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STEVEN BANK

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in the complex domain**

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AN EXISTENCE THEOREM FOR SOLUTIONS  
OF SECOND ORDER NON-LINEAR  
ORDINARY DIFFERENTIAL EQUATIONS  
IN THE COMPLEX DOMAIN \*)

STEVEN BANK

**1. Introduction.**

In this paper we treat second order differential polynomials  $\Omega(y) = \sum_{m, j, k \geq 0} f_{mjk}(x) y^m (y')^j (y'')^k$ , where the coefficients  $f_{mjk}(x)$  are complex functions, defined and analytic in a sectorial region which is approximately of the form,

$$(1) \quad a < \arg(x - \lambda e^{i(a+b)/2}) < b$$

(for some  $\lambda \geq 0$ ) and where as  $x \rightarrow \infty$  in this region, each non-zero  $f_{mjk}(x)$  has an asymptotic expansion in terms of *logarithmic monomials* (i. e. functions of the form

$$M(x) = Kx^{\alpha_0} (\log x)^{\alpha_1} (\log \log x)^{\alpha_2} \dots (\log_q x)^{\alpha_q} \text{ for complex } K \neq 0$$

and real  $\alpha_j$ ). Thus the class of differential polynomials we treat contains, in particular, those having rational functions for coefficients). In [1; § 43] and [5; § 122], existence theorems were proved for solu-

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Indirizzo dell'A: University of Illinois. Urbana, Illinois USA.

tions of  $\Omega(y) = 0$  which are asymptotically equivalent to logarithmic monomials as  $x \rightarrow \infty$  over a filter base consisting essentially of the sectors (1) as  $\lambda \rightarrow \infty$ . In this paper, we prove an existence theorem (§ 3 below) for solutions of  $\Omega(y) = 0$  which are of larger rate of growth than all logarithmic monomials as  $x \rightarrow \infty$  over such a filter base. (We are using here the concepts of asymptotic equivalence ( $\sim$ ) and larger rate of growth ( $>$ ) as  $x \rightarrow \infty$ , introduced in [4; § 13]. For the reader's convenience, these concepts are reviewed in § 2 below).

For a given  $\Omega$ , the corresponding first order differential polynomial,  $G(z) = \sum_{m+j+k=p} f_{mjk}(x) z^j (z' + z^2)^k$ , where  $p = \max\{m + j + k : f_{mjk} \neq 0\}$ , plays an important role in our existence theorem. More specifically, we are interested in the *critical* monomials of  $G(z)$  (i. e. those logarithmic monomials  $N$  for which there is a function  $h \sim N$  such that  $G(h)$  is not  $\sim G(N)$ ). In [1; §§ 21, 26], an algorithm was introduced for finding the set of *all* critical monomials of a given differential polynomial. To apply our existence theorem here, we look for critical monomials  $N$  of  $G$  such that  $N > x^{-1}$ . (Thus  $N = cx^{-1+\nu_0} (\log x)^{\gamma_1} \dots (\log_t x)^{\gamma_t}$  where  $(\gamma_0, \gamma_1, \dots, \gamma_t)$  is lexicographically greater than  $(0, 0, \dots, 0)$ ). Then if  $(a, b)$  is an interval on which the function  $I(\varphi) = \cos(\gamma_0 \varphi + \text{arcc})$  is positive, the theorem in § 3 below asserts the existence, in sectorial subregions of the region (1), of at least a one-parameter family of solutions of  $\Omega(y) = 0$ , each having the form  $\exp \int W$  for some  $W \sim N$ , provided certain subsidiary conditions are fulfilled. (The solutions given by § 3 are shown to be automatically of larger rate of growth than all logarithmic monomials). The subsidiary conditions are of two main types. One type requires that  $N$  *not* be a critical monomial of certain other first order differential polynomials. This type of condition is fulfilled in general, since by [1; § 29 (b), 21, 17 (Remark (2))], we see that for any non-zero first order differential polynomial, there are *finitely* many critical monomials such that any other critical monomial is a constant multiple of one of these. The other type of condition is also seen to be fulfilled in general since it requires that one or two logarithmic monomials which arise, do not have certain special forms. (In

this connection, see § 2 (d) and Remark (c) after § 3). It should be noted that it is easy to test the conditions in any given example, since those of the first type can be tested using the algorithm in [1; §§ 21, 26], while those of the second type can be tested by inspection. Also it should be remarked that these types of conditions are similar to those imposed in the existence theorems in [1; § 42] and [4; § 127] for solutions of first order equations which are  $\sim$  to logarithmic monomials. In fact, some of our conditions make it possible to apply, at the outset, a result in [4] to assert the existence of a solution  $\sim N$  of the first order differential equation  $G(z) = 0$ . The remaining conditions then enable us to use this solution of  $G(z) = 0$  to transform the second order differential equation  $\Omega(y) = 0$  into a quasi-linear form. Our conditions play an essential role in effecting this transformation, since they permit us, at a crucial stage, to assert the existence of a particular type of solution of a certain second order non-homogeneous linear differential equation (see § 4).

Of course one can obtain information on the existence of solutions of  $\Omega(y) = 0$  which are of *smaller* rate of growth than all logarithmic monomials, by making the change of variable  $y = w^{-1}$ , multiplying by a suitable power of  $w$  and then applying the theorem in § 3 to the resulting equation.

In § 7, we apply our results to an example.

## 2. Concepts from [4, 6].

(a) [4; § 94]. Let  $-\pi \leq a < b \leq \pi$ . For each non-negative real-valued function  $g$  on  $(0, (b-a)/2)$ , let  $E(g)$  be the union (over  $\delta \in (0, (b-a)/2)$ ) of all sectors,  $a + \delta < \arg(x - h(\delta)) < b - \delta$  where  $h(\delta) = g(\delta) \exp(i(a+b)/2)$ . The set of all  $E(g)$  (for all choices of  $g$ ) is denoted  $F(a, b)$  and is a filter base which converges to  $\infty$ . Each  $E(g)$  is simply-connected by [4; § 93]. If  $W$  is analytic in  $E(g)$  then the symbol  $\int W$  will stand for a primitive of  $W$  in  $E(g)$ . If  $x$  and  $x_0$  are in  $E(g)$ , then the contour of integration for  $\int_{x_0}^x W$  will be any

rectifiable path in  $E(g)$  from  $x_0$  to  $x$ . A statement is said to hold *except in finitely many directions* (briefly, *e. f. d.*) in  $F(a, b)$ , if there are finitely many points  $r_1 < \dots < r_q$  in  $(a, b)$  such that the statement holds in each of  $F(a, r_1), F(r_1, r_2), \dots, F(r_q, b)$  separately.

(b) [4; § 13]. If  $f$  is analytic in some  $E(g)$ , then  $f \rightarrow 0$  in  $F(a, b)$  means that for any  $\varepsilon > 0$ , there is a  $g_1$  such that  $|f(x)| < \varepsilon$  for all  $x \in E(g_1)$ .  $f < 1$  in  $F(a, b)$  means that in addition to  $f \rightarrow 0$ , all functions  $\theta_j^k f \rightarrow 0$  where  $\theta_j f = (x \log x \dots \log_{j-1} x) f'$ . Then  $f_1 < f_2, f_1 \sim f_2, f_1 \approx f_2$  and  $f \lesssim f_2$  mean respectively,  $f_1/f_2 < 1, f_1 - f_2 < f_2, f_1 \sim cf_2$  for some constant  $c \neq 0$ , and finally either  $f_1 < f_2$  or  $f_1 \approx f_2$ . If  $f < 1$ , then by [4; § 28],  $(x \log x \dots \log_q x) f' < 1$  for all  $q \geq 0$ . If  $M = Kx^{\alpha_0} (\log x)^{\alpha_1} \dots (\log_t x)^{\alpha_t}$ , then by simple calculation,  $M'/M \lesssim x^{-1}$ . If  $M$  is not constant, then it follows from [4; § 28], that  $\psi < M$  implies  $\psi' < M'$ . If for every real  $\alpha, f < x^\alpha$ , we say  $f$  is *trivial* in  $F(a, b)$ .

