Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 42 (1969), p. 201-207

<http://www.numdam.org/item?id=RSMUP_1969__42__201_0>

ON MULTIPLICITY FUNCTIONS AND LEBESGUE AREA *)

by D. J. SCHAEFER

1. Introduction.

Let $T : Q \rightarrow E_3$ be a continuons transformation from the unit square Q in the uv-plane into Euclidean 3-space E_3. Many writers have been concerned with the problem of finding formulas which express the Lebesgue area $A(T)$ in terms of multiplicity functions. This paper will show relationships between some of the results of Mickle [3] and Federer [2], and will present, for the case $A(T) < \infty$, a modified definition of significant maximal model continua (s.m.m.c.'s) (see Mickle [3]) which is more natural if one is interested in the tangential properties of the Frechet surface defined by T.

2. Plane transformations.

2.1. Throughout section 2, T will denote a plane transformation. Let $T : \Delta \rightarrow E_2$ be a continuous, bounded transformation from $\Delta \subset Q$ into E_2, where Δ is connected and open relative to Q. We write $T : (B, A) \rightarrow (D, C)$ when $A \subset B$, $C \subset D$, $T(A) \subset C$, and $T(B) \subset D$. For $y \in E_2$ and $r > 0$, let $C(y, r) = \{z : z \in E_2, |z - y| < r\}$. Let $\boldsymbol{C}A$ denote the complement of set A. It is easily shown that if $y \in T(\Delta)$ and V is a component of $T^{-1}[C(y, r)]$, then $T : (\boldsymbol{Cl}_\Delta V, \boldsymbol{B}_\Delta V) \rightarrow (E_2, \boldsymbol{C}C(y, r))$, where $\boldsymbol{Cl}_\Delta V$ denotes the closure of V relative to Δ, and $\boldsymbol{B}_\Delta V$ the boundary of V relative to Δ.

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*) Research supported in part by Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.

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Then (see [1]) T induces a homomorphism h_T on the 2-dimensional Čech cohomology groups with integer coefficients and based on locally finite coverings:

$$h_T : K^2[E_2, CC(y, r)] \to K^2[\boldsymbol{Cl}_\Delta V, \boldsymbol{B}_\Delta V].$$

Definite subsets of E_2 for $0<r<1$ as follows. $A(y, r)=\{z : |z-y| \leq 1/r\}$, $B(y, r)=\{z : r\leq |z-y| \leq 1/r\}$, and $U(y, r)=\{z : |z-y| > 1/r\}$.

By the excision theorem [1, p. 243], the following isomorphism holds.

$$K^2[E_2, CC(y, r)] \approx K^2[A(y, r), B(y, r)].$$

Suppose V is a 2-manifold whose closure relative to Δ is compact. Then $\boldsymbol{Cl}_\Delta V=\boldsymbol{Cl}V$ and $\boldsymbol{B}_\Delta V=\boldsymbol{B}V$. If $K^{*3}[A(y, r), B(y, r)]$ and $K^*[\boldsymbol{Cl}V, BV]$ denote the cohomology groups for the indicated pairs as defined in [5], we have the following isomorphisms (see [5, pp. 63-64], [1, pp. 253-254]).

$$K^2[A(y, r), B(y, r)] \approx K^{*3}[A(y, r), B(y, r)]$$

$$K^2[\boldsymbol{Cl}V, \boldsymbol{B}V] \approx K^{*3}[\boldsymbol{Cl}V, \boldsymbol{B}V].$$

2.2. Let $F(r)$ be the family of components of $T^{-1}[C(y, r)]$. Let $V\in F(r)$ have compact closure relative to Δ. Let $D(T, r, V)$ and $\mu(y, T, V)$ be as defined in [2] and [5] respectively. From 2.1 it follows that $D(T, r, V)= |\mu(y, T, V)|$. Let (see [2])

$$M(T, \Delta, y)= \lim_{r\to 0} \Sigma\, D(T, r, V) \qquad \text{(sum over } V\in F(r)),$$

where $F(r)$ is the collection of components of $T^{-1}[C(y, r)]$. We use e.m.m.c. as the abreviation for essential maximal model continua as defined in [4].

