OLAF TAMASCHKE

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A GENERALIZATION OF CONJUGACY IN GROUPS

OLAF TAMASCHKE *)

The category of all groups can be embedded in the category of all S-semigroups [*1*]. The S-semigroup is a mathematical structure that is based on the group structure. The intention is to use it as a tool in group theory. The Homomorphism Theorem and the Isomorphism Theorems, the notions of normal subgroup and of subnormal subgroup, the Theorem of Jordan and Hölder, and some other statements of group theory have been generalized to S-semigroups ([*1*] and [*2*]). The direction of these generalizations, and the intention behind them, may become clearer by the remark that the double coset semigroups form a category properly between the category of all groups and the category of all S-semigroups. By the double coset semigroup G/H of a group G modulo an arbitrary subgroup H of G we mean the semigroup generated by all HgH, $g \in G$, with respect to the « complex » multiplication, considered as an S-semigroup.

Yet, so far the generalization of one group theoretical concept is missing in this theory, namely a generalization of conjugacy. We show in this paper that a notion of conjugacy can be appropriately defined for a certain class of S-semigroups which we call *CS-semigroups* (Definitions 1.1 and 1.2).

*) Indirizzo dell'Autore: Mathematisches Institut der Universität Tübingen, W. Germany.

The intersection of the class of all CS-semigroups with the class of all double coset semigroups gives rise to a new class of subgroups. A subgroup H of a group G is called *prenormal* if the double coset semigroup G/H is a CS-semigroup. Every normal subgroup is prenormal. Every subgroup whose double coset semigroup G/H is commutative is prenormal too. Since it is easy to find groups G with non-normal subgroups H such that G/H is commutative (take any doubly transitive permutation group G of degree greater than 2, and let H be the stabilizer of one letter), the class of normal subgroups is properly contained in the class of prenormal subgroups. On the other hand the class of prenormal subgroups is properly contained in the class of all subgroups since we can show the existence of a finite group that has non-prenormal subgroups (Section 4). Every group is a CS semigroup, and the conjugacy of Definition 1.2 then coincides with the conjugacy in the ordinary sense.

Therefore the class of all CS-semigroups is properly contained in the class of all S-semigroups, and properly contains the class of all groups. Moreover, every homomorphic image of a CS-semigroup is a CS-semigroup (Corollary 2.5). Therefore the CS-semigroups form a category properly contained in the category of all S-semigroups, and properly containing the category of all groups.

Among other properties of CS-semigroups we prove the following. If T is a CS-semigroup on a group G, then the set of all T-normal subgroups of G is a modular sublattice of the lattice of all subgroups of G (Theorem 2.2).

If Σ is an S-semigroup on a group F, and T is an S-semigroup on a group G, then we define the external direct product $\Sigma \times \mathrm{T}$ as an S semigroup on the external direct product $F \times G$ (Definition 3.1). There exists an analogue for S-semigroups to the fact that the direct product of groups can be expressed in terms of trivially intersecting normal subgroups (Theorem 3.3).

If T_i is a CS-semigroup on the group G_i for $i = 1, \dots, n$, then the external direct product $\mathrm{T}_1 \times \dots \times \mathrm{T}_n$ is a CS-semigroup on the external direct product $G_1 \times \dots \times G_n$ (Theorem 3.5). If K_i is a prenormal subgroup of the group G_i for $i = 1, \dots, n$, then $K_1 \times \dots \times K_n$ is prenormal in $G_1 \times \dots \times G_n$ (Corollary 3.6).

Before we set out for our investigations we briefly recall the basic definitions and facts of the theory of S-semigroups [*1*].

Let G be a group. The set $\overline{G}$ of all non-empty subsets of G is a semigroup with respect to the subset (or complex) multiplication.

A subsemigroup T of $\overline{G}$ is called an *S-semigroup* on G if it has a unit element and if there exists a set $\mathfrak{T} \subseteq \overline{G}$ such that

(1) $G = \bigcup_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}.$

(2) $\mathcal{S} = \mathcal{T}$ *or* $\mathcal{S} \cap \mathcal{T} = \emptyset$ *for all* $\mathcal{S}, \mathcal{T} \in \mathfrak{T}.$

(3) $\mathcal{T}^{-1} = \{g^{-1} \mid g \in \mathcal{T}\} \in \mathfrak{T}$ *for all* $\mathcal{T} \in \mathfrak{T}.$

(4) $X = \bigcup_{\substack{\mathcal{T} \in \mathfrak{T} \\ \mathcal{T} \cap X \neq \emptyset}} \mathcal{T}$ *for all* $X \in \mathsf{T}.$

(5) T *is generated by* $\mathfrak{T}$, *that is every element of* T *is the product of a finite number of elements of* $\mathfrak{T}$.

The elements of $\mathfrak{T}$ are called the T-*classes* of G. The T-class containing $g \in G$ is denoted by $\mathcal{T}_g$. The T-class $\mathcal{T}_1$ containing the unit element $1 \in G$ is the unit element of T; it is a subgroup of G.

A subgroup H of G is called a T-*subgroup* of G if H is a union of T-classes. Then the T-classes of G contained in H define an S-semigroup T_H on H. Furthermore there exists an S-semigroup $\mathsf{T}_{G/H}$ on G with $\{H\mathcal{T}H \mid \mathcal{T} \in \mathfrak{T}\}$ as the set of all $\mathsf{T}_{G/H}$-classes of G. The set of all T-subgroups of G is a sublattice of the lattice of all subgroups of G.