(c) [6; p. 247]. A logarithmic differential field (briefly an *LDf*) over  $F(a, b)$ , is a differential field  $D$  of functions (each analytic in some  $E(g)$ ), for which there is an integer  $q \geq 0$  such that  $D$  contains all logarithmic monomials of rank  $\leq q$  (i. e. those of the form  $Kx^{\alpha_0} (\log x)^{\alpha_1} \dots (\log_q x)^{\alpha_q}$ ), and such that every non-zero element of  $D$  is  $\sim$  to a logarithmic monomial of rank  $\leq q$ . (For a fixed  $q$ , the set of rational combinations of logarithmic monomials of rank  $\leq q$ , is the simplest example of an *LDf*).

(d) [4; § 43, 100]. If in  $F(a, b)$ ,  $W$  is  $\sim$  to a monomial of the form,

$$Kx^{-1} (\log x)^{-1} \dots (\log_{k-1} x)^{-1} (\log_k x)^{-1+t} (\log_{k+1} x)^{c_1} \dots (\log_{k+s} x)^{c_s}$$

where  $k \geq 0$  and  $t > 0$ , then we say  $W$  is in the *divergence class* in  $F(a, b)$ . The *indicial function* of  $W$  is the function on  $(a, b)$  defined by  $IF(W)(\varphi) = \cos(\delta_{0k} t\varphi + \arg K)$  where  $\delta_{0k}$  is the Kronecker delta. Clearly  $IF(W)$  has at most finitely many zeros unless  $k > 0$  and  $K$  is purely imaginary.

### 3. The Main Theorem :

$$\text{Let } \Omega(y) = \sum_{m, j, k \geq 0} f_{mjk}(x) y^m (y')^j (y'')^k$$

be a second order differential polynomial with coefficients in an  $LDF$  over  $F(a, b)$ . Let  $A = \{m + j + k : f_{mjk} \neq 0\}$ . Let

$$G(z) = \sum_{m+j+k=p} f_{mjk}(x) z^j (z' + z^2)^k \text{ where } p = \max A.$$

Let  $N$  be a critical monomial of  $G$  such that (i)  $N > x^{-1}$ , (ii)  $IF(N) > 0$  on  $(a, b)$ , (iii)  $N$  is not a critical monomial of  $z(\partial G/\partial z) + z'(\partial G/\partial z')$ , (iv)  $\partial G/\partial z' \neq 0$  and  $N$  is not a critical monomial of  $\partial G/\partial z'$ , and (v) if  $A - \{p\}$  is non-empty and  $r = \max A - \{p\}$ , then  $N$  is not a critical monomial of  $H(z) = \sum_{m+j+k=r} f_{mjk}(x) z^j (z' + z^2)^k$ .

Let  $T = (N'/N) + (\partial G(N)/\partial z)(\partial G(N)/\partial z')^{-1}$ . (Then under the assumptions (iii) and (iv),  $T$  is automatically in the divergence class by [1; § 40 (b)]). We assume  $T$  satisfies the following three conditions: (vi)  $IF(T) \neq 0$ , (vii) if it is the case that  $T \approx N$ , say  $T \sim \sigma N$  where  $\sigma$  is a non-zero constant, then we require that  $\sigma \notin \{p - r, 2(p - r)\}$  and  $IF((\sigma + (r - p))N) \neq 0$ , (viii) if it is the case that  $T \approx N'/N$ , then we require that  $T$  is not  $\sim N'/N$ . (see Remark (c) after proof).

Then under these conditions, e. f. d. in  $F(a, b)$  there exists a function  $u_0 \sim N$  such that  $G(u_0) = 0$  and such that if  $x_0$  is in the domain of  $u_0$ , then the equation  $\Omega(y) = 0$  possesses solutions

$$y_c^* \sim c \exp \int_{x_0}^x u_0 \text{ for every constant } c \neq 0.$$

The solutions  $y_c^*$  have the following properties:

(A) For every real  $\alpha$ ,  $y_c^* > x^\alpha$ .

(B) For each  $c \neq 0$ , there is a function  $W_c \sim N$  such that  $y_c^*$  is of the form  $\exp \int W_c$ .

**PROOF:** In this proof, we will make use of the lemmas which are stated and proved in §§ 4, 5 and 6.

By conditions (iii) and (iv) (and § 6) it follows from [1; § 40 (b)] that there exists a logarithmic monomial  $Q(x)$ , such that

$Q(x)G(N + Nw)$  is normal, (i. e. has the form

$$(1) \quad Q(x)G(N + Nw) = \sum d_{jk}(x)w^j(w')^k$$

where in  $F(a, b)$ ,  $d_{00} < 1$ ,  $d_{10} \sim 1$ ,  $d_{j0} \lesssim 1$  for  $j > 1$ ,  $1/d_{01}$  is in the divergence class, and there exists  $q \geq 1$  such that for  $k \geq 1$  and  $j + k \geq 2$ , we have  $d_{jk} \lesssim d_{01}(L_q)^{k-1}$  where  $L_q(x) = x \log x \dots \log_{q-1} x$ . By differentiating (1) with respect to  $w$  and evaluating at  $w = 0$ , we obtain,  $Q(N(\partial G(N)/\partial z) + N'(\partial G(N)/\partial z')) \sim 1$  since  $d_{10} \sim 1$ . Similarly, differentiating (1) with respect to  $w'$  and evaluating at  $w = 0$ , we obtain  $QN(\partial G(N)/\partial z') \sim d_{01}$ . Thus clearly,

$$(2) \quad 1/d_{01} \sim T \text{ (where } T \text{ is as defined in the hypothesis).}$$

By assumption,  $IF(T) \not\equiv 0$ . Letting  $I$  be any open subinterval of  $(a, b)$  on which  $IF(T)$  is nowhere zero, it follows from [4; § 111] (if  $IF(T) > 0$ ) or [4; § 117] (if  $IF(T) < 0$ ) that there exists a function  $w_0 < 1$  in  $F(I)$  such that  $Q(x)G(N + Nw_0) = 0$ . Letting  $u_0 = N + Nw_0$ , we have,

$$(3) \quad u_0 \sim N \text{ in } F(I) \text{ and } G(u_0) = 0.$$

Let  $x_0$  be a point in the domain of  $u_0$  and let  $y_c = c \exp \int_{x_0}^x u_0$  for each constant  $c \neq 0$ . We now assert that

$$(4) \quad \text{For each real } \alpha, y_c > x^\alpha \text{ in } F(I).$$

To prove (4), let  $\alpha$  be given and let  $z_c = x^{-\alpha} y_c$ . Then by simple calculation it is easily seen that  $z_c$  is a solution of  $z - (u_0 - \alpha x^{-1})^{-1} z' = 0$ . But every non-zero solution of this equation is  $> 1$  by [3; p. 271, Lemma  $\delta$ ], since  $u_0 - \alpha x^{-1} \sim N$  and  $IF(N) > 0$  on  $I$ . Thus  $z_c > 1$  so  $y_c > x^\alpha$ .

Note that if  $A = \{p\}$ , the theorem is proved, for in this case  $\Omega(y_c) = y_c^p G(u_0) = 0$  and letting  $\delta$  be a value of  $\log c$ , we have

$y_c = \exp \left( \delta + \int_{x_0}^x u_0 \right)$  which is of the form  $y_c = \exp \int u_0$ . Thus we may assume  $A \neq \{p\}$ .

