta=0$. Thus, an antisymmetric tensor field of rank ρ corresponds to a ρ-dimensional complex.

The operators d and δ are generalized curl and generalized divergence operators, Δ is the generalized Laplacian. In that manner we can give the following definitions: *A field φ is non-vortical (solenoidal) if*

$$(1.6)\qquad d\varphi=0,\quad (\delta\varphi)=0.$$

A field $\varphi\in C^2(D)$, where D is an open subset of R, is said to be harmonic in D if

$$(1.7)\qquad 0=-\Delta\varphi=(d\delta+\delta d)\varphi.$$

For the work in sequel we need the Green-Stokes' theorem. Therefore, let $\mathfrak{S}^\rho$ be a differentiable simplex of the manifold R obtained as a topological image of a euclidian ρ-simple S^ρ. Then, the differentiable ρ-chain C^ρ of the manifold R is given by a linear combination

$$C^\rho = \sum_i k_i \mathfrak{S}_i^\rho \tag{1.8}$$

and is a ρ-dimensional algebraic complex, k_i are real coefficients. The boundary of the chain C^ρ is $(\rho-1)$-chain

$$\partial C^\rho = \sum_i k_i \partial \mathfrak{S}_i^\rho. \tag{1.9}$$

For each simplex $\mathfrak{S}^\rho$ we have an identity

$$\delta\delta\mathfrak{S}^\rho = 0.$$

Let α be a $(\rho-1)$-form defined on a domain containing ρ-chain C^ρ. Then the boundary operator ∂ and exterior derivative operator d are connected by a fundamental theorem of calculus, namely Green-Stokes' theorem

$$\int_{\partial C^\rho} \alpha = \int_{C^\rho} d\alpha. \tag{1.10}$$

Let $(u) = (u^1, \ldots, u^\rho)$ be a system of coordinates on the simplex S^ρ. Then points q are described as $q = q(u)$, and the local coordinates $x^i = x^i(q(u))$ of q are continuous functions of u and are of the class C^2 in each point of the simplex $\mathfrak{S}^\rho$. Hence the integral of the field α over the ρ-chain C^ρ is given by the expression

$$\langle \varphi^\rho, C^\rho \rangle = \frac{1}{\rho!} \int_{C^\rho} \varphi_{i_1, \ldots, i_\rho} dx^{i_1, \ldots, i_\rho}, \tag{1.11}$$

where

$$dx^{i_1, \ldots, i_\rho} = \frac{\partial(x^{i_1}, \ldots, x^{i_\rho})}{\partial(u^1, \ldots, u^\rho)} du^{1, \ldots, \rho}.$$

Thus, the Green-Stokes' theorem has the form

$$(1.12) \qquad \langle d\varphi^{p-1}, C^p \rangle = \langle \varphi^{p-1}, \partial C^p \rangle,$$

with $\varphi \in C^1$ in a certain neighborhood of C^p.

According to K. Kodaira [7] for the Green-Stokes' formula (1.12) an another expression can be obtained

$$(1.13) \qquad \delta c^p \times C^{n-p+1} = (-1)^p c^p \times \delta C^{n-p+1},$$

where we assumed that c^p is of the class C^1 in a certain neighborhood of C^p. By means of $\times$ we described the following form

$$(1.14) \qquad c^p \times C^{n-p} = \frac{1}{p!} \int_{C^{n-p}} c^{i_1, \dots, i_p} do_{i_1, \dots, i_p},$$

where $do_{i_1, \dots, i_p}$ are dual coordinates of the surface element $dx^{i_{p+1}, \dots, i_n}$ given as follows

$$(1.15) \qquad do_{i_1, \dots, i_p} = \frac{1}{(n-p)!} \operatorname{sgn} \left({}^{1, \dots, n}_{i_1, \dots, i_p i_{p+1}, \dots, i_n} \right) dx^{i_{p+1}, \dots, i_n},$$

and $\operatorname{sgn} \begin{pmatrix} pq, & \dots, & t \\ ij, & \dots, & m \end{pmatrix}$ means the sign of the permutation $\begin{pmatrix} pq, & \dots, & t \\ ij, & \dots, & m \end{pmatrix}$ if $i, j, \dots, m$ coincide with $p, q, \dots, t$ in a certain order, otherwise it means 0.

Let D be an open domain of the manifold R. Let us consider a class $C_0^k(D)$, $0 \leq k \leq \infty$, of all fields from C^k with a compact carrier. In such a class we introduce the scalar product setting

$$(1.16) \qquad (\varphi^p, \psi^p)_D = \frac{1}{p!} \int_D \varphi_{i_1, \dots, i_p} \psi^{i_1, \dots, i_p} \sqrt{g} dD.$$

Next we give the Green's formula. For the open domain D with the regular boundary ∂D, according to Kodaira the Green's formula has the form

$$(1.17) \qquad (d\varphi^p, \psi^{p+1})_D - (\varphi^p, \delta\psi^{p+1})_D = \int_{\partial D} \{\varphi, \psi\}^{i_n} \sqrt{g} do_{i_n},$$

where $\{\varphi, \psi\}^k$ is a bilinear form of variables φ and ψ, and $do_{i_{\rho+1},\ldots,i_n}$ are dual coordinates of the surface element $dx^{i_1,\ldots,i_\rho}$ given by (1.15).

For the operators d and δ one proves [2] that they are adjoint, namely that

$$(d\alpha, \beta)=(\alpha, \delta\beta),$$

respectively

$$(d\alpha^{\rho-1}, \beta^\rho)=(\alpha^{\rho-1}, \delta\beta^\rho), \tag{1.18}$$

where α and β are fields with compact carriers, and Δ is self-adjoint

$$(\Delta\alpha, \beta)=(\delta\alpha, \delta\beta)+(d\alpha, d\beta)=(\alpha, \Delta\beta). \tag{1.19}$$

Moreover, the operator Δ is permutable with operators d and δ, namely

$$d\Delta=\Delta d=d\delta d, \qquad \delta\Delta=\Delta\delta=\delta d\delta.$$

For the field φ is to be said that it is homologous (cohomologous) to zero if there exists a filed ψ such that

$$\varphi=d\psi, \qquad (\varphi=\delta\psi). \tag{1.20}$$

From (1.19) setting $\alpha=\beta$, we get, because of (1.7), that for a harmonic field it is necessary $d\alpha=0$ and $\delta\alpha=0$. Thus, every harmonic field is closed and coclosed. Every field cohomologous to zero is coclosed as $\delta\delta=0$. From (1.18) follows that every closed ρ-field α is orthogonal to every ρ-field cohomologous to zero. Contrary, if the field α is orthogonal to every field cohomologous to zero then it is orthogonal to $\delta d\alpha$, provided that $(\alpha, \delta d\alpha)=(d\alpha, d\alpha)$ it is closed. Also, in order that a ρ-field is closed it is necessary and sufficient that it is orthogonal to every ρ-field homologous to zero. Thus, in order that a ρ-field is harmonic it is necessary and sufficient that it is orthogonal to every ρ-field homologous to zero and every ρ-field cohomologous to zero. In that manner we can state the well-known theorem by Hodge and de Rham: *Every field on the manifold R can uniquelly be represented as the sum of three fields-the field homologous to zero, the field cohomologous to zero and a harmonic field.* That means, that for a determined ρ-field α on the manifold R exist fields λ and μ of rank $(\rho-1)$ and $(\rho+1)$ respectively and a harmonic field $H\alpha$, such that

$$\alpha = d\lambda + \delta\mu + H\alpha. \tag{1.21}$$

If the field α is closed, then $(\delta\mu, \alpha) = (\delta\mu, \delta\mu) = 0$, hence $\delta\mu = 0$. Thus

$$\alpha = d\lambda + H\alpha.$$

From here it is obvious that every closed field α is homologous to a unique harmonic field $H\alpha$.