A subgroup K of G is called T-*normal* if K is a T-subgroup of G such that

$$K\mathcal{T} = \mathcal{T}K \textit{ holds for all } \mathcal{T} \in \mathfrak{T}.$$

Let F be a group, and Σ be an S-semigroup on F. We denote by $\mathfrak{S}$ the set of all Σ-classes of F, and by $\mathcal{S}_f$ the Σ-class containing $f \in F$. A mapping φ of T into Σ is called a *homomorphism* of the S-semigroup T on G into the S-semigroup Σ on F if it has the following properties.

(1) $(XY)^{\varphi} = X^{\varphi} Y^{\varphi}$ *for all* $X, Y \in \mathsf{T}$.

(2) *For every* $\mathcal{T} \in \mathfrak{T}$ *there exists an* $\mathcal{S} \in \mathfrak{S}$ *such that*

$$\mathcal{T}^{\varphi} = \mathcal{S} \text{ and } (\mathcal{T}^{-1})^{\varphi} = \mathcal{S}^{-1}.$$

(3) $X^{\varphi} = \bigcup_{\substack{\mathcal{T} \in \mathfrak{T} \\ \mathcal{T} \subseteq X}} \mathcal{T}^{\varphi}$ *for all* $X \in \mathsf{T}$.

It is clear what is meant by an *epimorphism, monomorphism,* or *isomorphism* of T into, or onto, Σ.

The *kernel* of φ is defined as

$$\operatorname{Ker} \varphi = \bigcup_{\substack{\mathcal{T} \in \mathfrak{T} \\ \mathcal{T}^{\varphi} = \mathcal{S}_1}} \mathcal{T},$$

where $\mathcal{S}_1$ is the unit element of Σ. It has been shown [*1,* Theorem 2.7] that $\operatorname{Ker} \varphi$ is a T-normal subgroup of G, and that $\operatorname{Im} \varphi$ is isomorphic to the S-semigroup $\mathsf{T}_{G/\operatorname{Ker}\varphi}$ on G. Conversely, if K is a T-normal subgroup of G, then

$$\varphi_K : X \rightarrow XK$$

is an epimorphism of the S-semigroup T on G onto the S-semigroup $\mathsf{T}_{G/K}$ on G whose kernel is K [*1,* Theorem 2.8]. Therefore the S-semigroup $\mathsf{T}_{G/K}$ is considered as the *factor S-semigroup* of T modulo the T-normal subgroup K of G.

We denote by $\overline{\mathsf{T}}$ the semigroup of all those non-empty subsets of G which are unions of T-classes. Each homomorphism φ of an S-semigroup T on G into an S-semigroup Σ on F can be uniquely extended to a homomorphism $\overline{\varphi}$ of $\overline{\mathsf{T}}$ into $\overline{\Sigma}$ [*1,* Proposition 2.2]. Note that a T-subgroup of G is not necessarily an element of T, but it is an element of $\overline{\mathsf{T}}$, and so it has an image $H^{\overline{\varphi}}$ by $\overline{\varphi}$ which is a Σ-subgroup of F [*1,* Proposition 2.4 (1)].

Throughout the following let G denote a group, and let T be an S-semigroup on G.

1. CS-semigroups and T-conjugate Elements.

Let $\mathsf{Z}(\mathsf{T})$ be the centre of $\overline{\mathsf{T}}$. An element $C \in \overline{\mathsf{T}}$ belongs to $\mathsf{Z}(\mathsf{T})$ if and only if

$$C\mathcal{T} = \mathcal{T}C \text{ for all } \mathcal{T} \in \mathfrak{T}.$$

Therefore $\mathsf{Z}(\mathsf{T})$ is also the centralizer of T in $\overline{\mathsf{T}}$.

DEFINITION 1.1. *An S-semigroup* T *on G is called a CS*-semigroup *on G if there exists a subset $\mathfrak{C}(\mathsf{T})$ of $\mathsf{Z}(\mathsf{T})$ with the following properties.*

(1) $G = \bigcup_{\mathcal{C} \in \mathfrak{C}(\mathsf{T})} \mathcal{C}$.

(2) $\mathcal{C} = \mathcal{D}$ *or* $\mathcal{C} \cap \mathcal{D} = \varnothing$ *for all* $\mathcal{C}, \mathcal{D} \in \mathfrak{C}(\mathsf{T})$.

(3) $\mathcal{C}^{-1} = \{g^{-1} \mid g \in \mathcal{C}\} \in \mathfrak{C}(\mathsf{T})$ *for all* $\mathcal{C} \in \mathfrak{C}(\mathsf{T})$.

(4) $C = \bigcup_{\substack{\mathcal{C} \in \mathfrak{C}(\mathsf{T}) \\ \mathcal{C} \cap C \neq \varnothing}} \mathcal{C}$ *for all* $C \in \mathsf{Z}(\mathsf{T})$.

If T is a CS-semigroup on G, then the set $\mathfrak{C}(\mathsf{T})$ is uniquely determined by the properties (1) - (4). For every element $g \in G$ we denote by $\mathcal{C}_{\mathsf{T}}(g)$ the unique $\mathcal{C} \in \mathfrak{C}(\mathsf{T})$ containing g. The set $\mathfrak{C}(\mathsf{T})$ generates an S-semigroup which we denote by $\mathcal{C}(\mathsf{T})$.