We now write the equation  $\Omega(y) = 0$  in the form,

$$(5) \quad Q \sum_{m+j+k=p} f_{mjk} y^m (y')^j (y'')^k = - Q \sum_{m+j+k < p} f_{mjk} y^m (y')^j (y'')^k.$$

In what follows let  $c$  be a non-zero constant. Since  $y'_c = y_c u_0$  and  $y''_c = y_c (u'_0 + u_0^2)$ , it follows that under the change of dependent variable  $y = y_c (1 + v)$  and multiplication by  $y_c^{-p}$ , (5) is transformed into an equation of the form,

$$(6) \quad \sum g_{ijk} v^i (v')^j (v'')^k = \sum h_{ijk} v^i (v')^j (v'')^k$$

where it is easily verified that (7)-(13) below hold:

(7) Each  $h_{ijk}$  is trivial (by (4)),

(8) Each  $g_{ijk}$  is  $<$  some power of  $x$  (since it is true of  $u_0, u'_0$  and each  $f_{ijk}$ ),

$$(9) \quad g_{000} = Q \sum_{m+j+k=p} f_{mjk} u_0^j (u'_0 + u_0^2)^k,$$

$$(10) \quad h_{000} = - Q \sum_{m+j+k < p} f_{mjk} u_0^j (u'_0 + u_0^2)^k (y_c)^{m+j+k-p}$$

$$(11) \quad g_{100} = pQ \sum_{m+j+k=p} f_{mjk} u_0^j (u'_0 + u_0^2)^k,$$

$$(12) \quad g_{010} = Q \sum_{m+j+k=p} f_{mjk} (j u_0^{j-1} (u'_0 + u_0^2)^k + 2k u_0^{j+1} (u'_0 + u_0^2)^{k-1}),$$

$$(13) \quad g_{001} = Q \sum_{m+j+k=p} f_{mjk} k u_0^j (u'_0 + u_0^2)^{k-1}.$$

By (9) and (11),  $g_{000} = QG(u_0)$  and  $g_{100} = pQG(u_0)$ , so by (3),

$$(14) \quad g_{000} = 0 \text{ and } g_{100} = 0.$$



Now by definition of  $r$  as being  $\max(A - \{p\})$ , we have by (10),

$$(15) \quad h_{000} = -Q \sum_{t=0}^r \Gamma_t(u_0) y_c^{t-p}$$

where  $\Gamma_t(z) = \sum_{m+j+k=t} f_{mjk} z^j (z' + z^2)^k$ . By hypothesis,  $N$  is not a critical monomial of  $H(z) = \Gamma_r(z)$ . Thus  $\Gamma_r(N) \neq 0$  by [1; § 5], and since  $u_0 \sim N$ , we have  $\Gamma_r(u_0) \sim \Gamma_r(N)$ . Since  $\Gamma_r(N)$  clearly belongs to an *LDF* (namely the field generated by the original and set of logarithmic monomials of rank  $\leq \text{rank } N$ ), and since  $\Gamma_r(N)$  is not zero, there is a logarithmic monomial  $B$  such that  $\Gamma_r(N) \sim B$ . Hence,

$$(16) \quad \Gamma_r(u_0) \sim B \text{ in } F(I).$$

Now for each  $t \leq r$ , clearly  $\Gamma_t(u_0)$  is  $<$  some power of  $x$  (since it is true of  $u_0, u_0'$  and each  $f_{mjk}$ ). Thus by (16),  $(\Gamma_r(u_0))^{-1} \Gamma_t(u_0)$  is  $<$  some power of  $x$  for each  $t < r$ . But if  $t < r$ , then by (4),  $y_c^{r-t}$  is  $>$  all powers of  $x$ . Hence  $(\Gamma_r(u_0))^{-1} \Gamma_t(u_0) < y_c^{r-t}$  and so  $\Gamma_t(u_0) y_c^{t-p} < \Gamma_r(u_0) y_c^{r-p}$  for each  $t < r$ . Hence by (15) and (16),

$$(17) \quad h_{000} \sim -QBy_c^{r-p}.$$

From (13), we have,

$$(18) \quad g_{001} = Q\partial G(u_0)/\partial z'.$$

But from (1),

$$(19) \quad QG(z) = \sum d_{jk} ((z - N)/N)^j ((Nz' - zN')/N^2)^k.$$

Differentiating this relation with respect to  $z'$ , and then evaluating at  $z = u_0$ , we obtain (since  $u_0 = N + Nw_0$ ),

$$(20) \quad Q\partial G(u_0)/\partial z' = N^{-1} \sum d_{jk} kw_0^j (w_0')^{k-1}.$$

Now if  $j \geq 1$  and  $k \geq 1$ , then  $d_{jk} \lesssim (L_q)^{k-1} d_{01}$ . Since  $w_0 < 1$ , we have  $w_0' < L_q^{-1}$  (see § 2 (b)). Hence  $d_{jk} w_0^j (w_0')^{k-1} < d_{01}$  if  $j \geq 1$

and  $k \geq 1$ . Thus clearly

$$(21) \quad \sum d_{jk} k w_0^j (w_0')^{k-1} \sim d_{01}$$

so by (18), (19) and (20),

$$(22) \quad g_{001} \sim N^{-1} d_{01}.$$

Finally, we determine  $g_{010}$ . From (12), clearly

$$(23) \quad g_{010} = Q \partial G(u_0) / \partial z.$$

Differentiating (19) with respect to  $z$  and evaluating at  $z = u_0$ , we obtain (since  $u_0 = N + Nw_0$ ),

$$(24) \quad Q \partial G(u_0) / \partial z = \psi_1 + \psi_2, \text{ where } \psi_1 = (-N'/N^2) \sum d_{jk} k w_0^j (w_0')^{k-1}$$

and  $\psi_2 = N^{-1} \sum d_{jk} j w_0^{j-1} (w_0')^k$ . By (21), we have

$$(25) \quad \psi_1 \sim (-N'/N^2) d_{01}.$$

We now compute  $\psi_2$ . We have  $d_{10} \sim 1$  and  $d_{j0} \lesssim 1$  for  $j > 1$ . Thus since  $w_0 < 1$ , we have  $d_{j0} w_0^{j-1} < 1$  for  $j > 1$ . Now if  $j \geq 1$  and  $k \geq 1$ , then  $d_{jk} \lesssim d_{01} (L_q)^{k-1}$ . Since  $1/d_{01}$  is in the divergence class, there exists  $t > 0$  such that  $1/d_{01} > L_t^{-1}$ . Thus  $d_{01} < L_t$ . Letting  $m = \max\{t, q\}$ , we have  $w_0' < L_m^{-1}$  since  $w_0 < 1$  (see § 2 (b)). Thus for  $j \geq 1, k \geq 1$ , we have  $d_{jk} w_0^{j-1} (w_0')^k < 1$ , so clearly,

$$(26) \quad \psi_2 \sim N^{-1}.$$

Thus by (23) and (24), we have

$$(27) \quad g_{010} = \psi_1 + \psi_2 \text{ (where } \psi_1 \text{ satisfies (25) and } \psi_2 \text{ satisfies (26)).}$$

Now let  $S_{ijk} = g_{ijk} - h_{ijk}$ , so (6) may be written,

$$(28) \quad \sum S_{ijk} v^i (v')^j (v'')^k = 0,$$

where by (7), (8), (14), (17), (22), we have in  $F(I)$ ,

$$(29) \quad S_{000} \sim QBy_c^{r-p}, S_{100} \text{ is trivial and } S_{001} \sim N^{-1} d_{01}.$$

Now since  $d_{01}$  is  $\sim$  to a logarithmic monomial, we have in  $F(a, b)$  that one of the following must hold:  $d_{01} N'/N$  is  $> 1, \approx 1$  or  $< 1$ . We distinguish these three cases:

Case I:  $d_{01} N'/N > 1$ . Hence  $\psi_2 < \psi_1$  so we have,

$$(30) \quad S_{010} \sim g_{010} \sim (-N'/N^2) d_{01}.$$

We consider the equation,

$$(31) \quad S_{001} v'' + S_{010} v' + S_{000} = 0$$

which may be written  $(S_{010}/S_{001})^{-1} v'' + v' = -S_{000}/S_{010}$ . By

$$(29) \text{ and } (30), S_{010}/S_{001} \sim -N'/N \text{ and } -S_{000}/S_{010} \sim QBN^2(N'd_{01})^{-1}y_c^{r-p}.$$