Let us still introduce some notations and notions necessary for the further work. So, with $\mathfrak{L}^p$ we denote a class of fields φ^p such that the components $\varphi^{i_1, \dots, i_p}(q)$ are measurable functions of the coordinates $x^i(q)$, $i=1, 2, \dots, p$. The notations $C^k(D)$ and $C_0{}^k(D)$, $0 \le k \le \infty$ are customary and already known. With $C^k_{0,\delta}(D) = C_0{}^k(D) \cap C_\delta{}^1(D)$ we denote a class of all solenoidal fields from $C_0{}^k(D)$.

Next, $\mathfrak{H}^p$ means the linear space consisting of all fields of the class $\mathfrak{L}^p$ such that

$$(\varphi^p, \varphi^p) < +\infty. \tag{1.22}$$

Thus, $\mathfrak{H}^p$ costitues a real Hilbert space having (1.16) as inner product. Then the norm in $\mathfrak{H}^p$ is

$$\| \varphi^p \| = \sqrt{(\varphi^p, \varphi^p)}. \tag{1.23}$$

Furthermore, we denote by $\mathfrak{H}_\nu{}^p(D)$ the subspace of $\mathfrak{H}^p(D)$ consisting of all fields belonging to the class $C^\nu(D)$, and put

$$\mathfrak{H}_\nu{}^p = \mathfrak{H}^p \cap C^\nu. \tag{1.24}$$

Let l^p be the set of regular p-fields with compact carrier. Then the notation $l_\nu{}^p = l^p \cap \mathfrak{H}_\nu{}^p$ is obvious. Following Kodaira $\mathfrak{H}^p$ can be decomposed into three mutually orthogonal subspaces

$$\mathfrak{H}^p = \mathfrak{G}^p \oplus {}_1\mathfrak{H}^p \oplus {}_2\mathfrak{H}^p, \tag{1.25}$$

where $\mathfrak{G}^p$ consists of all regular harmonic fields and

$${}_1\mathfrak{H}_\nu{}^p = \overline{\delta l_\nu^{p+1}}, \qquad {}_2\mathfrak{H}_\nu{}^p = \overline{d l_\nu^{p-1}}. \tag{1.26}$$

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Moreover, for every $\nu=1, 2, \ldots$, $\mathfrak{Z}_\nu^\rho$ and $^*\mathfrak{Z}_\nu^\rho$ consist respectively of all fields satisfying conditions $\delta\varphi=0$ and $d\varphi=0$. $\mathfrak{Z}_\nu^\rho$ and $^*\mathfrak{Z}_\nu^\rho$ are decomposed as follows

$$\mathfrak{Z}_\nu^\rho=\mathfrak{E}^\rho \oplus {}_1\mathfrak{H}_\nu^\rho,$$
(1.27)
$$^*\mathfrak{Z}_\nu^\rho=\mathfrak{E}^\rho \oplus {}_2\mathfrak{H}_\nu^\rho,$$

where

$${}_1\mathfrak{H}_\nu^\rho=\mathfrak{H}_\nu^\rho \ominus {}^*\mathfrak{Z}_\nu^\rho,$$

$${}_2\mathfrak{H}_\nu^\rho=\mathfrak{H}_\nu^\rho \ominus \mathfrak{Z}_\nu^\rho.$$

Next, we consider measurable fields from the class $\mathfrak{L}^\rho$ of a fixed rank ρ defined in an open subset D of R. Then, we introduce the absolute value

$$(1.28)\qquad |\varphi(x)| = \sqrt{\varphi^2(x)}=\left|\frac{1}{\rho!}\varphi_{i_1,\ldots,i_\rho}(x)\varphi^{i_1,\ldots,i_\rho}(x)\right|,$$

of tensor φ in a point x, and define by its means the norm $\|\ \|_p$, where $1\leq p\leq\infty$, as follows:

$$\|\varphi\|_p=\|\varphi\|_{p,D}=\left\{\int_D|\varphi(x)|^p\sqrt{g}dD\right\}^{1/p},\qquad 1\leq p<\infty,$$
(1.29)
$$\|\varphi\|_\infty=\|\varphi\|_{\infty,D}=\sup_{x\in D}|\varphi(x)|.$$

If $\|\varphi\|_p$ and $\|\psi\|_q$, where by q is denoted the real number associated with p by the relation $1/p+1/q=1$, are finite, the integral

$$(\varphi, \psi)_D=\int_D\varphi\psi\sqrt{g}dD$$

converges absolutely and satisfies the inequality

$$(1.30)\qquad |(\varphi, \psi)_D| \leq \|\varphi\|_p\|\psi\|_q.$$

We denote the Hilbert space with the above norm by $L_p(D)$. With $\tilde{\mathcal{H}}_0^k(D)$ we denote the Hilbert space obtained by completion of the set of fields of the class $C_0^k(D)$ with the norm

$$(1.31) \qquad |||\varphi||| = ||\nabla^l\varphi||_{p,D}, \quad l\leq k.$$

As a set it is identical with the completion of $C_0^k(D)$ with the norm

$$(1.32) \qquad |||\varphi||| = \sum_{i=0}^{l} ||\nabla^{l-i}\varphi||_{p,D}, \quad l\leq k,$$

if D is bounded. For the case $k=0$ the space $\widetilde{\mathcal{H}}_0{}^0(D)$ is simply the space $L_p(D)$ with possible decomposition according to the general formula (1.25). In that manner we obtain the subspace $\widetilde{\mathcal{H}}_{0,\,\delta}^0(D)$ respectively $\widetilde{\mathcal{H}}_{0,\,\delta}^k(D)$ with the norm (1.31). If $k=\infty$ we introduce the notation $\widetilde{\mathcal{H}}_0(D)$ respectively $\widetilde{\mathcal{H}}_{0,\,\delta}(D)$.

By the identification mapping we can consider $\widetilde{\mathcal{H}}_0{}^k(D)$ as a linear subset of $L_p(D)$. A field $\varphi\in\mathcal{H}_0{}^k(D)$ possesses generalized derivatives up to the order k in $L_p(D)$, while $\varphi\in\widetilde{\mathcal{H}}_{0,\delta}^k(D)$ beside that satisfies the condition $\delta\varphi=0$.

Moreover, we have the notations $K\prec\!\!\!\!\prec D$ if $\bar{K}\subset \mathrm{Int}\,(D)$. The point set $\omega(2\gamma)=\omega(2\gamma,\, D)=\{x\,|\,x\in D,\ \mathrm{dist}\,(x,\, \partial D)<2\gamma\}$, where γ is a positive constant, is the boundary strip of D with the width 2γ. Then the notation $D(2\gamma)=D-\omega(2\gamma)$ is usual. The space consisting of the fields belonging to $L_p(K)$ for any compact subset $K\prec\!\!\!\!\prec D$ will be $L_p^{\mathrm{loc}}(D)$. For a fixed constant T we have notations $\Omega=D\times(0,\, T)$ and $\bar{\Omega}=\bar{D}\times[0,\, T]$. Another needed notations will occur in the course of the work.