2.3. Lemma. $M(T, \Delta, y)\geq 1$ implies that y is the image of an e.m.m.c. under T.

Proof. From the definition of $M(T, \Delta, y)$ and the relation between D and μ, there is an r such that $0<r<1$ and a component V_0 of

$T^{-1}[C(y, r)]$ such that $\mu(y, T, V_0)\neq 0$. Such a V_0 is a domain with closure in Δ, and $T^{-1}(y)\cap V_0=T^{-1}(y)\cap \mathbf{Cl}\, V_0$. Let $0<r'<r$ and let $\Omega_{r'}$ be the collection of components of $T^{-1}[C(y, r')]$ that lie in V_0. Then

$$T^{-1}(y)\cap[\cup V']=T^{-1}(y)\cap V_0=T^{-1}(y)\cap \mathbf{Cl}\, V_0 \, . \; (V'\in\Omega_{r'})$$

Therefore the class $\Omega_{r'}$ is (y, T, V_0) complete, i.e. $\mathbf{Cl}\, V_0 \cap T^{-1}(y)\subset \subset \cup V'$, the union taken over $V'\in\Omega_{r'}$. By [5, p. 126 theorem 3], $\mu(y, T, V_0)=\Sigma\, \mu(y, T, V')$, the sum taken over $V'\in\Omega_{r'}$. Furthemore, $\mathbf{Cl}\, V_0'\subset V_0$ because $\mathbf{Cl}\, V_0'\subset T^{-1}[\mathbf{Cl}\, C(y, r')]\subset T^{-1}[C(y, r)]$ and V_0' and its closure lie in the same component of $T^{-1}(C(y, r)]$. Therefore V_0' is an indicator region of T and y is the image an e.m.m.c. [5, p. 165].

3. i-fold essential and significant maximum model continua.

3.1. Let Q be the unit square in E_2 and $T: Q\to E_3$ denote a continuous transformation. Let $T=lm$ be a monotone-light factorization of T and denote the middle-space by $\boldsymbol{M}$. For a point $x\in E_3$ and a maximum model continuum (m.m.c.) $\gamma\subset T^{-1}(x)$, let $\Delta(\gamma, r)$ denote the component of $T^{-1}[S(x, r)]$ which contains γ, where $S(x, r)$ is the open sphere in E_3 with center x, radius r. Let $A[T, \Delta(\gamma, r)]$ denote the Lebesgue area of $T\,|\,\Delta(\gamma, r)$. Let $a\in\boldsymbol{M}$ be such that $a=m(\gamma)$. Then we also denote $\Delta(\gamma, r)$ by $\Delta(a, r)$. Let $L_2^*(T, a)$, $L_{*2}^*(T, a)$, and $E_2(T, a)$ be as defined in [2]. We define sets as follows.

$$Z_2=\{z: z\in Q, L_2^*(T, mz)=L_{*2}(T, mz)=E_2(T, mz)=0\},$$

$$Z_1=\{z: z\in Q, L_2^*(T, mz)=L_{*2}(T, mz)=E_2(T, mz)=1\},$$

$$Z_3=Q-Z_1\cup Z_2\, .$$

Denoting the Hausdorff 2-measure in $\boldsymbol{M}$ by H_T^2, the Hausdorff 2-measure in E_3 by H^2, and number of m.m.c.'s having non-empty intersections with $T^{-1}(x)\cap Z_1$ by $N^*[x, T, Z_1]$, we can state of the following.

3.2. THEOREM. If $A(T)<\infty$, then $A(T)=\int N^*[x, T, Z_1]dH^2$.

PROOF. From [2, 8.17] we have

$$A(T)=\int\sigma(x)dH^2, \tag{1}$$

where $\sigma(x)=\Sigma L_2^*(T, a)$, the sum $a\in M$ such that $l(a)=x$. We will show that

$$\sigma(x)=N^*[x, T, Z_1] \text{ for } H^2 \text{ a.e. } x\in E_3. \tag{2}$$

Note that Z_1 is the union of m.m.c.'s under T and let γ be any m.m.c. in $T^{-1}(x)\cap Z_1$. Letting $a=m(\gamma)$, we have $l(a)=x$ and $L_2^*(T, a)=1$. Hence

$$\sigma(x)\geq N^*[x, T, Z_1]. \tag{3}$$

Suppose inequality (3) to be strict. Then there is an $a\in M$ such that $l(a)=x$, $L_2^*(T, a)>0$ and $a\notin mZ_1$. Therefore $a\in m(Z_3)$ and $x\in T(Z_3)$. But [2, 8.16] gives $H^2{}_T[m(Z_3)]=0$ under our assumptions. Since $H^2{}_T[m(Z_3)]\geq H^2[T(Z_3)]$, the latter value is zero. Therefore strict inequality in (3) holds only on a set of H^2-measure zero. (1) and (2) imply the theorem.