Now  $y_c^{r-p}$  is of the form  $\exp \int (r-p) u_0$ , and since  $N$  is a logarithmic monomial,  $N'/N \lesssim x^{-1}$  (see § 2 (b)), so  $N'/N < u_0$ . Since  $IF((r-p)u_0) < 0$  on  $I$ , it follows by § 4, that e. f. d. in  $F(I)$ , equation (31) has a solution  $v_0 = R_1 y_c^{r-p}$  where,

$$(32) \quad R_1 \sim -QB/(r-p)^2 Nd_{01}.$$

Thus by (4) (since  $r < p$ ),  $v_0$  is trivial, and hence,

$$(33) \quad v_0^m \text{ is trivial for } m \geq 1.$$

Let  $J$  be any open subinterval of  $I$ , such that  $v_0$  exists on  $F(J)$ . A simple calculation shows that  $v_0'/v_0 = M$  where  $M = (r-p)u_0 + (R_1'/R_1)$  and hence  $M \sim (r-p)u_0$  (see § 2 (b)). Thus since  $v_0$  solves (31), clearly,

$$(34) \quad S_{001}(M^2 + M')v_0 + S_{010}Mv_0 + S_{000} = 0.$$

We now subject equation (28) to the change of variable  $v = v_0 + v_0 u$ , and divide the result by  $v_0$ . Since  $v' = v_0 \Phi_1(u)$  where  $\Phi_1(u) = M + Mu + u'$  and  $v'' = v_0 \Phi_2(u)$  where  $\Phi_2(u) = M' + M^2 + (M' + M^2)u + 2Mu' + u''$ , we obtain

$$(35) \quad \sum S_{ijk} v_0^{i+j+k-1} (1+u)^i (\Phi_1(u))^j (\Phi_2(u))^k = 0.$$

We denote this equation by

$$(36) \quad \sum t_{ijk} u^i (u')^j (u'')^k = 0.$$

Now clearly,

$$(37) \quad t_{000} = \sum S_{ijk} v_0^{i+j+k-1} M^j (M' + M^2)^k.$$

In view of (33), (34) and the fact that  $S_{100}$  is trivial we see that  $t_{000}$  is trivial in  $F(J)$ . Now,

$$(38) \quad t_{100} = \sum S_{ijk} v_0^{i+j+k-1} (i+j+k) M^j (M' + M^2)^k.$$

By (33) and (34), we may write  $t_{100} = (-S_{000}/v_0) + q_1$  where  $q_1$  is trivial in  $F(J)$ . By (29) and the fact that  $v_0 = R_1 y e^{r-p}$  where  $R_1$  satisfies (32), we obtain  $t_{100} \sim (r-p)^2 N d_{01}$ . Now,

$$(39) \quad t_{010} = \sum S_{ijk} v_0^{i+j+k-1} (2kM^{j+1}(M' + M^2)^{k-1} + jM^{j-1}(M' + M^2)^k).$$

By (33) therefore  $t_{010} = S_{010} + 2MS_{001} + q_2$ , where  $q_2$  is trivial in  $F(J)$ . But by (29) (and the fact that  $M \sim (r-p)N$ ), we see that  $2MS_{001} \sim 2(r-p)d_{01}$ . But  $N'/N \lesssim x^{-1}$  (see § 2 (b)) and so  $N' < N^2$ . Thus by (30), clearly  $t_{010} \sim 2(r-p)d_{01}$ . Now,

$$(40) \quad t_{001} = \sum S_{ijk} v_0^{i+j+k-1} kM^j (M' + M^2)^{k-1}$$

and so by (33) and (29), we have  $t_{001} \sim N^{-1}d_{01}$ . Finally in view of (33), (and the fact that each  $S_{ijk}$  is  $<$  some power of  $x$ , by (7) and (8)), we clearly see that if  $i+j+k \geq 2$  then  $t_{ijk}$  is trivial in

$F(J)$ . Hence if we divide equation (36) by  $(r - p)^2 N d_{01}$ , we obtain,

$$(41) \quad \sum a_{ijk} u^i (u')^j (u'')^k = 0, \text{ where in } F(J),$$

$$(42) \quad a_{100} \sim 1, a_{010} \sim 2/(r - p) N, a_{001} \sim 1/(r - p)^2 N^2, \text{ and } a_{ijk}$$

is trivial if  $i + j + k \neq 1$ .

Now let  $W_1 = W_2 = (p - r) N$ . Define  $E_2 = a_{001} W_1 W_2 - 1$ . From (42),  $a_{001} W_1 W_2 \sim 1$ , so  $E_2 < 1$  in  $F(J)$ . Now define  $E_1 = -W_1 (a_{010} + (1 + E_2) (W_1^{-1} + W_2^{-1} + W_1' (W_2 W_1^2)^{-1})$ . Since  $W_1' = (p - r) N'$ , we see that  $W_1' (W_2 W_1^2)^{-1} = N'/(p - r)^2 N^3$  which is  $< 1/N$  since  $N'/N \lesssim x^{-1}$  (see § 2 (b)). But  $W_1^{-1} + W_2^{-1} = 2/(p - r) N$  and  $a_{010} \sim -2/(p - r) N$  by (42) so clearly  $E_1 < -W_1/N$ . Thus  $E_1 < 1$  in  $F(J)$ . Finally if we define  $E_0 = a_{100} - (1 + E_2 + E_1)$ , then  $E_0 < 1$  since  $a_{100} \sim 1$ . Since  $IF(W_j) > 0$  on  $J$  for  $j = 1, 2$ , it follows from § 5, that equation (41) possesses a solution  $u^* < 1$  in  $F(J)$ . Thus the function  $v^* = v_0 + v_0 u^*$  is a solution of (28), and so the function  $y_c^* = y_c (1 + v^*)$  is a solution of  $\Omega(y) = 0$ . In view of what  $I$  and  $J$  represent, it is clear that such a  $y_c^*$  exists e. f. d. in  $F(a, b)$ . Since  $v_0$  is trivial by 33),  $v^*$  is trivial in  $F(J)$ . Thus clearly  $y_c^* \sim y_c$  and Part (A) follows from (4). Since  $y_c^* = y_c (1 + v^*)$ , clearly  $(y_c^*)' = y_c^* (u_0 + (v^*)'/(1 + v^*))$ . Since  $v^*$  is trivial in  $F(J)$ , so is  $(v^*)'$  (see § 2 (b)). Thus we may write  $(y_c^*)' = y_c^* W_c'$ , where  $W_c \sim u_0$ . Thus if  $x_1$  is a point in the domain of  $W_c$ , then for

some constant  $K$ ,  $y_c^* = K \exp \int_{x_1}^x W_c$ . Since  $y_c^* \neq 0$ ,  $K \neq 0$ . Thus for

for any value of  $\log K$ ,  $y_c^* = \exp \left( \log K + \int_{x_1}^x W_c \right)$  which is of the form  $\exp \int W_c$ . This proves Part (B) in Case I.

Case II:  $d_{01} N'/N \approx 1$ . In this case,  $d_{01} N'/N \sim \delta$  where  $\delta$  is a non-zero constant. By (2) and hypothesis (viii),  $\delta \neq 1$ . Thus in (27), we have  $g_{010} \sim (1 - \delta)/N$  so by (7),

$$(43) \quad S_{010} \sim (1 - \delta)/N.$$

We again consider the equation (31). In this case, by (29) and (43),  $S_{010}/S_{001} \sim (1 - \delta)/d_{01}$  and  $-S_{000}/S_{010} \sim -QB(1 - \delta)^{-1}y_e^{r-p}$ .