2. Setting the problem.

Let D be a connected domain with regular boundary ∂D of the manifold R. In the domain D, covered by domains of the family $\mathfrak{C}$, is defined non-stationary flow of fluids by the following system of partial equations given in a system of local coordinates (x) as

$$(2.1) \qquad \frac{\partial u}{\partial t}+(u\cdot\nabla)u=f-\frac{1}{\rho}\nabla p+\nu\Delta u,$$

$$\delta u=0,$$

with initial

$$u\,|_{t=0}=a(x), \qquad x\in D, \tag{2.2}$$

and boundary condition

$$u\,|_{\partial D}=b(x,\ t), \qquad t>0, \qquad x\in\partial D, \tag{2.3}$$

where: $u=u(x,\ t)$ is the velocity field on D, $f=f(x,\ t)$ is the external force field, both of rank 1; $p=p(x,\ t)$ is the pressure field of rank 0; ρ is the constant density, and ν is the kinematic viscosity.

For solving the above stated problem we take some assumptions. Namely, we shall consider that D is a bounded domain of the manifold R, that the initial field $a(x)$ belongs to $\widehat{\mathcal{H}}_0^2(D)\cap\widehat{\mathcal{H}}_{0,\delta}(D)$ and that the field $f(x,\ t)$ and its time derivative belong $L_p(D)$.

In this section we give an indispensable lemma for the work in sequel. On its proof we shall not stay considering it simple and well-known. We assume the existence of a field u of rank 1, such that

$$u\in C^1(D)\cap C^0(\overline{D}) \text{ and that } \int_D \delta u dD$$

converges absolutely. Then, from the Green-Stokes' formula given by (1.13), considering that n-chain C^n is an open domain D with the regular boundary ∂D, we obtain $\partial u\times D=-u\times\partial D$, respectively

$$\int_D \delta u dD=\int_{\partial D} u^i do_i\,. \tag{2.4}$$

Hence we have the following

LEMMA 1. A field $u\in C_\delta^1(D)\cap C^0(\overline{D})$ of rank 1 satisfies the condition

$$\int_{\partial D} u^i do_i=0. \tag{2.5}$$

But, if the field $h\in C^1(\partial D)$ of the same rank satisfies the above condition, then h is the boundary value of a field $u\in C_\delta^3(D)\cap C^0(\overline{D})$.

According to the above lemma the field b satisfies the condition

$$\int_{\partial D} b^i do_i = 0 \text{ for any } t > 0. \tag{2.6}$$

At the end of this section we still mention some assumptions concerning the field $a(x)$. Namely, we assume

$$a\,|_{\partial D} = 0 \quad \text{and} \quad \delta a = 0. \tag{2.7}$$

These assumptions are naturally assumed.

3. Lemmas.

In this section we first give lemmas which are commonly a corollary of well-known results given in Section 1 and Lemma 1. Next, we establish the existence of a coclosed (solenoidal) field u_*^ρ of rank $\rho = 1$ which will be used for a reduction of equations (2.1) and conditions (2.2)-(2.3) to the form suitable for further considerations. Then, we give still two lemmas for the convenience of a later reference.

LEMMA 2. If a field $\alpha \in \mathfrak{H}_1^1$ is coclosed then it is cohomologous to an unique harmonic field φ, namely

$$\alpha = \delta\lambda + \varphi, \tag{3.1}$$

belongs to $\mathfrak{Z}_1^1$.

LEMMA 3. If there exists a scalar field $\psi \in C^2(D)$ such that

$$\varphi = d\psi \quad \text{in} \quad D, \tag{3.2}$$

then ψ must satisfy the equation

$$\Delta\psi = 0 \quad \text{in} \quad D. \tag{3.3}$$

As it is already mentioned, the results of these lemmas are contained in Section 1.

LEMMA 4. A field $\alpha \in C^1(D) \cap C^0(\bar{D})$ belongs to $\mathfrak{Z}$ if and only if

$$\delta\alpha = 0 \quad \text{in } D \quad \text{and } \alpha\,|_{\partial D} = b. \tag{3.4}$$

Proof. The first part of lemma is contained in Lemma 2 and the second in Lemma 1.

According to Lemmas 3 and 4 the scalar field ψ is to be determined as the solution of the second boundary value problem

$$\Delta\psi=0 \quad \text{in } D, \quad d\psi\,|_{\partial D}=b. \tag{3.5}$$

Such a solution exists and is unique up to additive constant. Namely, we can say that there exists a unique field $\psi\in L_r(D)$ such that ψ has in D continuous derivatives of any order and satisfies (3.5). Let us consider a symmetric tensor field $K^\rho(x, \xi)$ of rank $\rho=2$. Let $K^\rho(x, \xi)$ be the kernel field of the second boundary value problem (3.5), which can be determined by means of the fundamental solution of Laplace equation in D given in [1]. Then, not staying on deails, as it is not our main goal, we only emphasize that, taking into account the condition (2.6), there exists a field $\psi=\psi(x, t)\in C^1(\bar{D})$ determined as follows:

$$\psi(x, t)=\int\limits_{\partial D} K^{ij}(x, \xi)(d\psi(\xi, t))_i\sqrt{g}do_j\,, \tag{3.6}$$

where dO_j are dual coordinates of surface element given in (1.15), and such that

$$\psi(\cdot,t)\in C^2(\bar{D})\cap C^\infty(D),\ \Delta\psi=0 \text{ in } D \text{ and } d\psi\,|_{\partial D}=b_n n \tag{3.7}$$

for any $t>0$, where $b_n=b^i n_i$.

Now, we set, without any proof, a lemma which is concerned with the Hölder continuity of the field ψ. We have

Lemma 5. If the field $\psi(x, t)$ is such that $\|\psi\|<\infty$, for any $t>0$, then $d\psi$ is Hölder continuous in the interior of Ω.

Other lemmas concerning the questions of regularity and Hölder continuity of the field ψ can also be stated.

Let us assume now the existence of an antisymmetric field λ^ρ of rank $\rho=2$ in the following form

$$\lambda^{ij}=\delta^{ij}_{st}(\mu^s\eta^t), \tag{3.8}$$

where fields $\mu=\mu(x, t)\in C^0(\bar{\Omega})$ and $\eta=\eta(x)\in C^2(\bar{D})\cap C^\infty(D)$ are assumed as known. Namely, we assume that the fields μ^ρ and η^ρ both of rank $\rho=1$ are such that; the field $\mu(\cdot, t)\in C^2(\bar{D})\cap C^\infty(D)$ and satisfies

(3.9) $$\Delta\mu=0 \text{ in } D \text{ and } \mu\,|_{\partial D}=b-b_n n \text{ for any } t\geq 0,$$

while the field η is determined to satisfy

(3.10) $$\Delta\eta=0 \text{ in } D \text{ and } \eta\,|_{\partial D}=n,$$

where n^i is the normal unit vector. Thus, we consider the field λ^ρ completely determined. For a cohomologous to zero field $\beta=\delta\lambda$ of rank $\rho=1$ we state the following.

LEMMA 6. Under an assumption that there exists a scalar field $\xi\in C^3(\bar{D})\cap C^4(D)$ such that on the boundary ∂D are satisfied conditions $\xi=0$ and $(d\psi)_i n^i=1$, then the field

(3.11) $$\beta_*=\delta\lambda_*\,,$$

where λ_* is determined as follows

(3.12) $$\lambda_*^{ij}=\delta_{st}^{ij}(\mu^s(\xi\eta^t)),$$

satisfies the condition

(3.13) $$\beta_*|_{\partial D}=b-d\psi\,|_{\partial D}\,.$$

PROOF. Developing the expression (3.11) taking into account (3.12), (3.9) and (3.10) one easily gets (3.13).