DEFINITION 1.2. *Let* T *be a CS-semigroup on G. Two elements $x, y \in G$ are called* T-conjugate *if* $\mathcal{C}_{\mathsf{T}}(x) = \mathcal{C}_{\mathsf{T}}(y)$. *The elements of* $\mathfrak{C}(\mathsf{T})$ *are called the* T-conjugacy classes *of G.*

Let T now denote a CS-semigroup on G.

LEMMA 1.3. $\mathcal{C}_{\mathsf{T}}(1) = \mathcal{T}_1$.

PROOF. $\mathcal{T}_1 \in \mathsf{Z}(\mathsf{T})$ since $\mathcal{T}_1$ is the unit element T and of $\overline{\mathsf{T}}$. Therefore

$$\mathcal{T}_1 = \bigcup_{\substack{\mathcal{C} \in \mathfrak{C}(\mathsf{T}) \\ \mathcal{C} \cap \mathcal{T}_1 \neq \varnothing}} \mathcal{C}$$

by 1.1 (4) But $\mathcal{C} \cap \mathcal{T}_1 \neq \emptyset$ implies $\mathcal{T}_1 \subseteq \mathcal{C}$ because every $\mathcal{C} \in \mathfrak{C}(\mathsf{T})$ is a union of T-classes. It follows that $\mathcal{T}_1 \in \mathfrak{C}(\mathsf{T})$, and hence $\mathcal{T}_1 = \mathcal{C}_{\mathsf{T}}(1)$.

Lemma 1.4. *Two elements $x, y \in G$ are T-conjugate if and only if x^{-1} and y^{-1} are T-conjugate.*

Proof. 1.1(3) and 1.2.

Definition 1.5. *A subgroup H of G is called* prenormal *in G if the double coset semigroup G/H is a CS-semigroup on G.*

I thank Professor Wielandt for suggesting the name « prenormal » to me.

Lemma 1.6. *Let H be a normal subgroup of G. Then H is prenormal in G, and two elements $x, y \in G$ are G/H-conjugate if and only if the cosets Hx, Hy are conjugate in the factor group G/H. Especially G-conjugacy is the conjugacy in the ordinary sense.*

Proof. The set of all $\mathcal{C}_{\mathcal{K}} = \bigcup_{Hg \in \mathcal{K}} Hg$, where $\mathcal{K}$ runs through all conjugacy classes of the factor group G/H satisfies 1.1.

Lemma 1.7. *Let H be a subgroup of G such that the double coset semigroup G/H is commutative. Then H is prenormal in G, and two elements $x, y \in G$ are G/H-conjugate if and only if $HxH = HyH$.*

Proof. $\overline{G/H}$ coincides with its centre if G/H is commutative. Therefore the set $\{HgH \mid g \in G\}$ satisfies 1.1.

2. CS-semigroups and T-normal Subgroups.

Proposition 2.1. *Let T be a CS-semigroup on G. Then a subgroup K of G is T-normal if and only if K is the union of T-conjugacy classes.*

Proof. A subgroup K of G is T-normal if and only if $K \in \mathsf{Z}(\mathsf{T})$. If, in addition, T is a CS-semigroup on G, then $K \in \mathsf{Z}(\mathsf{T})$ if and only if $K = \bigcup_{k \in K} \mathcal{C}_{\mathsf{T}}(k)$ by 1.1 (4).

THEOREM 2.2. *Let* T *be a CS-semigroup on G. If K and L are* T*-normal subgroups of G, then KL and* $K \cap L$ *are* T*-normal subgroups of G, and therefore the set of all* T*-normal subgroups of G is a modular sublattice of the lattice of all subgroups of G.*

PROOF. Let K and L be T-normal subgroups of G. Then KL is T-normal by [*1*, Proposition 1.12]. Since T is a CS-semigroup, K and L, and hence $K \cap L$ too, are unions of T-conjugacy classes by 2.1. Therefore $K \cap L$ is T-normal, again by 2.1. It follows that the set of all T-normal subgroups of G is a sublattice of the lattice of all subgroups of G, which is modular since any two T-normal subgroups of G are permutable [*1*, Theorem 1.11 (1)].

LEMMA 2.3. *Let* T *be an S-semigroup on G, and let K be a* T*-normal subgroup of G. Then* $\mathsf{Z}(\mathsf{T}_{G/K}) \subseteq \mathsf{Z}(\mathsf{T})$.

PROOF. Take any $C \in \mathsf{Z}(\mathsf{T}_{G/K})$. Then

$$C = KC = CK = KCK$$

since every element of $\mathsf{T}_{G/K}$ is a union of elements $\mathcal{T}K = K\mathcal{T} = K\mathcal{T}K, \mathcal{T} \in \mathfrak{T}$. Therefore

$$C\mathcal{T} = CK\mathcal{T} = C(K\mathcal{T}K) = (K\mathcal{T}K)\,C = \mathcal{T}KC = \mathcal{T}C \text{ for all } \mathcal{T} \in \mathfrak{T},$$

which means that $C \in \mathsf{Z}(\mathsf{T})$.

THEOREM 2.4. *Let* T *be a CS-semigroup on G, and let K be a* T*-normal subgroup of G. Then* $\mathsf{T}_{G/K}$ *is a CS-semigroup on G.*

PROOF. Set

$$\mathfrak{C} = \mathfrak{C}(\mathsf{T}) \text{ and } \mathfrak{D} = \{\mathcal{C}K \mid \mathcal{C} \in \mathfrak{C}\}.$$

Then $\mathfrak{D} \subseteq \mathsf{Z}(\mathsf{T}_{G/K})$. We show that $\mathfrak{D}$ satisfies Definition 1.1.