As in Case I,  $y_e^{r-p}$  is of the form  $\exp \int (r - p) u_0$ , and in this case,  $1/d_{01} < N$  since  $N'/N \lesssim x^{-1}$  (see § 2 (b)). Hence by § 4, e. f. d. in  $F(I)$ , equation (31) possesses a solution  $v_0 = R_2 y_e^{r-p}$ , where,

$$(44) \quad R_2 \sim -QB/(r - p)^2 N d_{01} .$$

As in Case I, (33) holds. Let  $J$  be any open subinterval of  $I$  such that  $v_0$  exists on  $F(J)$ . A simple calculation shows that  $v_0'/v_0 = M$  where  $M = (r - p) u_0 + R_2'/R_2$ . Thus, as in Case I,  $M \sim (r - p) u_0$  and since  $v_0$  solves (31), we have that (34) holds. We now subject equation (28) to the change of variable  $v = v_0 + v_0 u$ , and then divide by  $v_0$ . If we denote the result by

$$(45) \quad \sum t_{ijk} u^i (u')^j (u'')^k = 0,$$

then as in Case I, the coefficients  $t_{000}$ ,  $t_{100}$ ,  $t_{010}$  and  $t_{001}$  are given by (37), (38), (39) and (40) respectively, and (in view of (33)), we also have

$$(46) \quad t_{ijk} \text{ is trivial in } F(J) \text{ for } i + j + k \geq 2.$$

In view of (29), (33) and (34), it follows from the representation in (37) that  $t_{000}$  is trivial in  $F(J)$ , and it follows from the representation in (38), that  $t_{100} = (-S_{000}/v_0) + q_3$  where  $q_3$  is trivial in  $F(J)$ . Since  $v_0 = R_2 y_e^{r-p}$  where  $R_2$  is as in (44), we have using (29) that  $t_{100} \sim (r - p)^2 N d_{01}$ . In view of (29) and (33), it follows from the representation in (39) that  $t_{010} = S_{010} + 2MS_{001} + q_4$  where  $q_4$  is trivial in  $F(J)$ . Now  $2MS_{001} \sim 2(r - p) d_{01}$  by (29). But in this case  $d_{01} > 1/N$  since  $1/d_{01} \approx N'/N \lesssim x^{-1}$  (see § 2 (b)). Thus clearly from (43),  $t_{010} \sim 2(r - p) d_{01}$ . Finally from (29), (33) and the representation in (40), we see that  $t_{001} \sim N^{-1} d_{01}$ . Hence if we di-

vide equation (45) by  $(r - p)^2 N d_{01}$ , we obtain an equation,

$$(47) \quad \sum a_{ijk} u^i (u')^j (u'')^k = 0, \text{ where in } F(J),$$

$$(48) \quad a_{100} \sim 1, a_{010} \sim 2/(r - p) N, a_{001} \sim 1/(r - p)^2 N^2 \text{ and } a_{ijk}$$

is trivial if  $i + j + k \neq 1$ . But these are *precisely* the same asymptotic properties we found for the coefficients of (41) in Case I (see (42)). Thus letting  $W_1 = W_2 = (p - r) N$ , we see that if we define  $E_2, E_1, E_0$  as in Case I, we will find that  $E_j < 1$  in  $F(J)$  ( $j = 0, 1, 2$ ), and so by § 5, it follows that equation (47) possesses a solution  $u^* < 1$  in  $F(J)$ . Thus as in Case I, the equation  $\Omega(y) = 0$  possesses the solution  $y_c^* = y_c(1 + v_0 + v_0 u^*)$  e. f. d. in  $F(a, b)$ . The proof the solution  $y_c^*$  has the desired properties in this case follows *exactly* as in Case I.

*Case III:*  $d_{01} N'/N < 1$ . In this case, in (27) we have  $g_{010} \sim N^{-1}$ , so we have

$$(49) \quad S_{010} \sim N^{-1}.$$

We again consider the equation (31). In this case by (29) and (49),  $S_{010}/S_{001} \sim 1/d_{01}$  and  $-S_{000}/S_{010} \sim -QB N y_c^{r-p}$ . As in Case I,  $y_c^{r-p}$  is of the form  $\exp \int (r - p) u_0$ . Let  $U = (1/d_{01}) + (r - p) N$ . Since  $1/d_{01}$  and  $N$  are both  $\sim$  to logarithmic monomials, one of the following three cases must hold:

(a)  $1/d_{01} < N$ , (b)  $1/d_{01} > N$  and (c)  $1/d_{01} \approx N$ . In Case (a),  $U \sim (r - p) N$  so  $IF(U) < 0$  on  $I$ . In Case (b),  $U \sim 1/d_{01}$  so by (2),  $IF(U)$  is nowhere zero on  $I$ . In Case (c),  $1/d_{01} \sim \sigma N$  for some constant  $\sigma \neq 0$ . By (2) and hypothesis (vii),  $\sigma \neq p - r$  so  $U \sim (\sigma + (r - p)) N$  and  $IF(U) \neq 0$  on  $I$ . Thus in all three cases  $IF(U) \neq 0$ . Hence by § 4, e. f. d. in  $F(I)$ , equation (31) possesses a solution  $v_0 = R_3 y_c^{r-p}$ , where

$$(50) \quad R_3 \sim -QB/(r - p) d_{01} U.$$

As in Case I (33) holds. Let  $J$  be any open subinterval of  $I$  such that  $v_0$  exist on  $F(J)$ . A simple calculation shows that  $v_0'/v_0 = M$

where  $M = (r - p) u_0 + R_3'/R_3$ . Thus  $M \sim (r - p) u_0$ , and since  $v_0$  solves (31), we have that (34) holds. We now subject equation (28) to the change variable  $v = v_0 + v_0 u$  and then divide by  $v_0$ . If we denote the result by

$$(51) \quad \sum t_{ijk} u^i (u')^j (u'')^k = 0,$$

then as in Case I, the coefficients  $t_{000}, t_{100}, t_{010}$  and  $t_{001}$  are given by (37), (38), (39), and (40) respectively, and (in view of (33)), we also have

$$(52) \quad t_{ijk} \text{ is trivial in } F(J) \text{ if } i + j + k \geq 2.$$

In view of (29), (33) and (34), it follows from the representation in (37) that

$$(53) \quad t_{000} \text{ is trivial in } F(J),$$

and it follows from (38) that  $t_{100} = (-S_{000}/v_0) + q_5$  where  $q_5$  is trivial in  $F(J)$ . Since  $v_0 = R_3 y_c^{r-p}$  where  $R_3$  is as in (50), it follows from (29) that

$$(54) \quad t_{100} \sim (r - p) d_{01} U \text{ in } F(J).$$

In view of (29) and (33) it follows from (39) that  $t_{010} = S_{010} + 2MS_{001} + q_6$ , where  $q_6$  is trivial in  $F(J)$ . By (29),  $2MS_{001} \sim 2(r-p) d_{01}$ , and by (49),  $S_{010} \sim N^{-1}$ . Let  $U^* = (1/N) + 2(r-p) d_{01}$ . By hypothesis (vii),  $1/d_{01}$  is not  $\sim 2(p-r)N$ , so  $U^*$  is  $\approx$  to either  $N^{-1}$  or  $d_{01}$  and so is nontrivial. Hence clearly,

$$(55) \quad t_{010} \sim U^* \text{ in } F(J).$$

Finally from (29) and (33), it follows from (40), that

$$(56) \quad t_{001} \sim d_{01}/N \text{ in } F(J).$$

Hence if we divide equation (51) by  $(r - p) d_{01} U$ , we obtain an equation

$$(57) \quad \sum a_{ijk} u^i (u')^j (u'')^k = 0,$$



where by (52)-(56), we have in  $F(J)$ ,

$$(58) \quad a_{100} \sim 1, a_{010} \sim U^*/(r-p) d_{01} U, a_{001} \sim 1/(r-p) NU,$$

$$(59) \quad a_{ijk} \text{ is trivial if } i + j + k \neq 1.$$

To proceed any further, we must consider each of the possibilities for the asymptotic behavior of  $U$ . Since  $1/d_{01}$  and  $N$  are  $\sim$  to logarithmic monomials, one of the following cases must hold in  $F(a, b)$ :  $1/d_{01} < N$ ,  $1/d_{01} > N$ ,  $1/d_{01} \approx N$ . We distinguish these three subcases.

*Subcase A:*  $1/d_{01} < N$ . Thus in this subcase,  $U \sim (r-p)N$  and  $U^* \sim 2(r-p)d_{01}$ . Hence by (58) and (59),  $a_{100} \sim 1, a_{010} \sim 2/(r-p)N, a_{001} \sim 1/(r-p)^2 N^2$  and  $a_{ijk}$  is trivial if  $i + j + k \neq 1$ . But these are *precisely* the same asymptotic properties we found for the coefficients of (41) in Case I (see (42)). Thus exactly as in Case I (i. e. by letting  $W_1 = W_2 = (p-r)N$ , and defining  $E_1, E_2, E_0$  as in Case I), we can apply § 5 to show that equation (57) possesses a solution  $u^* < 1$  in  $F(J)$ . Thus  $\Omega(y) = 0$  possesses the solution  $y_c^* = y_c(1 + v_0 + v_0 u^*)$  in  $F(J)$  and hence e. f. d. in  $F(a, b)$ . The proof that  $y_c^*$  has the desired properties in this case follows exactly as in Case I.