We have reached a stage to state the principal lemma in this section. We have

LEMMA 7. Let Ω be a bounded domain. Then there exists a field $u^\rho_*=u^\rho_*(x, t)\in C^0(\bar{\Omega})\cap\mathfrak{Z}$ of rank $\rho=1$ with the following properties:

i) $u_*(\cdot, t)\in C^3(D)$, $u_*(x, o)=0$, $u_*|_{\partial D}=b(x, t)$ for $t\geq 0$ and $\delta u_*= =0$;

ii) u_*, ∇u_*, $\partial_t u_*$ $\partial_t\nabla u^*\in L_r(\Omega)$;

iii) $u_*(\cdot, t)$, $\nabla u_*(\cdot, t)$, $\partial_t u_*(\cdot, t)$, $\partial_t\nabla u_*(\cdot, t)\in L_r(D)$ for each $t\in \in(0, T)$ and their L_r norms are bounded in t;

iv) u_* and ∇u_* are Hölder continuous in the interior of Ω.

PROOF. A field $u'_*(x, t)$ determined according to (3.1) namely in the form

$$u'_* = \beta_* + \varphi. \tag{3.14}$$

where β_* is given by (3.11) and (3.12) and φ by (3.2) and (3.6)-(3.7), will satisfy the condition $u'_* \in \mathfrak{Z}$ and according to Lemma 4 conditions $\delta u'_* = 0$ and $u'_*|_{\partial D} = b$.

Let there exists a function $\zeta \in C^\infty([0, T])$ such that

$$\zeta = \begin{cases} 0, & t=0, \\ 1, & t>0, \end{cases} \tag{3.15}$$

then the field

$$u_* = \zeta u'_* \tag{3.16}$$

satisfies also the condition $u_*(x, o) = 0$. Thus all properties i) hold for a field u_* determined according to (3.16).

ii)-iv) From the properties of fields β_* and φ follow all properties of the field u_*. We assume that β_* and φ and their derivatives quoted in the lemma belong to $L_r(D)$ for each $t>0$, and find that u_* also belongs to $L_r(D)$. The boundeness and Hölder continuity of u_* also follow from the boundeness and Hölder continuity of β_* and φ. The formality of these proofs essentially correspond to those carried out later in the case of the field v.

As we have done all preparations we can reduce Navier-Stokes equations to the new form. Setting

$$u = u_* + v, \tag{3.17}$$

into (2.1) we obtain

$$\begin{gathered} \frac{\partial v}{\partial t} = \mathfrak{F} - \frac{1}{\rho} \nabla p + \nu \Delta v, \\ \delta v = 0, \end{gathered} \tag{3.18}$$

with initial

$$v\,|_{t=0}=a(x), \qquad x\in D, \tag{3.19}$$

and boundary condition

$$v\,|_{\partial D}=0, \qquad t>0. \tag{3.20}$$

In the equation (3.18), $\mathfrak{F}$ represents a nonlinear mapping defined by

$$\mathfrak{F}=\mathfrak{F}(v^p)=\mathfrak{F}_*-(u_*\cdot\nabla)v-(v\cdot\nabla)(u_*+v), \tag{3.21}$$

and

$$\mathfrak{F}_*=\nu\Delta u_*+f-\partial_t u_*-(u_*\cdot\nabla)u_*.$$

For the necessity of a later reference we still give here two lemmas, which appear essentially known.

LEMMA 8. Let D be a domain of the manifold R. Let α be any field in $\widehat{\mathfrak{H}}_0^1(D)$ and set $\alpha^*(\xi)=\alpha(\xi)/s_{x\xi}$ where $s_{x\xi}$ is the smallest distance between the points x and ξ for an arbitrary but fixed x. Then we have

$$\|\alpha^*\|_{2,D}\leq k\,\|d\alpha\|_{2,D}, \tag{3.22}$$

where k is a constant.

PROOF. It suffices to consider the case of a scalar field. Therefore, we assume such a field $\alpha\in C_0^1(D)$. We can extend this field over the whole space R setting $\alpha\equiv 0$ outside D. Let D be the annular domain $D^*(r)-D^*(\varepsilon)$, where r and ε are any positive constants such that $r>\varepsilon$ and $D^*(\varepsilon)=D^*(x,\,\varepsilon)$. The Green's formula (1.17) for the case of a scalar field φ and a field c^p of rank $p=1$ over the domain D^* with the boundary ∂D^* has the form

$$(d\varphi,\,c)_{D^*}-(\varphi,\,\delta c)_{D^*}=\int_{\partial D^*}\varphi c^i\sqrt{g}do_i\,.$$

Let c be a homologous to zero field, namely a field of the form $c=d\psi$

where ψ is a scalar field, then the above formula gets

$$(d\varphi,\ d\psi)_{D^*} - (\varphi,\ \delta d\psi)_{D^*} = \int\limits_{\partial D^*} \varphi (d\psi)^i \sqrt{g}\, do_i\,.$$

We consider, at first, the case $n=2$ and assume that fields φ and ψ have forms $\varphi=\alpha^2$ and $\psi=\log s$, then we obtain

$$(3.23)\qquad \int\limits_{D^*} |\,\alpha^*\,|^2 \sqrt{g} dD^* \leq 2 \int\limits_{D^*} |\,\alpha^*\,|\,|\,d\alpha\,| \sqrt{g} dD^* + rA(r) - \varepsilon A(\varepsilon),$$

where

$$A(r) = \frac{1}{r^2} \int\limits_{\partial D^*(r)} |\,\alpha^2\,| \sqrt{g} dD.$$

Allowing $r \to \infty$ and $\varepsilon \to 0$ in (3.23) we obtain (3.22) on account of Schwarz' inequality, with $k=2$.

For the case $n \geq 2$ we assume that fields φ and ψ are of the forms $\varphi = \alpha^{\lambda+2}$ and $\psi = \frac{1}{s^\lambda}$ where $\lambda = n-2$. By the same procedure we arrive at (3.22) with $k = \frac{\lambda+2}{\lambda+1}$, respectively $k = n/(n-1)$. Thus, the lemma is proved for all $n \geq 2$.

From (3.22) for any bounded subdomain $K \subset D$ with diameter less than a we have immediately

$$(3.24)\qquad \|\,\alpha\,\|_{2,K} \leq \frac{n}{n-1}\, a\, \|\,d\alpha\,\|_{2,D}\,,$$

for any $\alpha \in \widehat{\mathcal{H}}_0^1(D)$. The inequality (3.23) implies that the strong convergence in $\widehat{\mathcal{H}}_0^1(D)$ ensures the strong convergence in $L_p(K)$.

LEMMA 9. Let D be a domain of the manifold R. Then there exists a constant C such that

$$(3.25)\qquad \|\,\alpha\,\|_{2(\lambda+2)} \leq C\, \|\,d\alpha\,\|_2\,,$$

for any $\alpha \in \widehat{\mathcal{H}}_0^1(D)$.