I. 1.1 (1) for $\mathfrak{D}$ follows from 1.1 (1) for $\mathfrak{C}$.

II. Assume that $\mathcal{C}K \cap \mathcal{D}K \neq \emptyset$ for $\mathcal{C}, \mathcal{D} \in \mathfrak{C}$. Take any

$g \in \mathcal{C}K \cap \mathcal{D}K$. There exist $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $k, k' \in K$ such that

$$g = ck = dk'.$$

Therefore $c = dk' k^{-1} \in \mathcal{D}K$. But $\mathcal{D}K \in Z(\mathsf{T})$ since $\mathcal{D} \in Z(\mathsf{T})$ and $K \in Z(\mathsf{T})$ because K is T-normal. It follows from 1.1(4) that $\mathcal{C} \subseteqq \mathcal{D}K$, and hence $\mathcal{C}K \subseteqq \mathcal{D}K$. In the same way we show that $\mathcal{D}K \subseteqq \mathcal{C}K$. Therefore $\mathcal{C}K = \mathcal{D}K$.

III. 1.1(3) is satisfied since

$$(\mathcal{C}K)^{-1} = K^{-1}\,\mathcal{C}^{-1} = K\mathcal{C}^{-1} = \mathcal{C}^{-1}\,K \in \mathfrak{D} \quad \text{for all } \mathcal{C} \in \mathfrak{C}.$$

IV. Since $Z(\mathsf{T}_{G/K}) \subseteqq Z(\mathsf{T})$ by 2.3,

$$Z = \bigcup_{\substack{\mathcal{C} \in \mathfrak{C} \\ \mathcal{C} \cap Z \neq \emptyset}} \mathcal{C} \qquad \text{for all } Z \in Z(\mathsf{T}_{G/K}).$$

by 1.1(4). Therefore

$$Z = ZK = \bigcup_{\substack{\mathcal{C}K \in \mathfrak{D} \\ \mathcal{C}K \cap Z \neq \emptyset}} \mathcal{C}K \qquad \text{for all } Z \in Z(\mathsf{T}_{G/K}),$$

and 1.1(4) is satisfied by $\mathfrak{D}$. We have proved that $\mathsf{T}_{G/K}$ is a CS-semigroup on G.

COROLLARY 2.5. *Every homomorphic image of a CS-semigroup is a CS-semigroup.*

PROOF. Let T be an S-semigroup on G, and let φ be a homomorphism of T into an S semigroup Σ on F. Then $\overline{\text{Ker } \varphi}$ is a T-normal subgroup of G, and Im φ is an S-semigroup on G^{φ} isomorphic to $\mathsf{T}_{G/\text{Ker } \varphi}$ [*1*, Theorem 2.7], and so 2.5 follows from Theorem 2.4.

COROLLARY 2.6. *Let T be a CS-semigroup on G, and let K be a T-normal subgroup of G. Then two elements $x, y \in G$ which are T-conjugate are also $\mathsf{T}_{G/K}$-conjugate, and therefore each $\mathsf{T}_{G/K}$-conjugacy class is the union of T-conjugacy classes.*

PROOF. If $\mathcal{C}$ is a T-conjugacy class, then $\mathcal{C}K$ is a $\mathsf{T}_{G/K}$-conjugacy class, and $\mathcal{C} \subseteqq \mathcal{C}K$.

3. Direct Products of S-semigroups.

Let Σ be an S-semigroup on the group F. Denote by $\mathfrak{S}$ the set of all Σ-classes of F. Let $F \times G$ be the external direct product of the groups F and G, and let $M \times N$ be the cartesian product of two subsets $M \subseteqq F$ and $N \subseteqq G$. If also $M' \subseteqq F$ and $N' \subseteqq G$, then

$$(M \times N)(M' \times N') = (MM') \times (NN').$$

For the set

$$\mathbb{R} = \{\mathcal{S} \times \mathcal{T} \mid \mathcal{S} \in \mathfrak{S} \text{ and } \mathcal{T} \in \mathfrak{T}\}$$

the following statements hold.

(1) $$F \times G = \bigcup_{\mathcal{R} \in \mathbb{R}} \mathcal{R}.$$

(2) $$\mathcal{R} = \mathcal{R}' \text{ or } \mathcal{R} \cap \mathcal{R}' = \varnothing \qquad \text{for all } \mathcal{R}, \mathcal{R}' \in \mathbb{R}.$$

(3) $$\mathcal{R}^{-1} \in \mathbb{R} \text{ for all } \mathcal{R} \in \mathbb{R}.$$

Denote by P the subsemigroup of $\overline{F \times G}$ which is generated by $\mathbb{R}$. For every $X \in \mathsf{P}$ there exist $\mathcal{S}^{(1)}, \dots, \mathcal{S}^{(n)} \in \mathfrak{S}$ and $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(n)} \in \mathfrak{T}$ such that

(4) $$\begin{aligned} X &= (\mathcal{S}^{(1)} \times \mathcal{T}^{(1)}) \dots (\mathcal{S}^{(n)} \times \mathcal{T}^{(n)}) = (\mathcal{S}^{(1)} \dots \mathcal{S}^{(n)}) \times (\mathcal{T}^{(1)} \dots \mathcal{T}^{(n)}) \\ &= \Big(\bigcup_{\mathcal{S} \subseteqq \mathcal{S}^{(1)} \dots \mathcal{S}^{(n)}} \mathcal{S}\Big) \times \Big(\bigcup_{\mathcal{T} \subseteqq \mathcal{T}^{(1)} \dots \mathcal{T}^{(n)}} \mathcal{T}\Big) \\ &= \bigcup_{\substack{\mathcal{S} \subseteqq \mathcal{S}^{(1)} \dots \mathcal{S}^{(n)} \\ \mathcal{T} \subseteqq \mathcal{T}^{(1)} \dots \mathcal{T}^{(n)}}} \mathcal{S} \times \mathcal{T} \\ &= \bigcup_{\substack{\mathcal{R} \in \mathbb{R} \\ \mathcal{R} \cap X \neq \varnothing}} \mathcal{R}. \end{aligned}$$

Therefore P is an S-semigroup on $F \times G$ with $\mathfrak{R}$ as the set of all P-classes of $F \times G$.

