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A note on Fourier coefficients

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A NOTE ON FOURIER COEFFICIENTS

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1. Riesz in his famous paper [4] posed a problem whether exists a continuous function f of bounded variation for which the sequences (na_n) and (nb_n), a_n, b_n being the Fourier coefficients [7] of f, converge and at least one of the two limits be different from zero. Moreover Steinhaus [5] proved that if f is a continuous functions of bounded variation and $na_n \to a$ and $nb_n \to b$, then $a=b=0$, which was later improved by Alexits [1] who proved that if f is a function which has only removable discontinuities i.e. $f(x+0)=f(x-0)$ and if $na_n \to a$, $nb_n \to b$, then $a=b=0$.

The object of this note is to further improve the results of Steinhaus an Alexits and prove the following theorem.

THEOREM. If a_n, b_n are the Fourier coefficients of a 2π-periodic and L-integrable function f. If the sequences (na_n) and (nb_n) are summable to a and b respectively by a regular Nörlund method (N, p) [6] satisfying the conditions that p_n's *are* non-negative, non-increasing, $p_0=1$ and

$$(1.1) \qquad \frac{p_n+1}{p_n} \geq \frac{p_n}{p_n-1} (n>0)$$

and if f has a removable discontinuity at $x=0$, then $a=b=0$.

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2. To prove the theorem we require the following lemma.

LEMMA. If (N, p) is a regular Nörlund method where (p_n) is a non-negative and non-increasing sequence, then so is (N, q) where $q_k=(n-k)\Delta p_k(\Delta p_k=p_k-p_{k+1})$. Moreover

$$Q_n=\sum_{k=0}^{n}(n-k)\Delta p_k=\sum_{k=0}^{n}p^k=P_n\,.$$

PROOF.

$$Q_n=\sum_{k=0}^{n}(n-k)\Delta p_k=\sum_{k=0}^{n}k\Delta p_{n-k}$$

$$=b\sum_{k=0}^{n}(p_{n-k}-p_{n-k-1})=\sum_{k=0}^{n}kp_{n-k}-\sum_{k=1}^{n+1}(k-1)p_{n-k}$$

$$=-(n+1)p_{-1}+\sum_{k=1}^{n+1}p_{n-k}$$

$$=P_n(\text{regarding } p_{-1}=0).$$

Then

$$\frac{q_n}{Q_n}=\frac{q_n}{P_n}=\frac{(n-n)\Delta p_n}{P_n}\to 0 \text{ as } n\to\infty\,.$$

Also

$$q_k=(n-k)\Delta p_k=(n-k)(p_k-p_{k+1})\geq 0,$$

which shows that (N, q) is *a* regular method.

3. Proof of the theorem: Since f has *a* removable discontinuity at $x=0$, by Fejer's theorem [7], the Fourier series of f is $(C, 1)$-summable to $\frac{1}{2}\{f(0+0)-f(0-0)\}=1$ at $x=0$ i. e. the series

$$\sum a_k \tag{3.1}$$

is $(C, 1)$ summable to L. Let σ_n and s_n denote the *n-th* Cesaro mean and partial sum of (3.1) respectively. Then by Abel's lemma we have

$$\begin{aligned}\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}ka_k &= \frac{1}{P_n}\sum_{k=0}^{n-1}\Delta p_{n-k}\sum_{v=0}^{k} va_v + p_0\sum_{v=0}^{n} va_v \\ &= \frac{1}{P_n}\sum_{k=0}^{n}\Delta p_{n+k}\sum_{v=0}^{k} va_v (p_{-1}=0) \\ &= \frac{1}{P_n}\sum_{k=0}^{n}\Delta p_{n-k}\left\{ks_k - \sum_{v=0}^{k-1} s_v\right\} \\ &= \frac{1}{P_n}\sum_{k=0}^{n} k\Delta p_{n-k}s_k - \frac{1}{P_n}\sum_{k=0}^{n}\Delta p_{n-k}\sum_{v=0}^{k-1} s_v \\ &= \frac{1}{P_n}\sum_{k=0}^{n}\Delta p_{n-k}s_k - \frac{1}{P_n}\sum_{k=0}^{n} q_{n-k}\sigma k \\ &= \frac{1}{Q_n}\sum_{k=0}^{n} q_{n-k} - s_k\frac{1}{Q_n}\sum_{k=0}^{n} k\Delta p_{n-k}\sigma k\end{aligned}$$

where $q_k=(n-k)\Delta p_k$ and $Q_n=\sum_{k=0}^{n} q_k=p_n$. By lemma (N, q) is a regular Nörlund method hence making $n\to\infty$ we get

$$a=\lim_{n\to\infty}\frac{1}{Q_n}\sum_{k=0}^{n} q_{n-k}s_k - L.$$

Since any two Nörlund methods are consisten, the method (N, q) is consistent with $(C, 1)$ and hence

$$a=L-L=0.$$

Now by [6, pp. 69, Th. 23] and condition (1.1) it follows that (nb_n) is $(C, 1)$-summable to b i.e.

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} kb_k=b. \tag{3.2}$$

Lukacs [2] has proved that at the point where $g(x\pm 0)$ exists

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{\infty}(b_k\cos kx - a_k\sin kx)=\frac{f(x+0)-f(x-0)}{\pi}$$

and in this case at $x=0$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{\infty} b_k = \frac{f(0+0)-f(0-0)}{x} = 0. \tag{3.3}$$

It follows from (3.2), (3.3) and the inequalities

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{\infty} k u_k \leq \lim_{\infty\leftarrow u} \frac{1}{\log n} \sum_{k=1}^{\infty} u_k$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{n=1}^{\infty} k u_k \geq \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{\infty} u_k$$

that

$$b = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} k b_k = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} b_k = 0,$$

which completes the proof of the theorem.

4. Special Cases: Lastly we note that making the choice of p_n in (N, p) to be

$$p_k = 1 \quad \text{then} \quad P_n = n+1$$

or

$$p_k = \frac{1}{k+1} \quad \text{then } P_k \sim \log n \text{ as } n \to \infty$$

we see that the result of this note holds in particular if the sequences (na_n) and (nb_n) are $(C, 1)$-summable or harmonic summable to a and b respectively.

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REFERENCES

[1] ALEXITS, C.: *Zwei Säze über Fourier-koeffizient,* Math. Zeit. vol. 27 (1928), pp. 65-67.

[2] LUKAS, F.: *Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe,* Journal für die reine und angewandte Matematik vol. 150 (1920), pp. 107-112.

[3] NEDER, L.: *Über die Fourier-koeffizienten der Funktionen von beschränkter Schwankung,* Math. Zeit. vol. 6 (1920), pp. 270-273.

[4] RIESEZ, F.: *Über die Fourier-koeffizienten einer stetigen Funktion von beschänkter Schwankung,* Math. Zeit. vol. 12 (1918), pp. 312-315.

[5] STEINHAUS, H.: Math. Zeit. vol. 8 (1920), pp. 320-322.

[6] HARDY, C. H.: *Divergent Series,* Oxford University Press, London.

[7] ZYGMUND, A.: *Trigonometric Series,* Camb. Univ. Press, Cambridge 1959.

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