PROOF. As in the proof of Lemma 8 also here it is suffices to deal with the scalar field $\alpha(x) \in C_0^1(D)$. We extend this field over the whole space R, so that $\alpha \equiv 0$ outside D. Let be given a field $\beta = \beta(x, \xi)$ such that the field $\alpha_*(x) \in C_0^1(D) \cap \mathfrak{Z}$ is defined as follows:

$$\alpha_* = k(\beta, \Delta\alpha_*),$$

where k is a constant. Then because of (1.5) and (1.18) we have

$$\alpha_* = k(d\beta, d\alpha_*). \tag{3.26}$$

For $n=2$ we have $\beta(x, \xi) = \log s_{x\xi}$ and $k = -1/4\pi$, and for $n>2$ is $\beta(x, \xi) = \frac{1}{s_{x\xi}^{\lambda}}$ with $\lambda = n-2$, and $k = 1/(n-2)\tau_n$, where $\tau_n = 2\tau^{n/2}/\Gamma(n/2)$. Let $\alpha_*(x)$ be given as $\alpha_*(x) = \alpha^{\lambda+3}(x)$ for the case $n>2$. Then we have

$$\alpha^{\lambda+3}(x) = -l(s^{-\lambda-1}ds, \alpha^{\lambda+2}d\alpha),$$

where $l = k\lambda(\lambda+3)$. Multiplying both sides of the above equation by $\alpha^{\lambda+1}(x)$ and integrating over R with respect to the point x, after the interchange of the integration order, we obtain

$$\| \alpha \|_{2(\lambda+2)}^{2(\lambda+2)} \leq l \, \| \alpha/s \|_2^{\lambda+1} \int_R | \alpha |^{\lambda+2} \, | d\alpha | \sqrt{g} dR.$$

Next, after an application of Lemma 8 we arrive at (3.25) setting $C = m^{(\lambda+2)^{-1}}$, where $m = l(\lambda+1)^{-1}(\lambda+2)$.

For the case $n=2$, applying (3.26) to $\alpha_*(x) = \alpha^3(x)$ by the same procedure as in the case $n>2$, we obtain (3.25) with $C = (6k)^{1/2} = (3/2\pi)^{1/2}$.

Lemma 9 can be extended. So, if K is a bounded subdomain of D, then for any $\alpha \in \widehat{\mathfrak{H}}_0^1(D)$ we have

$$\| \alpha \|_{\frac{4}{3}(\lambda+2), K} \leq C_K \| d\alpha \|_{2, D}, \tag{3.27}$$

with the constant C_k depending on K.

The above inequality is obtained from (3.25) by means of Hölder inequality as

$$\| \alpha \|_{\frac{4}{3}(\lambda+2)} \leq \| \alpha \|_{2(\lambda+2)} \| 1 \|_{4(\lambda+2)} = (\mathrm{mes} \cdot (K))^{\frac{1}{4}(\lambda+2)} \| \alpha \|_{2(\lambda+2)} .$$

If D is a bounded domain, then from (3.27) setting $K=D$ we have

$$\| \alpha \|_{\frac{4}{3}(\lambda+2), D} \leq C^* \| d\alpha \|_{2, D} ,$$

with a constant C^*. Hence strong convergence in $\widehat{\mathcal{H}}_0^1(D)$ implies strong convergence in L_p, where $p=\frac{4}{3}(\lambda+2)$, when D is bounded. The inequality (3.27) shows that strong convergence in $\widehat{\mathcal{H}}_0^1(D)$ ensure locally strong convergence in $L_p(D)$. Next if $\alpha \in \widehat{\mathcal{H}}_0^1(D)$ and K is a bounded subdomain of D, then we have $\alpha \in L_p(K)$, $d\alpha \in L_2(K)$, hence $(\alpha \cdot d)\alpha \in L_q(K)$, where $q = \dfrac{4(\lambda+2)}{2(\lambda+2)+3}$, by means of Hölder inequality.

From the properties of $u_*(x, t)$ given in Lemma 7, if $\| f \| < \infty$ we have $\| \mathfrak{F}_* \| < \infty$ for any $t>0$, and $\mathfrak{F}_*(x, t)$ is Hölder continuous in Ω if $f(x, t)$ is Hölder continuous in the same domain.

4. Definition of Generalized solution.

Suppose that fields u and p of rank 1 and 0 respectively are sufficiently smooth and obey the system of equations (2.1) in a domain D of R. Furthermore, let χ be a sufficiently smooth, solenoidal field with the compact carrier. Multiplying the first of (2.1) scalary by χ according to (1.16) and then integrating over the interval $[0, T]$, we obtain

$$\text{(4.1)} \qquad \int_0^T (-(\partial_t + \nu\Delta)\chi,\ u)dt + \int_0^T (\chi,\ (u \cdot \nabla)u)dt = \int_0^T (\chi,\ f)dt.$$

If the above equation is valid for any $\chi \in C^2_{0,\delta}(\Omega)$, then there exists a scalar field p which satisfies (2.1) in D together with u. Hence one says that a field u satisfies (2.1) weakly in Ω or satiesfies the weak equation (4.1) of the equation (2.1) in Ω, if u and $(u \cdot \nabla)u$ are locally integrable in Ω, and (4.1) holds for any $\chi \in C^2_{0,\delta}(\Omega)$. The field χ in (4.1) is a test field of the weak equation.

Thus, we can state the definition of the generalized solution of the Navier-Stokes equations in a domain Ω as follows.

DEFINITION. Let D be a bounded domain of the manifold R and let T be a positive constant. Then a vector field $u \in \Omega$ is called the generalized solution $u(t)=u(x, t)$ of the Navier-Stokes equations if the following conditions i)-iv) are all satisfied:

i) $u-u_*$ belongs to $\widetilde{\mathfrak{H}}_{0,\delta}(D)$ for each $t \in (0, T)$ and some u_* such that

$$u_* \in \mathfrak{Z} \text{ and } u_*=0 \text{ as } t=0; \tag{4.2}$$

ii) u satisfies (4.1) weakly in Ω;

iii) u and its derivatives ∇u, $\partial_t u$ and $\partial_t \nabla u$ belong to $L_r(\Omega)$;

iv) $u(t)$, $\nabla u(t)$, $\partial_t u(t)$ and $\partial_t \nabla u(t)$ belong to $L_r(D)$ for each $t \in (0, T)$ and their L_r norms are bounded in t.

The existence of such a solution we shall establish in the next chapter.

CHAPTER II.

EXISTENCE AND REGULARITY THEOREMS

The purpose of this chapter is to prove the theorems concerned with existence and regularity of the generalized solution of the Navier-Stokes equations in a bounded domain Ω.

5. Fundamental solution of the parabolic equation.

Under the fundamental solution of a parabolic type equation

$$(\partial_t - \nu A_x) Q^p = 0, \tag{5.1}$$

where $A_x = b^{ij}\partial_i\partial_j + a^i\partial_i + c$ with coefficients depending on x is an elliptic operator which is the Laplace operator in our case, we understand a symmetric tensor field $P^p = P^p(x,\ t;\ \xi,\ \tau)$ defined for each $(x,\ t) \in \Omega$ $(\xi,\ \tau) \in \Omega$ and $t > \tau$ and which satisfies the following conditions;

i) for fixed $(\xi,\ \tau) \in \Omega$ it satisfies, as a function of $(x,\ t)$, where $x \in D$ and $\tau < t \leq T$, the equation (5.1);

ii) when $t \to \tau$ and the square of geodesic distance $\Gamma \to 0$ the field P^p allows the principal singularity given by the representation

$$\pi^{-n/2}(t-\tau)^{-n/2} \exp\left(-\frac{\Gamma(x,\ \xi)}{4\nu(t-\tau)}\right) U^p(x,\ t;\ \xi,\ \tau), \tag{5.2}$$

where $U^p(x,\ t;\ \xi,\ \tau)$ is a regular field in Ω and of the class C^∞.