On the other hand we can form the direct product $\Sigma \times \mathsf{T}$ of Σ and T qua semigroups. For every element $(X, Y) \in \Sigma \times \mathsf{T}$ there exist

$$\mathcal{S}^{(1)}, \dots, \mathcal{S}^{(m)} \in \mathfrak{S} \text{ and } \mathcal{T}^{(1)}, \dots, \mathcal{T}^{(n)} \in \mathfrak{T}$$

such that

$$X = \mathcal{S}^{(1)} \dots \mathcal{S}^{(m)} \text{ and } Y = \mathcal{T}^{(1)} \dots \mathcal{T}^{(n)}.$$

Set $k = \max(m, n)$ and

$$\mathcal{S}^{(\mu)} = \mathcal{S}_1 \text{ for } \mu = m + 1, \dots, k,$$

$$\mathcal{T}^{(\nu)} = \mathcal{T}_1 \text{ for } \nu = n + 1, \dots, k,$$

so that

$$(X, Y) = (\mathcal{S}^{(1)}, \mathcal{T}^{(1)}) \dots (\mathcal{S}^{(k)}, \mathcal{T}^{(k)}).$$

Therefore the mapping

$$(X, Y) \to X \times Y$$

is a semigroup isomorphism of $\Sigma \times \mathsf{T}$ onto P.

DEFINITION 3.1. *The S-semigroup P on $F \times G$ is called the* external direct product *of the S-semigroup Σ on F and the S-semigroup T on G. We write $\mathsf{P} = \Sigma \times \mathsf{T}$ (by identification).*

We define S-semigroup homomorphisms

$\iota_1 : X \to X \times \mathcal{T}_1$ which is a monomorphism of Σ into $\Sigma \times \mathsf{T}$,

$\iota_2 : Y \to \mathcal{S}_1 \times Y$ which is a monomorphism of T into $\Sigma \times \mathsf{T}$,

$\pi_1 : X \times Y \to X$ which is an epimorphism of $\Sigma \times \mathsf{T}$ onto Σ,

$\pi_2 : X \times Y \to Y$ which is an epimorphism of $\Sigma \times \mathsf{T}$ onto T,

and we have the direct product diagram

$$\Sigma \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} \Sigma \times \mathsf{T} \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} \mathsf{T}$$

with the equations

$$\iota_1 \pi_1 = 1_\Sigma, \quad \iota_1 \pi_2 = 0,$$

$$\iota_2 \pi_1 = 0, \quad \iota_2 \pi_2 = 1_\mathsf{T},$$

$$\pi_1 \iota_1 + \pi_2 \iota_2 = 1_{\Sigma \times \mathsf{T}}.$$

PROPOSITION 3.2. *Let Δ be an S-semigroup on a group H, and assume that the diagram*

$$\Sigma \underset{\pi_1'}{\overset{\iota_1'}{\rightleftarrows}} \Delta \underset{\pi_2'}{\overset{\iota_2'}{\leftrightarrows}} \mathsf{T}$$

and the above equations hold with 1_Δ, ι_j', π_j' $(j = 1, 2)$ instead of $1_{\Sigma \times \mathsf{T}}$, ι_j, π_j $(j = 1, 2)$. Then there exists a unique isomorphism φ of the S-semigroup Δ on H onto the S-semigroup $\Sigma \times \mathsf{T}$ on $F \times G$ such that

$$\iota_j' \varphi = \iota_j \text{ and } \varphi \pi_j = \pi_j' \qquad (j = 1, 2).$$

PROOF. It is easy to show that $\varphi = \pi_1' \iota_1 + \pi_2' \iota_2$ is the unique homomorphism satisfying the above conditions, and that $\psi = \pi_1 \iota_1' + \pi_2 \iota_2'$ is its inverse.

THEOREM 3.3. *Let Δ be an S-semigroup on a group H. Let $\mathfrak{D}$ be the set of all Δ-classes of H, and denote by $\mathcal{D}_h$ the Δ-class of H containing $h \in H$. The following statements are equivalent.*

(1) *There exists a diagram*

$$\Sigma \underset{\pi_1'}{\overset{\iota_1'}{\rightleftarrows}} \Delta \underset{\pi_2'}{\overset{\iota_2'}{\leftrightarrows}} \mathsf{T}$$

with the homomorphisms 1_Δ, ι_j', π_j' $(j = 1, 2)$ satisfying the above equations and $\iota_1' \neq 0$, $\iota_2' \neq 0$.