*Subcase B:*  $1/d_{01} > N$ . Thus in this subcase,  $U \sim 1/d_{01}$  and  $U^* \sim 1/N$ . Thus by (58) and (59), we have  $a_{100} \sim 1, a_{010} \sim 1/(r-p)N, a_{001} \sim d_{01}/(r-p)N$ . Let  $W_1 = (p-r)N$  and  $W_2 = -1/d_{01}$ . Define  $E_2 = a_{001} W_1 W_2 - 1$ . Then  $E_2 < 1$ , in  $F(J)$  since clearly  $a_{001} W_1 W_2 \sim 1$ . Now define  $E_1 = -W_1(a_{010} + (1 + E_2)(W_1^{-1} + W_2^{-1} + W_1'(W_2 W_1^2)^{-1}))$ . By this subcase,  $W_2^{-1} < W_1^{-1}$ . Also,  $W_1'(W_2 W_1^2)^{-1} = d_{01} N'/(r-p) N^2$  which is also  $< W_1^{-1}$  since by this case  $d_{01} N'/N < 1$ . Hence since  $a_{010} \sim -W_1^{-1}$ , we see that  $E_1 < -W_1 W_1^{-1}$  so  $E_1 < 1$  in  $F(J)$ . Finally if we define  $E_0 = a_{100} - (1 + E_2 + E_1)$  then  $E_0 < 1$  in  $F(J)$  since  $a_{100} \sim 1$ . Thus since  $IF(W_1) > 0$  on  $J$  while  $IF(W_2) \not\equiv 0$  on  $J$  (since  $IF(W_2) \equiv -IF(T)$  by (2) and  $IF(T) \not\equiv 0$  by assumption (vi)), it follows from § 5 that e. f. d. in  $F(J)$ , equation (57) possesses a solution  $u^* < 1$ . In view of what  $J$

and  $I$  represent, clearly such a  $u^*$  exists e. f. d. in  $F(a, b)$ . Thus (28) possesses the solution  $v^* = v_0 + v_0 u^*$ , and hence  $\Omega(y) = 0$  possesses, e. f. d. in  $F(a, b)$ , the solution  $y_c^* = y_c(1 + v_0 + v_0 u^*)$ . The proof that the solution  $y_c^*$  has the desired properties follows exactly as in Case I.

*Subcase C*;  $1/d_{01} \approx N$ . Thus there is a non-zero constant  $\sigma$  such that  $1/d_{01} \sim \sigma N$ . Since  $T \sim 1/d_{01}$  we have by assumption (vii) that,

$$(60) \quad \sigma \notin \{p - r, 2(p - r)\} \text{ and } IF(\lambda N) \neq 0 \text{ where } \lambda = \sigma + (r - p).$$

In this subcase, clearly  $U \sim \lambda N$  and  $U^* \sim (\sigma + 2(r - p))/\sigma N$ , so by (58) and (59) we have  $a_{100} \sim 1$ ,  $a_{010} \sim (\sigma + 2(r - p))/(r - p)\lambda N$ ,  $a_{001} \sim 1/\lambda(r - p)N^2$  and  $a_{ijk}$  is trivial for  $i + j + k \neq 1$ . Let  $W_1 = (p - r)N$  and  $W_2 = -\lambda N$ . Define  $E_2 = a_{001} W_1 W_2 - 1$ . Then  $E_2 < 1$  in  $F(J)$  since  $a_{001} W_1 W_2 \sim 1$ . Now define  $E_1 = -W_1(a_{010} + (1 + E_2)(W_1^{-1} + W_2^{-1} + W_1'(W_2 W_1^2)^{-1}))$ . Clearly, since  $\lambda = \sigma + (r - p)$ , we have  $W_1^{-1} + W_2^{-1} = (\sigma + 2(r - p))/\lambda(p - r)N$ . Now  $W_1'(W_2 W_1^2)^{-1} = N'/\lambda(r - p)N^3$  which is  $< W_1^{-1} + W_2^{-1}$  since by this subcase  $N'/N^3 \approx d_{01} N'/N^2$  which is  $< 1/N$  by this case. Since  $a_{010} \sim -(W_1^{-1} + W_2^{-1})$  clearly, we have that  $E_1 < -W_1(W_1^{-1} + W_2^{-1})$  so  $E_1 < 1$  in  $F(J)$ . Finally if we define  $E_0 = a_{100} - (1 + E_2 + E_1)$ , then  $E_0 < 1$  in  $F(J)$  since  $a_{100} \sim 1$ . Thus since  $IF(W_1) > 0$  on  $J$ , while  $IF(W_2) \neq 0$  on  $J$  (in view of (60) and the fact that  $IF(W_2) = -IF(\lambda N)$ ), it follows from § 5 that e. f. d. in  $F(J)$ , equation (57) possesses a solution  $u^* < 1$ . In view of what  $J$  and  $I$  represent, clearly such a  $u^*$  exists e. f. d. in  $F(a, b)$ . Thus (28) possesses the solution  $v^* = v_0 + v_0 u^*$ , and hence  $\Omega(y) = 0$  possesses, e. f. d. in  $F(a, b)$ , the solution  $y_c^* = y_c(1 + v_0 + v_0 u^*)$ . The proof that the solution  $y_c^*$  has the desired properties follows exactly as in Case I. This concludes the proof of the theorem.

REMARKS: (a) If  $I$  is any open subinterval of  $(a, b)$  on which  $IF(T) > 0$ , then in view of (2), it follows from [4; § 111] that the equation  $Q(x)G(N + Nw) = 0$  actually possesses a whole one-parameter family of solutions  $< 1$  in  $F(I)$ . By using this family

for  $w_0$  in the proof, it is clear that e. f. d. in  $F(I)$ , we can actually expect a *two-parameter* family of solutions of  $\Omega(y) = 0$ , each having the form  $\exp \int W$  for some  $W \sim N$ .

(b) In each case in the proof we showed that for each  $c \neq 0$ , the equation  $\Omega(y) = 0$  possesses an exact solution  $y_c^* = y_c(1 + v_c + v_c u_c^*)$  where  $y_c = c \exp \int_{x_0}^x u_0$ ,  $u_c^* < 1$  and  $v_c = R_c y_c^{r-p}$  for some  $R_c \sim$  to a logarithmic monomial. This representation clearly gives more information about  $y_c^*$  than either of the statements  $y_c^* \sim y_c$  or  $y_c^* = \exp \int W_c$  for some  $W_c \sim N$ , and furthermore, is useful in estimating how good an approximation  $y_c$  is to the exact solution  $y_c^*$ .

(c) In general one can make no specific statement about the relation between  $N$  and its corresponding  $T$ . In fact for *any* given logarithmic monomials  $N$  and  $T^*$ , where  $T^*$  is in the divergence class, one can always produce a second order  $\Omega(y)$ , whose corresponding  $G(z)$  has  $N$  as a critical monomial (which automatically satisfies (iii), (iv) and (v) of § 3), and where  $T^*$  is the corresponding  $T$  for  $N$ . To see this, consider  $\Omega(y) = (1/T^* N) yy'' - (1/T^* N) (y')^2 + ((1/N) - (N'/T^* N^2)) yy' - y^2$ . Here  $G(z) = (1/T^* N) z' + ((1/N) - (N'/T^* N^2)) z - 1$ , and hence  $G(Nw) = (w'/T^*) + w - 1$ . Since  $T^*$  is in the divergence class, an easy application of [1; § 26] to  $G(Nw)$ , followed by [1; § 30 (b)] shows that  $N$  is a critical monomial of  $G$  satisfying (iii) by § 6. Clearly (iv) and (v) are also satisfied. A simple calculation shows that  $T^* = (N'/N) + (\partial G(N)/\partial z)(\partial G(N)/\partial z')^{-1}$ . In view of this example, one can view the situations where  $T \approx N$  or  $T \approx N'/N$  (i. e. those where one would have to verify (vii) or (viii) of § 3), as being somewhat special situations.