A fundamental solution of the parabolic equation with time dependent coefficients was given by K. Yosida [17]. Here, we shall give a slightly different construction. We first apply the operator Δ to the product of a scalar field $h(\Gamma,\ t,\ \tau)$ and a tensor field $X^p(x,\ t,\ \xi,\ \tau)$ of rank $\rho = 2$ and get

$$(\partial_t - \nu\Delta)(hX) = \partial_t(hX) - \tag{5.3}$$

$$-\nu\left[4\Gamma\frac{\partial^2 h}{\partial\Gamma^2}X + \frac{\partial h}{\partial\Gamma}(\Delta\Gamma + 2V^i\Gamma_i + 2g^{ij}\Gamma_i\partial_j)X + h\Delta X\right] = 0,$$

where X is the matrix of the tensor X^{ij}.

Now we quote the well-known identities which can be found in [16], [17] and so on. If x and ξ are two close points on the manifold R and $s = s(x,\ \xi)$ the geodesic distance between them we have

$$g^{ij}\Gamma_i\Gamma_j = 4\Gamma, \qquad g^{ij}\Gamma_i\partial_j = 2s\frac{\partial}{\partial s}, \tag{5.4}$$

where $\Gamma_i = \partial_i\Gamma$. Moreover, we have

$$\Delta\Gamma \to 2n \quad \text{as} \quad x \to \xi.$$

Let us define a new operator as

$$\overset{M}{\Delta}=M^{-1}\Delta M, \tag{5.5}$$

with

$$M=\exp\left(\int_s \frac{H}{s}\,ds\right), \qquad 4H=2n-\Delta\Gamma-V^i\Gamma_i\,.$$

Then, on account of the above quoted identities and the definition of the new operator (5.5), the equation (5.3) is reduced to the form

$$\nu\left[4\Gamma\frac{\partial^2 h}{\partial\Gamma^2}X+\frac{\partial h}{\partial\Gamma}\left(2n+4s\frac{\partial}{\partial s}\right)X+h\overset{M}{\Delta}X\right]-\frac{\partial}{\partial t}(hX)=0. \tag{5.6}$$

Furthermore, as $h=\exp\left(-\dfrac{\Gamma}{4\nu(t-\tau)}\right)$ we have

$$-\frac{1}{4(t-\tau)}\left[2n+4s\frac{\partial}{\partial s}\right]X+\overset{M}{\Delta}X-\frac{\partial X}{\partial t}=0. \tag{5.7}$$

Let a positive integer k be $>2+n/2$ and let X be formally given in the form of a power series

$$X=\sum_{\mu=0}^{k}U_\mu(t-\tau)^{\mu-n/2}, \tag{5.8}$$

then by the substitution of (5.8) into (5.7) we obtain

$$\sum_\mu\left[-\frac{1}{4}\left(2n+4s\frac{\partial}{\partial s}\right)U_\mu+\overset{M}{\Delta}U_{\mu-1}-(\mu-n/2)U_\mu\right](t-\tau)^{\mu-n/2-1}=0.$$

From here, for $t>\tau$, we determine succesively $U_\mu(x,\xi)$ so that

$$\left(\mu+s\frac{\partial}{\partial s}\right)U_\mu-\overset{M}{\Delta}U_{\mu-1}=0,$$

starting from $U_0(x, \xi)=1$. Hence we have

$$U_\mu = \frac{1}{s^\mu} \int_0^s r^{\mu-1} \overset{M}{\Delta} U_{\mu-1} dr, \qquad \mu = 1, \ldots, k. \tag{5.9}$$

Thus, we can set the following.

PROPOSITION 1. A symmetric tensor field of rank $\rho=2$ determined as follows

$$P^\rho(x, t; \xi, \tau)= \tag{5.10}$$

$$=\pi^{-n/2}(t-\tau)^{-n/2} \exp\left(-\frac{\Gamma(x, \xi)}{4\nu(t-\tau)}\right) \sum_{\mu=0}^{k} U_\mu{}^\rho(x, \xi)(t-\tau)^\mu,$$

where k is an integer $>2+n/2$ and $U_\mu{}^\rho$ is given by (5.9) represents a fundamental solution of the parabolic equation (5.1) in Ω.

The proof that this fundamental solution is also the fundamental solution of the adjoint equation to the equation (5.1) can be found in [17].

If $h(x, t)$ is a field in Ω we define the operator P with the kernel $P(x, t; \xi, \tau)$ by

$$(P \circ h)(x, t) = \int_0^t d\tau \int_D P(x, t; \xi, \tau) h(\xi, \tau) \sqrt{g} dD. \tag{5.11}$$

Then, we consider a field

$$f(x, t)=(P \circ h)(x, t). \tag{5.12}$$

Let us study some properties of this field. To show its existence we give a limitation of the field P^ρ by separated singularities. It is easy to show that

$$|P^\rho(x, t; \xi, \tau)| \leq \frac{C^\rho}{(t-\tau)^\mu} \frac{1}{s_{x\xi}^{n-2\mu}}, \qquad \mu \in (0, 1), \tag{5.13}$$

where C^{ρ} is a constant field and $s_{x\xi}$ is the geodesic distance between x and ξ. Choosing the positive constant μ arbitrarily in the given interval we see that the field P^{ρ} allows the weak singularity as $t \rightarrow \tau$ and $x \rightarrow \xi$. A singularity, however, is integrable. Thus we conclude the existence of the integral (5.11) for $x \in D$ and $0 \leq t \leq T$, which is absolutely and uniformly convergent for an integrable $h(\xi, \tau)$, where $(\xi, \tau) \in \Omega$.

Moreover, one shows that $\nabla_{\alpha} P^{\rho}(x, t; \xi, \tau)$ satisfies the inequality

$$(5.14) \qquad |\nabla_{\alpha} P^{\rho}(x, t; \xi, \tau)| \leq \frac{C_{\alpha}^{\rho}}{(t-\tau)^{\mu}} \frac{1}{s_{x\xi}^{n+1-2\mu}}, \qquad \mu \in (1/2, 1),$$

and that there exists the field $e(x, t) = \nabla_{\alpha} f(x, t)$ given by

$$(5.15) \qquad e(x, t) = (K \circ h)(x, t),$$

where $K = \nabla_{\alpha} P$.

Now we give two lemmas concerning the fields $f(x, t)$ and $e(x, t)$.

Lemma 10. Let $h(x, t)$ be locally bounded in Ω. Then $f(x, t)$ is Hölder continuous with respect to $t \in (0, T)$, and $e(x, t)$ is Hölder continous in a subdomain Ω^* with respect to (x, t).

Lemma 11. Let $h(x, t)$ be locally Hölder continuous in Ω with respect to (x, t). Then $f(x, t)$ is twice differentiable with respect to x and once differentiable with respect to t in a subdomain Ω^*, and these derivatives are continuous in (x, t).