(2) *There exist Δ-normal subgroups K and L of H different from the unit element $\mathcal{D}_1$ of Δ such that*

$$H = KL, \quad K \cap L = \mathcal{D}_1,$$

$$\Delta = \Delta_K \Delta_L, \quad \Delta_K \cap \Delta_L = \{\mathcal{D}_1\},$$

$$\mathfrak{D} = \{\mathcal{D}_k \mathcal{D}_l \mid k \in K \text{ and } l \in L\}.$$

PROOF. I. Assume that (1) holds. Set

$$K = F^{\bar{\iota}_1'} \text{ and } L = G^{\bar{\iota}_2'}.$$

Then $K \neq \mathcal{D}_1 \neq L$ because of $\iota_1' \neq 0 \neq \iota_2'$. Furthermore $K \leq \operatorname{Ker} \pi_2'$ since $\iota_1' \pi_2' = 0$. If, on other hand, $x \in \operatorname{Ker} \pi_2'$, then

$$x \in \mathcal{S}_x = \mathcal{S}_x^{\pi_1' \iota_1' + \pi_2' \iota_2'} = (\mathcal{S}_x^{\pi_1'})^{\iota_1'} \subseteq F^{\bar{\iota}_1'} = K.$$

Therefore $K = \operatorname{Ker} \pi_2'$ which implies that K is a Δ-normal subgroup of H [*1*, Theorem 2.7 (2)]. Similarly $L = \operatorname{Ker} \pi_1'$ is a Δ-normal subgroup of H. For every $Y \in \Delta$ we have

$$Y = Y^{\pi_1' \iota_1'} Y^{\pi_2' \iota_2'} \in \Delta_K \Delta_L.$$

Therefore

$$\Delta = \Delta_K \Delta_L \qquad \text{and} \qquad H = KL.$$

If $Z \in \Delta_K \cap \Delta_L$, then

$$Z = Z^{\pi_1' \iota_1'} Z^{\pi_2' \iota_2'} = \mathcal{S}_1^{\iota_1'} \mathcal{T}_1^{\iota_2'} = \mathcal{D}_1 \mathcal{D}_1 = \mathcal{D}_1,$$

and hence

$$\Delta_K \cap \Delta_L = \{\mathcal{D}_1\} \qquad \text{and} \qquad K \cap L = \mathcal{D}_1.$$

Take any $k \in K$ and any $l \in L$. Then

$$\mathcal{D}_{kl}^{\pi_1' \iota_1'} \subseteq K.$$

Therefore there exists $k' \in K$ such that

$$\mathcal{D}_{kl}^{\pi_1' \iota_1'} = \mathcal{D}_{k'} = \mathcal{D}_{k'}^{\pi_1' \iota_1'}.$$

But $\operatorname{Ker} \pi_1' \iota_1' = L$ and hence [*1*, Theorem 2.7 (1)]

$$\mathcal{D}_{kl} L = \mathcal{D}_{k'} L.$$

There exists $k'' \in \mathcal{D}_{k'}$ and $l' \in L$ such that

$$kl = k''l'.$$

It follows that

$$k^{-1} k'' \in K \cap L = \mathcal{D}_1,$$

$$k \mathcal{D}_1 = k'' \mathcal{D}_1 \subseteqq \mathcal{D}_k \cap \mathcal{D}_{k'},$$

$$\mathcal{D}_k = \mathcal{D}_{k'},$$

and hence

$$\mathcal{D}_{kl}^{\pi_1' \iota_1'} = \mathcal{D}_k.$$

Similarly

$$\mathcal{D}_{kl}^{\pi_2' \iota_2'} = \mathcal{D}_l,$$

and we obtain

$$\mathcal{D}_{kl} = \mathcal{D}_{kl}^{\pi_1' \iota_1'} \mathcal{D}_{kl}^{\pi_2' \iota_2'} = \mathcal{D}_k \mathcal{D}_l.$$

Since $H = KL$, every Δ-class $\mathcal{D}$ can be written as $\mathcal{D}_{kl} = \mathcal{D}_k \mathcal{D}_l$ with $k \in K$ and $l \in L$. Therefore (2) holds.

II. Assume that (2) holds. Set

$$\Sigma = \Delta_K \qquad \text{and} \qquad \mathrm{T} = \Delta_L,$$

and let ι_1' be the injection of Σ into Δ, and ι_2' be the injection of T into Δ. Then $\iota_j' \neq 0$ for $j = 1, 2$, since $K \neq \mathcal{D}_1 \neq L$.

Because of $H = KL$, every $h \in H$ can be written as $h = kl$, $k \in K$, $l \in L$. By our assumption we have

$$\mathcal{D}_h = \mathcal{D}_{kl} = \mathcal{D}_k \mathcal{D}_l.$$

If $\mathcal{D}_h = \mathcal{D}_{k'} \mathcal{D}_{l'}$, $k' \in K$, $l' \in L$, then

$$kl = k''l'' \quad \text{where} \quad k'' \in \mathcal{D}_{k'} \subseteqq K \quad \text{and} \quad l'' \in \mathcal{D}_{l'} \subseteqq L,$$

$$k''^{-1} k = l''l^{-1} \in K \cap L = \mathcal{D}_1,$$

$$k \mathcal{D}_1 = k''\mathcal{D}_1 \subseteqq \mathcal{D}_k \cap \mathcal{D}_{k''} \quad \text{and} \quad \mathcal{D}_1 l'' = \mathcal{D}_1 l \subseteqq \mathcal{D}_l \cap \mathcal{D}_{l''}$$

and hence

$$\mathcal{D}_k = \mathcal{D}_{k''} = \mathcal{D}_{k'} \quad \text{and} \quad \mathcal{D}_l = \mathcal{D}_{l''} = \mathcal{D}_{l'}.$$

Therefore

$$\mathcal{D}_{kl} \rightarrow \mathcal{D}_k$$

is a surjectice mapping of the set of all Δ-classes of H onto the set of all Δ_K-classes of K.