4. LEMMA: Let  $W, V$  and  $H$  be functions  $\sim$  to logarithmic monomials in some  $F(I)$ . Let  $V > x^{-1}$  and  $W \sim -V$ . (Then clearly  $W + V \gtrsim V$ , so  $W + V$  is also in the divergence class). Let  $IF(V) \not\equiv 0$  and  $IF(W + V) \not\equiv 0$ . Let  $w_0$  be a function of the form  $\exp \int V$ . Then e. f. d. in  $F(I)$ , the equation  $(v''/W) + v' = Hw_0$  possesses a solution  $v_0 = R w_0$  where  $R \sim WH/V(W + V)$ .

PROOF: Let  $z_0 = Hw_0$ . Under the change of variable  $w = v'$ , the equation

$$(a) \quad (v''/W) + v' = z_0 \text{ becomes}$$

$$(b) \quad (w'/W) + w = z_0.$$

Under the change of dependent variable  $w = xWz_0 u$ , followed by division by  $xz_0$ , equation (b) becomes,

$$(c) \quad u' + Uu = x^{-1},$$

where  $U = W + V + (H'/H) + (W'/W) + x^{-1}$ . Now since  $W \sim -V$ , clearly  $W + V$  is  $\sim$  to a logarithmic monomial which is  $\gtrsim V$  and hence  $> x^{-1}$ . Since  $H$  and  $W$  are  $\sim$  to logarithmic monomials,  $H'/H$  and  $W'/W$  are  $\lesssim x^{-1}$ . (see § 2 (b)). Thus,

$$(d) \quad U \sim W + V > x^{-1}.$$

Hence in some element of  $F(I)$ ,  $U$  is nowhere zero, so (c) may be written  $u + u'/U = 1/xU$ . By (d),  $1/xU < 1$  and  $IF(U) \neq 0$ . Letting  $J$  be any open subinterval of  $I$  on which  $IF(U)$  is nowhere zero, it follows from [3; p. 271, Lemma  $\delta$ ] that there is a function  $u_1 < 1$  in  $F(J)$  such that  $u_1 + u_1'/U = 1/xU$ . But  $xu_1' < 1$  (see § 2 (b)), so  $u_1 \sim 1/xU$ . Now clearly the function  $w_1 = xWu_1 z_0$  is a solution of (b). Let  $J_1$  be any open subinterval of  $J$  on which  $IF(V)$  is nowhere zero. Since  $z_0$  is of the form  $H \exp \int V$ , and  $xWu_1 \sim W/W + V$  (by (d)), it follows by [2; § 10 (b)] that the equation  $v' = w_1$  possesses a solution  $v_0 = R w_0$  where  $R \sim WH/V(W + V)$  in  $F(J_1)$ . (Note: [2; § 10 (b)] was proved under the hypothesis that  $V \approx x^{-1+t}$  for some  $t > 0$ , but it is easily seen that the exact same proof holds if the condition  $V \approx x^{-1+t}$  is replaced by the condition that  $V$  be  $\approx$  to a logarithmic monomial which is  $> x^{-1}$ ). Then  $v_0$  is clearly a solution of (a) proving the lemma.

5. LEMMA: (a) Let  $\Phi(y) = A_2 y'' + A_1 y' + A_0 y$  where  $A_0, A_1$  and  $A_2$  are analytic in some  $F(I)$ . Let  $W_1, W_2$  be  $\sim$  to logarithmic

monomials is  $F(I)$ . Define  $E_2, E_1$  and  $E_0$  as follows:  $E_2 = A_2 W_1 W_2 - 1$ ,  $E_1 = -W_1 (A_1 + (1 + E_2) (W_1^{-1} + W_2^{-1} + W_1' (W_2 W_1^2)^{-1}))$ , and  $E_0 = A_0 - (1 + E_2 + E_1)$ . Then,

$$(1) \quad \Phi = \dot{W}_2 \dot{W}_1 + E_2 \dot{W}_2 \dot{W}_1 + E_1 \dot{W}_1 + E_0$$

where  $\dot{W}_j$  is the operator  $\dot{W}_j y = y - y' / W_j$ .

$$(b). \quad \text{Let } \Psi(y) = a_{000} + \Phi(y) + \sum_{i+j+k \geq 2} a_{ijk} y^i (y')^j (y'')^k,$$

where  $a_{000}$  and  $a_{ijk}$  (for  $i + j + k \geq 2$ ) are trivial in  $F(I)$ , and where  $\Phi(y)$  is a second order linear differential polynomial for which there exist functions  $W_1$  and  $W_2$  in the divergence class in  $F(I)$  (see § 2 (d)) and functions  $E_0, E_1, E_2$  each  $< 1$  in  $F(I)$ , such that (1) above holds. Then for any open subinterval  $J$  of  $I$  on which both  $IF(W_1)$  and  $IF(W_2)$  are nowhere zero, the equation  $\Psi(y) = 0$  possesses at least one solution  $y^* < 1$  in  $F(J)$ .

PROOF: Part (a) is proved by directly computing the right side of (1) and showing that (1) holds if  $E_2, E_1$  and  $E_0$  are defined as in the statement.

To prove (b), let  $A_0 y = y$ ,  $A_1 y = \dot{W}_1 y$  and  $A_2 y = \dot{W}_2 \dot{W}_1 y$ . A simple calculation shows that  $y'$  and  $y''$  are linear polynomials in  $A_0 y, A_1 y$ , and  $A_2 y$  (with coefficients which are each  $<$  some power of  $x$ ). Thus when  $\Psi(y)$  is written as a polynomial in the  $A_j y$ , we obtain (using (1)),

$$\begin{aligned} \Psi(y) = & a_{000} + (1 + E_2) A_2 y + E_1 A_1 y + E_0 A_0 y + \\ & + \sum_{m+n+q \geq 2} b_{mnq} (A_0 y)^m (A_1 y)^n (A_2 y)^q, \end{aligned}$$

where each  $b_{mnq}$  is trivial (since each  $a_{ijk}$  is trivial). Since  $E_0, E_1, E_2$  are each  $< 1$  in  $F(I)$ , it follows directly from the remark after [5; § 99] (which states that formulas (99.6) and (99.7) are sufficient for a strong factorization sequence), that  $(W_1, W_2)$  is a *strong factorization sequence* (see [5; § 88 (b)]) for  $\Psi(y)$ . Hence if  $J$  is any open subinterval of  $I$  on which both  $IF(W_1)$  and  $IF(W_2)$  are nowhere

zero, then by [5; § 99, Part 5], the equation  $\Psi(y) = 0$  possesses at least one solution  $y^* < 1$  in  $F(J)$ , proving (b).

6. LEMMA: Let  $G(z) = \sum g_{ij} z^i (z')^j$  have coefficients in an LDF over  $F(a, b)$ . Let  $N$  be a critical monomial of  $G(z)$ . Then  $N$  is a simple non-parametric critical monomial of  $G$  (as defined in [1; §§ 28 (d), 14]) if and only if  $N$  is not a critical monomial of  $\Gamma(z) = z(\partial G/\partial z) + z'(\partial G/\partial z')$ .