The proofs of these lemmas are clear and straightforward but cumbersome. Therefore, we shall only, on account of an illustration, give brief features of the proof of Lemma 10. Let us write

$$f(x, t) - f(x, t_1) = \int_{t_1}^{t} d\tau \int_D P(x, t; \xi, \tau) h(\xi, \tau) \sqrt{g} dD +$$

$$+ \int_0^{t_1} d\tau \int_D [P(x, t; \xi, \tau) - P(x, t_1; \xi, \tau)] h(\xi, \tau) \sqrt{g} dD = j_1 + j_2 .$$

For the first integral according to (5.13) and boundedness of h we have the inequality

$$|j_1| < C_1 |t-t_1|^{1-\mu}, \qquad \mu\in(0, 1)$$

Now we circumscribe a geodesic spheres with center at the point x and with the radius r_0, then divide the integral j_2 into the sum of integrals over the set S' laying in the interior of S and over the complementary part $D\setminus S'$. According to (5.13) we have

$$|j_2^{S'}| < C_2 r_0^{2\mu}, \qquad \mu\in(0, 1).$$

Taking into account the estimation

$$|P(x, t; \xi, \tau)-P(x, t_1; \xi, \tau)| < \frac{C}{(t-\tau)^{\mu}} \frac{1}{s_{x\xi}^{n+2-2\mu}},$$

we have the following inequality

$$|j_2^{D\setminus S'}| < C_3 r_0^{2(\mu-1)} |t-t_1|.$$

Setting $r_0=B|t-t_1|^{\nu}$, then choosing the positive constant ν so that $1-2(1-\mu)\nu=2\mu\nu$ we obtain $\nu=1/2$, and thus the inequality

$$(5.16) \qquad |f(x, t)-f(x, t_1)| < C_4 |t-t_1|^{\theta},$$

where θ is a positive number less than unity.

To prove the second assertion of the lemma we consider two close points x and x_1 whose smallest distance is $r_{x\xi}$ and write

$$e(x, t)-e(x_1, t)=$$
$$=\int_0^t d\tau \int_D [K(x, t; \xi, \tau)-K(x_1, t; \xi, \tau)]h(\xi, \tau)\sqrt{g}dD.$$

Divide the integral into the sum of integrals over the domain O' belonging to the interior of a geodesic sphere O and over the complementary part $D\backslash O'$. According to (5.14) we obtain

$$|j^0|<C_5 r_{xx_1}^{2\mu-1}, \qquad \mu\in(1/2,\ 1).$$

For the second integral because of

$$|K(x,\ t;\ \xi,\ \tau)-K(x_1,\ t;\ \xi,\ \tau)|<\frac{C}{(t-\tau)^{\mu}}\frac{1}{r_{x\xi}^{n+2-2\mu}},$$

we have the same estimation. Thus

$$|e(x,\ t)-e(x_1,\ t)|<C_6 r_{xx_1}^{\theta}, \tag{5.17}$$

where θ is a positive constant less than unity. By the similar procedure as before we get the Hölder continuity with respect to t in the form

$$|e(x,\ t)-e(x,\ t_1)|<C_7|t-t_1|^{\theta'/2}. \tag{5.18}$$

On the base of estimations (5.17) and (5.18) we obtain the inequality

$$|e(x,\ t)-e(x_1,\ t_1)|<C_8[r_{xx_1}^{\theta}+|t-t_1|^{\theta'/2}], \tag{5.19}$$

with θ and θ' positive constants less than unity.

6. Parametrix of the parabolic x elliptic equation.

As for establishing the existence of the generalized solution we shall use the parametrix of a parabolic x elleptic equation, namely an equation of the form

$$L_{tx}N^{\rho}=0, \tag{6.1}$$

where $L_{tx}=(\partial_t+\nu\Delta_x)\Delta_x$ and N^{ρ} is a tensor field of rank ρ on the domain Ω, then we shall give it here. Let us fix a truncating function $\delta(x,\ t;\ \xi,\ \tau)=\delta(s,\ t-\tau)$, where $s=s(x,\ \xi)$ is the smallest distance between points x and ξ, such that $\delta(x,\ t)$ is an even C_0^{∞} function with properties

$$\delta(x,\ t)=\begin{cases} 1, & s\leq\gamma, \quad |t|\leq\gamma, \\ 0, & s\geq 2\gamma, \quad |t|\geq 2\gamma, \end{cases} \tag{6.2}$$

where γ is a positive constant. Then the tensor field

$$P_\delta(x,\ t;\ \xi,\ \tau)=P(x,\ t;\ \xi,\ \tau)\delta(x,\ t) \tag{6.3}$$

is defined everywhere. Namely, P_δ vanishes identically for $s>2\gamma$ and $|t|>2\gamma$ and coincides with P for $s\leq\gamma$ and $|t|\leq\gamma$. The behaviour of P_δ for $\gamma<s<2\gamma$ and $\gamma<|t|<2\gamma$ may be freely determined.

Now, we intend to establish the existence of the parametrix of the equation (6.1), namely a field $F^\rho(x,\ t;\ \xi,\ \tau)$ with the following properties:

i) F belongs to C^∞ and is defined everywhere;

ii) the form $\Phi(x,\ t;\ \xi,\ \tau)=L_{tx}F(x,\ t;\ \xi,\ \tau)$ belongs to C^∞ and is bounded everywhere;

iii) if a field $\chi(x,\ t)\in C_0^\infty$ then

$$\psi(x,\ t)=(F\circ\chi)(x,\ t) \text{ is } C^\infty, \tag{6.4}$$

and we have

$$L_{tx}\psi(x,\ t)=\chi(x,\ t)-(\Phi\circ\chi)(x,\ t). \tag{6.5}$$

If $H^\rho(x,\ \xi)$ is the fundamental solution of the Laplace equation given in [1] we have the following

Proposition 2. A tensor field F of rank ρ determined as follows

$$F(x,\ t;\ \xi,\ \tau)=\int_D H(x,\ \eta)P_\delta(\eta,\ t;\ \xi,\ \tau)\sqrt{g}dD, \tag{6.6}$$

is the parametrix of the equation (6.1).

Proof. All properties of the parametrix F are obvious. We only give a formal proof of (6.5). If we apply the operator Δ to (6.4) we obtain

$$\Delta\psi(x,\ t)=-(P_\delta\circ\chi)(x,\ t),$$

because of $\Delta H=0$. Now, an application of the parabolic operator to the obtained result gives (6.5).

7. Existence of a solution of Navier-Stokes equations.

In this section we establish the existence of a generalized solution of Navier-Stokes equations by using the parametrix given in the previous section. Namely, we only derive integral representation of it, then study properties of integral operators. We have

THEOREM 1. Let Ω be bounded. Then there exists a generalized solution $u(t)=u(x, t)$ of Navier-Stokes equations.

PROOF. It suffices to show the existence of a coclosed field $v^\rho(t)=v^\rho(x, t)$ of rank $\rho=1$ satisfying the equation (3.18) and conditions (3.19) and (3.20). Thus, the weak equation (4.1) is reduced to

$$\int_0^T (-(\partial_t+\nu\Delta)\chi, v)dt=\int_0^T (\chi, \mathfrak{F})dt. \tag{7.1}$$

Assume $\chi=\delta d\psi$, where the field $\psi\in C_0^\infty(\Omega)$. Then (7.1), because of (1.18) and cocloseness of the field v^ρ, gets

$$\int_0^T ((\partial_t+\nu\Delta)\Delta\psi, v)dt=\int_0^T (\delta d\psi, \mathfrak{F})dt. \tag{7.2}$$