Next we want to show that $\mathcal{D}_k \mathcal{D}_l = \mathcal{D}_l \mathcal{D}_k$. Take any Δ-class $\mathcal{D}_x \subseteqq \mathcal{D}_l \mathcal{D}_k$. Then

$$x = l_1 k_1, \quad k_1 \in \mathcal{D}_k, \quad l_1 \in \mathcal{D}_l.$$

The Δ-normality of K and L implies

$$K \mathcal{D}_l = \mathcal{D}_l K \qquad \text{and} \qquad L \mathcal{D}_k = \mathcal{D}_k L.$$

Hence there exist $k_2 \in K$, $l_2 \in \mathcal{D}_l$, $k_3 \in \mathcal{D}_k$, $l_3 \in L$ such that

$$x = k_2 l_2 = k_3 l_3.$$

Therefore

$$k_3^{-1} k_2 = l_3 l_2^{-1} \in K \cap L = \mathcal{D}_1,$$

$$k_2 \in k_3 \mathcal{D}_1 \cap \mathcal{D}_{k_2} \subseteqq \mathcal{D}_k \cap \mathcal{D}_{k_2},$$

$$l_3 \in \mathcal{D}_1 l_2 \cap \mathcal{D}_{l_3} \subseteqq \mathcal{D}_l \cap \mathcal{D}_{l_3}.$$

It follows that

$$\mathcal{D}_{k_2} = \mathcal{D}_k \qquad \text{and} \qquad \mathcal{D}_{l_3} = \mathcal{D}_l,$$

$$\mathcal{D}_x = \mathcal{D}_{k_2} \mathcal{D}_{l_2} = \mathcal{D}_k \mathcal{D}_l = \mathcal{D}_{kl},$$

and we have proved that

$$\mathcal{D}_k \mathcal{D}_l = \mathcal{D}_l \mathcal{D}_k \quad \text{for all} \quad k \in K \quad \text{and all} \quad l \in L.$$

Every element $X \in \Delta$ is a finite product

$$X = \mathcal{D}_{k_1 l_1} \dots \mathcal{D}_{k_n l_n} = \mathcal{D}_{k_1} \mathcal{D}_{l_1} \dots \mathcal{D}_{k_n} \mathcal{D}_{l_n} = (\mathcal{D}_{k_1} \dots \mathcal{D}_{k_n})(\mathcal{D}_{l_1} \dots \mathcal{D}_{l_n})$$

$$= \Big(\bigcup_{\mathcal{D}_k \subseteq \mathcal{D}_{k_1} \dots \mathcal{D}_{k_n}} \mathcal{D}_k \Big) \Big(\bigcup_{\mathcal{D}_l \subseteq \mathcal{D}_{l_1} \dots \mathcal{D}_{l_n}} \mathcal{D}_l \Big)$$

$$= \bigcup_{\substack{\mathcal{D}_k \subseteq \mathcal{D}_{k_1} \dots \mathcal{D}_{k_n} \\ \mathcal{D}_l \subseteq \mathcal{D}_{l_1} \dots \mathcal{D}_{l_n}}} \mathcal{D}_k \mathcal{D}_l$$

of Δ-classes of H, and the mapping

$$\pi_1' : X \longrightarrow \mathcal{D}_{k_1} \dots \mathcal{D}_{k_n} = \bigcup_{\mathcal{D}_k \subseteq \mathcal{D}_{k_1} \dots \mathcal{D}_{k_n}} \mathcal{D}_k$$

is an epimorphism of the S-semigroup Δ on H onto the S-semigroup Δ_K on K, and

$$\pi_2' : X \longrightarrow \mathcal{D}_{l_1} \dots \mathcal{D}_{l_n} = \bigcup_{\mathcal{D}_l \subseteq \mathcal{D}_{l_1} \dots \mathcal{D}_{l_n}} \mathcal{D}_l$$

is an epimorphism of the S-semigroup Δ on H onto the S semigroup Δ_L on L. It is easy to check that the required equations for ι_j', π_j' are satisfied, and so Theorem 3 3 is proved.

REMARK 3.4. For every homomorphism $\alpha : H \longrightarrow F$ of a group H into a group F, and every homomorphism $\beta : H \longrightarrow G$ of H into a group G, the mapping

$$\gamma : h \longrightarrow (h^\alpha, h^\beta)$$

is a homomorphism of the group H into the external direct product $F \times G$. We warn the reader that the analogue for S-semigroups does not hold.