3.3. In [3], Mickle makes the following definitions. Γ denotes the collection of H^2-measurable sets of E_3. Let U denote the unit sphere in E_3. $\pi_p : E_3 \to E_2$ is the projection of E_3 onto the plane normal to the direction determined by $p\in U$. Let $\Gamma_p=\{E : E\in\Gamma,\ L_2\pi_p(E)=0\}$ where L_2 is the Lebesgue exterior planar measure. For each $E\in\Gamma$ *define*

$$H_p(E)=\inf H^2(E-E_p) \qquad (E_p\in\Gamma_p).$$

If $E\in\Gamma$, $p\in U$, and m and n are positive integers,

$$G_{nm}(E, p)=\{x : H_p[E\cap S(x, r)]>\pi r^2/n \text{ for some } r,\ 0<r<1/m\}.$$

Define

$$D^*(T, \boldsymbol{O})=\bigcup_n\bigcap_m\bigcup_p G_{nm}[T(\boldsymbol{O}\cap \boldsymbol{E}_p),p],\ (n, m=1, 2, \dots;\ p\in U)$$

where $\boldsymbol{O}$ is an open set in the uv-plane and $\boldsymbol{E}_p$ is the union of e.m.m.c.'s under $\pi_pT : Q\to E_2$. Let Ω denote the class of sets in the uv-plane.

An m.m.c. γ under T is called a significant m.m.c. (s.m.m.c.) if and only if for every open set $\boldsymbol{O}\in\Omega$ such that $\gamma\subset\boldsymbol{O}$ we have $T(\gamma)\in D^*(T, \boldsymbol{O})$. The set $\boldsymbol{S}=\boldsymbol{S}(T)$ is defined the union of all s.m.m.c.'s under T.

3.4. We make the following modification.
Define

$$D^{\#}(T, \boldsymbol{O})= \bigcup_p \bigcup_n \bigcap_m G_{nm}[T(\boldsymbol{O}\cap \boldsymbol{E}_p), p] \qquad (n, m=1, 2, ...; p\in U)$$

Let $\boldsymbol{S}^{\#}=\boldsymbol{S}^{\#}(T)$ be the union of all m.m.c.'s γ under T such that for each $\boldsymbol{O}\in\Omega$ such that $\gamma\subset\boldsymbol{O}$ we have $T(\gamma)\in D^{\#}(T, \boldsymbol{O})$. It is clear from the definition that $\boldsymbol{S}^{\#}\subset\boldsymbol{S}$, and that with $\boldsymbol{S}^{\#}$ we single out particular, though not unique, planes.

3.5. LEMMA. Let $T: Q\to E_3$ be a continous transformation. Let Z_1 and $\boldsymbol{S}^{\#}$ be as defined in 3.1 and 3.3. Then $Z_1\subset\boldsymbol{S}^{\#}$.

PROOF. Let γ be an m.m.c. under T in Z_1 and let $x=T(\gamma)$. Then there is a $p\in U$ such that

$$\text{(1)} \qquad \lim_{r\to 0} L_2\{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z\geq 1\}/\pi r^2=1.$$

By lemma 2.3 each z such that $M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z)\geq 1$ is the image of an e.m.m.c. γ_z under $\pi_p T \mid \Delta(\gamma, z)$. γ_z is also an e.m.m.c. under $\pi_p T: Q\to E_2$ (see [4]), so $z\in\pi_p T(\Delta(\gamma, r)\cap E_p)$. Therefore

$$\text{(2)} \qquad \{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z)\geq 1\}\subset\pi_p T(\Delta(y, r)\cap \boldsymbol{E}_p).$$

The middle space $\boldsymbol{M}$ is a separable metric space. Let $\{a_i\}$ be a countable dense set in $\boldsymbol{M}$ and $\boldsymbol{S}(a, r)$ the open sphere in $\boldsymbol{M}$ with center a, radius r. Define $\boldsymbol{O}_{ij}=m^{-1}\boldsymbol{S}(a_i, 1/j)$ for $i, j=1.2, ...$ Suppose $\gamma\subset\boldsymbol{O}_{ij}$. Then $m(\gamma)=a\in\boldsymbol{S}(a_i, 1/j)$. Let r_1 be small enough that $\boldsymbol{S}(a, 2r_1)\subset\boldsymbol{S}a_i, 1/j)$. Then for $0<r\leq r_1$, $m[\Delta(\gamma, r)]\subset\boldsymbol{S}(a, 2r)\subset\boldsymbol{S}(a_i, 1/j)$ and

$$\text{(3)} \qquad \Delta(\gamma, r)\subset\boldsymbol{O}_{ij}.$$

From (1) there exists an $r_2>0$ such that

$$\text{(4)} \qquad L_2\{z : M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z)\geq 1\}/\pi r^2\geq\frac{1}{2} \text{ for } 0<r\leq r_2 .$$