Let K be an arbitrary bounded subdomain of D such that $K \Subset D$ and γ a positive constant as in the previous section. Take a field $\varphi\in C_0^\infty(\Omega_K(2\gamma))$, where Ω_K means Ω with K instead of D, and consider a field of the form (6.4), namely

$$\psi(x, t)=(F\circ\varphi)(x, t), \tag{7.3}$$

where F^ρ is the parametrix of the equation (6.1), as a test field of the equation (7.2). From (7.3) on account of (6.5) we have

$$L_{tx}\psi(x, t)=\varphi(x, t)-(\Phi\circ\varphi)(x, t). \tag{7.4}$$

Furthermore, we denote by means of S^ρ a symmetric tensor field of rank

$\rho=2$ and of the form $S^\rho=-\Delta F^\rho-d\delta F^\rho$. Then we have

$$(7.5)\qquad \delta d\psi(x,\ t)=(S\circ\varphi)(x,\ t)+(T\circ\varphi)(x,\ t),$$

where S and T are operators with kernels S^ρ and T^ρ respectively, namely C_0^∞ functions vanishing identically near $x=\xi$, $t=\tau$. Substituting (7.4) and (7.5) into (7.2) we obtain

$$(7.6)\qquad \int_0^T((\varphi-(\Phi\circ\varphi)),\ \nu)dt=\int_0^T((S\circ\varphi),\ \mathfrak{F})dt+\int_0^T((T\circ\varphi),\ \mathfrak{F})dt.$$

By virtue of the integrability of the field $\mathfrak{F}$ and $\varphi\in C_0^\infty(\Omega_K(2\gamma))$ we may change the order of integration. Thus, we derive

$$(7.7)\qquad \int_0^T(\varphi,\ (\nu-(\Phi^*\circ\nu))dt=\int_0^T(\varphi,\ (S^*\circ\mathfrak{F}))dt+\int_0^T(\varphi,\ (T^*\circ\mathfrak{F}))dt,$$

where Φ^*, S^* and T^* are operators with kernels obtained from Φ^ρ, S^ρ and T^ρ by means of interchange of $(x,\ t)$ and $(\xi,\ \tau)$. The fact that (7.7) is valid for every $\varphi\in C_0^\infty(\Omega_K(2\gamma))$ implies

$$(7.8)\qquad \nu(x,\ t)=(S^*\circ\mathfrak{F})(x,\ t)+(\Phi^*\circ\nu)(x,\ t)+(T^*\circ\mathfrak{F})(x,\ t).$$

Now, if instead of the general element φ we take an element $\nabla_\alpha\varphi$, then by the same procedure we obtain

$$(7.9)\qquad \nabla_\alpha\nu(x,\ t)=(S^{**}\circ\mathfrak{F})(x,\ t)+(\Phi^{**}\circ\nu)(x,\ t)+(T^{**}\circ\mathfrak{F})(x,\ t)$$

where given operators have respective kernels obtained from $\nabla_\alpha S$, $\nabla_\alpha\Phi$, $\nabla_\alpha T$ by means of interchange of $(x,\ t)$ and $(\xi,\ \tau)$ belonging to C_0^∞ and vanishing near $x=\xi$, $t=\tau$. In that manner we have obtained the « local » integral representations for the generalized solution. Thus, we can state the following

LEMMA 12. Let $\nu(x,\ t)$ be a generalized solution of Navier-Stokes equations, and let K be an arbitrary bounded subdomain $K\Subset D$ and γ

any positive constant, Then for almost every $(x, t)\in\Omega_K(2\gamma)$ we have integral representations (7.8) and (7.9).

Setting $f(x, t)=S^*\mathfrak{F}(x, t)$ and $e(x, t)=S^{**}\mathfrak{F}(x, t)$ we see that these fields correspond to the fields (5.12) and (5.15) respectively. Some properties of these fields we have already studied. For the further work we still need a Sobolev type lemma [14], namely

LEMMA 13. Let be given a field $w(x, t)$ in the form

$$w(x, t)=\int_0^t k(t, \tau)d\tau\int_D \frac{h(\xi, \tau)}{s_{x\xi}^{\lambda}}\sqrt{g}dD.$$

If $h(t)=h(x, t)$ belongs to $L_p(D)$ for each $t\in(0, T)$ with its L_p norm bounded and if $\lambda\geq n(1-1/p)$ then for each $t\in(0, T)$ and for any fixed q such that $1/q\geq\lambda/n+1/p-1$, $w(t)=w(x, t)$ belongs to $L_q(D^*)$ and its L_q norm is bounded in t.

For each t from considered interval the above lemma is, in fact, the Sobolev lemma which proof can be found in [14].

8. Regularity of the generalized solution.

As we have already obtained the integral representations of the generalized solution and have studied some properties of the integral operators then we can state the main theorem, which says that this solution is regular, namely it is twice continuosly differentiable with respect to x and once to t. We have

THEOREM 2. The generalized solution $v(t)=v(x, t)$ is regular in any subdomain Ω^* of Ω in which the external force field is Hölder continuous with respect to (x, t).

PROOF. Let Ω^* be an arbitrary subdomain of Ω. Then on account of (7.8) we notice that the regularity of v in Ω^* is implied by that of terms in (7.8) of the form $(K^*\circ\mathfrak{F})(x, t)$, namely

$$\text{Reg}\,(K^*\circ\mathfrak{F})(x, t)\Rightarrow\text{Reg}\,v(x, t).$$

According to the property iv) of the solution $v(t)$ and the assumption on the force field $f(x, t)$ we have $\| v(t) \| \leq A$, $\| (v \cdot \nabla) v(t) \| \leq A$ and also $| f(x, t) | \leq A$. By virtue of Lemma 9 we have $\| v(t) \|_{2(\lambda+2)} \leq \| \nabla v(t) \|_2 \leq A$ and $(v \cdot \nabla) v \in L_q(K)$, where $q = \dfrac{2(\lambda+2)}{\lambda+3}$. Thus, we arrive at the conclusion that $\mathfrak{F} \in L_q(K)$ with the norm bounded in t.

Now, we achieve the wanted regularity by successive applications of Lemmas 13, 10 and 11. As conditions for application of Lemma 13 hold, then we apply it to (7.8) and find that $v \in L_q(K^*)$ with $\dfrac{1}{r} \geq \dfrac{\lambda - n}{n} + \dfrac{1}{q}$ and $q = \dfrac{2(\lambda+2)}{\lambda+3}$, where $\lambda = n - 2\mu$ and $\mu \in (0, 1)$. From here we see that $(v \cdot \nabla) v \in L_s(K^*)$, where $1/s = 1/r + 1/2$, hence $\mathfrak{F} \in L_s(K^*)$. Next we choose instead of K^* a suitable subset of K, for instance $K(2\gamma)$ and conclude that v is bounded in $K(2\gamma)$ and its bound is bounded in t. Thus, v is bounded in $K(2\gamma)$. The obtained result ensures that $(v \cdot \nabla) v \in L_s(K(2\gamma))$ and its L_s norm is bounded for each $t \in (2\gamma, T - 2\gamma)$. Further, we apply Lemma 13 to (7.9) and obtain $\nabla v \in L_k(K(4\gamma))$ with $\dfrac{1}{k} \geq \dfrac{\lambda + 1 - n}{n} + \dfrac{1}{s}$, and its norm is bounded in $t \in (4\gamma, T - 4\gamma)$. Thus, we arrive at the conclusion that the field v and its derivative ∇v are bounded in $\Omega_k(4\gamma)$.

The results so far obtained enable us to apply Lemma 10 to (7.8) and (7.9) and to conclude that v is Hölder continuous with respect to $t \in (2\gamma, T - 2\gamma)$ and that ∇v is Hölder continuous in $\Omega_k(6\gamma)$ with respect to (x, t). Finally we can apply Lemma 11 to (7.8) and see that the field v is twice continuously differentiable in x and once in t in the domain $\Omega_k(8\gamma)$. Thus the theorem is proved.

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Manoscritto pervenuto in redazione il 20-12-1968.