Let Σ, T, Δ be S-semigroups on the groups F, G, H respectively. Let

$$\varphi : \Delta \longrightarrow \Sigma \qquad \text{and} \qquad \psi : \Delta \longrightarrow \mathrm{T}$$

be homomorphisms of the relevant S-semigroups. Then the mapping

$$\chi : X \longrightarrow X^{\varphi} \times X^{\psi} = (X^{\varphi}, X^{\psi}) \qquad (= \text{by identification})$$

is a homomorphism of the semigroup of Δ into the semigroup of $\Sigma \times \mathrm{T}$ which also maps each Δ-class $\mathcal{D}$ of H onto a $\Sigma \times \mathrm{T}$-class of $F \times G$, namely

$$\mathcal{D}^{\chi} = \mathcal{D}^{\varphi} \times \mathcal{D}^{\psi} = (\mathcal{D}^{\varphi}, \mathcal{D}^{\psi}),$$

and therefore satisfies condition (2) of the definition of homomorphism of S-semigroups. But in general χ will fail to satisfy condition (3) of that definition, since there is no apparent reason why

$$X^{\chi} = X^{\varphi} \times X^{\psi} = \bigcup_{\substack{\mathcal{S} \subseteq X^{\varphi} \\ \mathcal{T} \subseteq X^{\psi}}} \mathcal{S} \times \mathcal{T}$$

should be equal to

$$\bigcup_{\mathcal{D} \subseteq X} \mathcal{D}^{\chi} = \bigcup_{\mathcal{D} \subseteq X} \mathcal{D}^{\varphi} \times \mathcal{D}^{\psi}.$$

This fact shows a similarity of the direct product of S-semigroups with the tensor product of algebras. If A, B, C are algebras over the same field, and if

$$\varphi : A \longrightarrow B \quad \text{and} \quad \psi : A \longrightarrow C$$

are homomorphisms, then

$$\chi : a \longrightarrow a^{\varphi} \otimes a^{\psi}$$

is a homomorphism of the multiplicative semigroup of A into the multiplicative semigroup of $B \otimes C$, but it will not be a homomorphism of the additive group of A into the additive group of $B \otimes C$.

THEOREM 3.5. *Let T_i be a CS-semigroup on the group G_i for $i = 1, \ldots, n$. Then the external direct product $\mathrm{T}_1 \times \ldots \times \mathrm{T}_n$ is a CS-semigroup on the external direct product $G_1 \times \ldots \times G_n$.*

PROOF. $\mathfrak{C}(\mathsf{T}_1 \times \dots \times \mathsf{T}_n) = \{\mathcal{C}^{(1)} \times \dots \times \mathcal{C}^{(n)} \mid \mathcal{C}^{(i)} \in \mathfrak{C}(\mathsf{T}_i)$ for $i=1,\dots,n\}$ satisfies Definition 1.1.

COROLLARY 3.6. *Let K_i be a prenormal subgroup of the group G_i for $i = 1, \dots, n$. Then the external direct product $K_1 \times \dots \times K_n$ is a prenormal subgroup of the external direct product $G_1 \times \dots \times G_n$.*

PROOF. The external direct product $(G_1/K_1) \times \dots \times (G_n/K_n)$ of the double coset semigroups G_i/K_i $(i = 1, \dots, n)$ is identical with the double coset semigroup $(G_1 \times \dots \times G_n)/(K_1 \times \dots \times K_n)$, and the Corollary follows from Theorem 3.5.

4. An Example of Non-prenormal Subgroups.

4.1. *There exists a group G of order $54 = 3^3 \cdot 2$ whose 2-Sylow-groups are non-prenormal in G.*

PROOF. Let K be the group of all unitriangular matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix}$$

with coefficients in the Galois-field $GF(3)$ of 3 elements. K has the order 27 and is generated by the elements

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

of order 3, whose commutator

$$z = [x, y] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

is in the centre

$$Z = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{pmatrix} \middle| \delta \in GF(3) \right\}$$

of K. The matrix

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

normalizes K because of

$$x^a = x^{-1}, \qquad y^a = y^{-1},$$

and centralizes Z. Let G be the group generated by K and by a, and set

$$H = \langle a \rangle, \qquad X = \langle x \rangle.$$

Z is in the centre of G. Therefore

$$HZ \in \mathsf{Z}(G/H).$$

XZ is a normal subgroup of G, and hence

$$N = HXZ \in \mathsf{Z}(G/H).$$

Assume that H is prenormal in G. Then, by 1.1 (4),

$$HZ = \bigcup_{\substack{\mathcal{C} \in \mathfrak{C}(G/H) \\ \mathcal{C} \cap HZ \neq \emptyset}} \mathcal{C} \quad \text{and} \quad N = \bigcup_{\substack{\mathcal{D} \in \mathfrak{C}(G/H) \\ \mathcal{D} \cap N \neq \emptyset}} \mathcal{D}.$$

Therefore

$$A = N \setminus HZ = (H \cup HxH)Z \setminus HZ = HxHZ =$$

$$= HxH \cup HxzH \cup Hxz^{-1}H \in \mathsf{Z}(G/H).$$

The conjugacy class

$$\mathcal{K}_a = \{a, ax, ax^{-1}, ay, ay^{-1}, axyz, axy^{-1}z^{-1}, ax^{-1}yz^{-1}, ax^{-1}y^{-1}z\}$$

of a in G yields the element

$$H\mathcal{K}_a H = H \cup HxH \cup HyH \cup HxyzH \cup Hx^{-1}yz^{-1}H \in Z(G/H).$$

By the assumption that H is prenormal in G the intersection of two elements of $Z(G/H)$ is again in $Z(G/H)$. Therefore

$$HxH = A \cap H\mathcal{K}_a H \in Z(G/H).$$

(Note that every element $g \in G$ has a unique representation

$$g = a^\alpha x^\xi y^\eta z^\zeta; \qquad \alpha \in \{0, 1\}; \qquad \xi, \eta, \zeta \in \{-1, 0, 1\}.)$$

But

$$(HxH)(HyH) = HxyH \cup Hx^{-1}yH \neq HyxH \cup Hyx^{-1}H = (HyH)(HxH)$$

is a contradiction to $HxH \in Z(G/H)$. Therefore H is non-prenormal in G.

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