PROOF: (For notation and definitions used in this proof, see [1; §§ 7, 9, 28]). Let  $N$  have exponent  $\beta$  and multiplicity  $m$  in  $G$ . Let  $H(z) = G(Nz)$ , say  $H(z) = \sum h_{ij}(x) z^i (z')^j$ . By [1; § 30 (b)], 1 is a critical monomial of  $H$  having exponent  $\beta$  and multiplicity  $m$ . Now for each  $s \geq 0$ , there is a logarithmic monomial  $Q(x)$  such that  $[1, s, H](u, v) = \sum r_{ij}(u) v^i (v')^j$ , where

$$(a) \quad r_{ij}(u) = h_{ij}(e_s(u)) (e_s(u) \dots e_1(u))^{-j} Q(e_s(u)),$$

(where  $e_j(u)$  is the  $j$ th iteration of the exponential function). Thus by [1; § 27], for  $s$  sufficiently large,

(b)  $r_{i\beta} \lesssim 1$ ,  $r_{ij}$  is trivial if  $j \neq \beta$  and 1 is a root of multiplicity  $m$  of  $C(v) = \sum r_{i\beta}(\infty) v^i$ , (where  $r_{i\beta}(\infty)$  is the constant defined by:  $r_{i\beta}(\infty) = 0$  if  $r_{i\beta} < 1$ , while if  $r_{i\beta} \approx 1$ ,  $r_{i\beta} \sim r_{i\beta}(\infty)$ ).

Now let  $A(z) = \Gamma(Nz)$ . Then clearly, we have  $A(z) = \sum (i+j) h_{ij} z^i (z')^j$ . Thus for all  $s \geq 0$ , there is a logarithmic monomial  $R(x)$  such that  $[1, s, A](u, v) = \sum t_{ij} v^i (v')^j$ , where

$$(c) \quad t_{ij}(u) = (i+j) h_{ij}(e_s(u)) (e_s(u) \dots e_1(u))^{-j} R(e_s(u)).$$

Letting  $B(u) = R(e_s(u))/Q(e_s(u))$ , we have by (a) and (c) that

$$(d) \quad [1, s, A](u, v) = B \sum (i+j) r_{ij}(u) v^i (v')^j.$$

Now assume  $N$  is not critical of  $\Gamma(z)$ . Then by [1; § 30 (b)], 1 is not critical of  $A$ . Now if  $\beta$  were  $> 0$ , then in view of (b) and (d), clearly  $[1, s, A](u, 1)$  would be trivial, so by [1; §§ 3, 11 (b)], 1 would be critical of  $A$  which is a contradiction. Thus  $\beta = 0$  proving that  $N$  is non-parametric by [1; § 28]. In view of (b) and the fact

that  $\beta = 0$ , we may write (d) as,

$$(e) \quad [1, s, A](u, v) = B(dC/dv) + \Psi(u, v)$$

where all coefficients of  $\Psi$  are trivial. If  $m > 1$ , then 1 is a root of  $dC/dv$ , so by (e) and [1; §§ 3, 11 (b)], we would again obtain the contradiction that 1 is critical of  $A$ . Thus  $m = 1$ , so  $N$  is simple also.

Conversely, suppose  $N$  is simple and non-parametric in  $G$ . Then  $\beta = 0$ , so (e) holds. Since  $m = 1$ ,  $dC/dv$  does not have 1 as a root. Thus by [1; § 12 (b)], 1 is not a critical monomials of  $A$ , so  $N$  is not critical of  $\Gamma$  by [1; § 11 (a)], proving the lemma.

7. EXAMPLE: The example we consider here illustrates the fact that in the theorem of § 3, only the terms of degree  $p$  (and in a negative way, those of degree  $r$ ) play an essential role, since in this example the terms of degree  $< r$  are essentially arbitrary.

Consider the example,

$$\begin{aligned} \Omega(y) = (y')^{n+2} - y^n (y'')^2 + x^{-3} (\log x)^t y^{n+1} y' + \\ + \psi(x) y^{r-(m+q)} (y')^m (y'')^q + A(y), \end{aligned}$$

where  $n$  is any integer  $> 2$ ,  $t$  is any strictly positive real number,  $\psi$  is any finite sum of logarithmic monomials,  $r$  is any non-negative integer  $\leq n + 1$ ,  $m$  and  $q$  are any non-negative integers such that  $m + q \leq r$  and where  $A$  is any second order differential polynomial with coefficients in an LDF over  $F(-\pi, \pi)$ , each of whose terms has total degree  $\leq r - 1$  in  $y, y', y''$ . In this case,  $G(z)$  and  $H(z)$  of § 3 are as follows:  $G(z) = z^{n+2} - (z')^2 - 2z^2 z' - z^4 + x^{-3} (\log x)^t z$ , and  $H(z) = \psi(x) z^m (z' + z^2)^q$ . Using [1; § 26], we find that  $G(z)$  has the  $(n + 1)$  simple, non-parametric (see § 6) critical monomials consisting of  $M_1 = x^{-1} (\log x)^{t/3}$ ,  $M_2 = e^{2\pi i/3} M_1$ ,  $M_3 = e^{4\pi i/3} M_1$  and the  $(n - 2)nd$  roots of unity. Using [1; §§ 21, 26], we find that  $H$  has the one critical monomial  $x^{-1}$  if  $q > 0$ , and no critical monomials if  $q = 0$ . Since  $\partial G/\partial z' = -2z' - 2z^2$ , we find that  $\partial G/\partial z'$  has the one critical monomial  $x^{-1}$ .

Now consider  $M_1, M_2$  and  $M_3$ . Clearly  $IF(M_1) \equiv 1$  while  $IF(M_2) = IF(M_3) \equiv -1/2$ . Thus  $M_1$  is the only one of these three

which satisfies (ii) of § 3. For  $M_1$ , we now compute the function  $T$  of § 3. Since  $\partial G/\partial z = (n+2)z^{n+1} - 4zz' - 4z^3 + x^{-3}(\log x)^t$ , an easy calculation shows that  $T \sim (3/2)x^{-1}(\log x)^{t/3}$ . Thus  $IF(T) \equiv 1$ . However, we see that  $T \approx M_1$  so we must verify (vii) of § 3. But here  $\sigma = 3/2$ , so clearly  $\sigma \notin \{p-r, 2(p-r)\}$  (since  $p-r$  is a positive integer), and  $IF((\sigma + (r-p))M_1)$  is either  $+1$  or  $-1$  depending on whether  $p-r < 3/2$  (i. e.  $p=r+1$ ) or  $p-r > 3/2$ . Thus by § 3, we can conclude that e. f. d. in  $F(-\pi, \pi)$  the equation  $\Omega(y) = 0$  possesses a one parameter family of solutions  $y_c^* = \exp \int W_c$  where  $W_c \sim M_1$ .

Now let  $N$  be any  $(n-2)nd$  root of unity, say  $N = e^{i\alpha}$  where  $-\pi < \alpha \leq \pi$ . For  $N$ , we now compute the function  $T$  of § 3. Since  $N^{n-2} = 1$ , an easy calculation shows that  $T \sim (1 - (n/2))N$ , so since  $N$  is a constant, clearly  $IF(T) \neq 0$ . However, since  $T \approx N$  we must again verify (vii) of § 3. But here  $\sigma = 1 - (n/2)$ , so  $\sigma < 0$  since  $n > 2$ . Thus clearly  $\sigma \notin \{p-r, 2(p-r)\}$ , and since  $(\sigma + (r-p))N$  is a constant monomial, clearly  $IF((\sigma + (r-p))N) \neq 0$ . Now  $IF(N)(\varphi) = \cos(\varphi + \alpha)$ . Thus by determining the largest open subintervals of  $(-\pi, \pi)$  on which  $IF(N)$  is positive, we obtain the following conclusions by § 3; If  $-\pi/2 \leq \alpha \leq \pi/2$ , then e. f. d. in  $F(-\alpha - \pi/2, -\alpha + \pi/2)$ , the equation  $\Omega(y) = 0$  possesses a one-parameter family of solutions  $\exp \int V_c$  where  $V_c \sim N$ . If  $\pi/2 \leq \alpha \leq \pi$  (respectively,  $-\pi < \alpha \leq -\pi/2$ ), then e. f. d. in each of  $F(-\pi, -\alpha + \pi/2)$  and  $F(-\alpha + 3\pi/2, \pi)$  separately (respectively,  $F(-\pi, -\alpha - 3\pi/2)$  and  $F(-\alpha - \pi/2, \pi)$  separately), the equation  $\Omega(y) = 0$  possesses a one parameter family of solutions  $\exp \int V_c$ , where  $V_c \sim N$ .

All solutions found in this example are  $>$  all logarithmic monomials by § 3 (A).



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