Observe that $T(\Delta(\gamma, r))\subset S(x, r)$ so from (3) we have

$$\pi_p[T(\boldsymbol{O}_{ij}\cap \boldsymbol{E}_p)\cap S(x, r)]\supset\pi_p T(\Delta(\gamma, r)\cap \boldsymbol{E}_p),$$

(5)

$$0<r\leq r_0=\min(r_1, r_2).$$

(5), (2), and (4) imply that if $0<r\leq r_0$,

$$\begin{aligned} H_p[T(\boldsymbol{O}_{ij}\cap \boldsymbol{E}_p)\cap S(x, r)]/\pi r^2 &\geq L_2\{\pi_p[T(\boldsymbol{O}_{ij}\boldsymbol{E}_p)\cap S(x, r)]\}/\pi r^2 \\ &\geq L_2[\pi_p T(\Delta(\gamma, r)\cap \boldsymbol{E}_p)]/\pi r^2 \\ &\geq L_2\{z: M(\pi_p T \mid \Delta(\gamma, r), \Delta(\gamma, r), z)\geq 1\}/\pi r^2\geq \frac{1}{2}. \end{aligned}$$

Therefore,

$$\limsup_{r\to 0} H_p[T(\boldsymbol{O}_{ij}\cap \boldsymbol{E}_p)\cap S(x, r)]/\pi r^2>0,$$

which implies $T(\gamma)\in D_p[T(\boldsymbol{O}_{ij}\cap \boldsymbol{E}_p)]$ and hence $\gamma\subset\boldsymbol{S}^{\#}$. γ was an arbitrary m.m.c. in Z_1, hence $Z_1\subset\boldsymbol{S}^{\#}$.

3.6. THEOREM. Let $T: Q\rightarrow E_3$ be a continuous transformation. Then $T(Z_1)\subset T(\boldsymbol{S}^{\#})\subset T(\boldsymbol{S})$ and if $A(T)<\infty$, then

$$H^2[T(Z_1)]=H^2[T(\boldsymbol{S}^{\#})]=H^2[T(\boldsymbol{S})].$$

PROOF. $T(Z_1)\subset T(\boldsymbol{S}^{\#})$ from lemma 3.5, and $T(\boldsymbol{S}^{\#})\subset T(\boldsymbol{S})$ from the definitions of $\boldsymbol{S}^{\#}$ and $\boldsymbol{S}$ in 3.3 and 3.4. Observe that $T(\boldsymbol{S})=T(\boldsymbol{S}-Z_1)\cup T(Z_1)$, so the theorem follows when it is shown that $H^2[T(\boldsymbol{S}-Z_1)]=0$. Since $\boldsymbol{S}-Z_1$ and Z_1 are disjoint unions of m.m.c.'s,

$$N^*[x, T, \boldsymbol{S}]=N^*[x, T, \boldsymbol{S}-Z_1]+N^*[x, T, Z_1], \qquad x\in E_3.$$

Therefore

$$\int N^*[x, T, \boldsymbol{S}]dH^2=\int N^*[x, T, \boldsymbol{S}-Z_1]\,dH^2+\int N^*[x, T, Z_1]dH^2. \tag{1}$$

But $A(T)<\infty$, so theorem 3.2, [3] and (1) imply that $0=\int N^*[x, T, \boldsymbol{S}-Z]dH^2$. Since $x\in T(\boldsymbol{S}-Z_1)$ implies $N^*[x, T, \boldsymbol{S}-Z_1]\geq 1$, it follows that $0=H^2[T(\boldsymbol{S}-Z_1)]$ and the theorem is proved.

3.7. Remark. By arguments essentially the same as those of Mickle [3] one can show that $\boldsymbol{S}^{\#}$ satisfies invariance under Frechet equivalence and that whenever $N^{*}[x, T, \boldsymbol{S}^{\#}]$ is measurable, $A(T)=\int N^{*}[x, T, \boldsymbol{S}^{\#}]dH^{2}$. In fact, when $A(T)<\infty$, $\boldsymbol{S}^{\#}=\boldsymbol{S}$. Measurability has not been established in case $A(T)=\infty$.

3.8. Remark. In view of theorem 3.4, one can apply the results on approximate tangential planes in [6] to the sets $T(\boldsymbol{S}^{\#})$ and $T(Z_1)$.

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Manoscritto pervenuto in redazione il 30 settembre 1968.