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SALVATORE CIAMPA

On compactness in uniform spaces

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ON COMPACTNESS IN UNIFORM SPACES

SALVATORE CIAMPA *)

1. Introductory remarks.

1.1. This paper [1]) stems from the following two considerations.

(*a*) in *SC* [2]) (of which this work may be considered a continuation) we proved that in uniform spaces completeness may be deduced from the convergence of all Cauchy nets defined on the directed set consisting of a base for the uniformity of the space under consideration, with no need, therefore, to refer to the convergence of all Cauchy nets in the space. A noteworthy consequence of this fact is the possibility of constructing a completion of the given space by means of a determined set of Cauchy nets (as in the case of metric spaces).

(*b*) net convergence in uniform spaces may be seen as constituted by two elements, the first depending more on the net (i. e. the Cauchy property) and the second linked more to the space structure than to net itself (i. e. the existence of a limit point).

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Indirizzo dell'A.: Scuola Normale Superiore, Pisa.

1) The results of this paper will be communicated at the « Internationale Spezialtagung über Erweiterungstheorie topologischer Strukturen und deren Anwendungen », Berlin, August 1967.

2) By *SC* we refer to our previous paper « *Successioni di Cauchy e completamento degli spazi uniformi* », Rend. Sem. Mat. Univ. Padova, vol. XXXIV, 1964, pp. 427-433.

1.2. One may wonder then whether similar considerations may be made with respect to compactness in uniform spaces. Since compactess is equivalent to completeness together with total boundedness (also called precompactness, which means having compact completion) [3]) and since completeness has already been examined, we may restrict ourselves to consideration of total boundedness.

It is easily seen that, as far as compactness is concerned, a solution similar to the one obtained for completeness cannot be hoped for. In other words, bearing in mind that a uniform space is totally bounded if and only if all universal nets enjoy the Cauchy property [4]), we could try to obtain a more useful characterization of total boundedness by showing that this property is equivalent to the Cauchy property of those universal nets which are defined on the directed set consisting of a base for the space uniformity. Such an endeavour cannot be successful: in a metric space (where a base for the uniformity can be ordered as the set of positive integers) every such universal net (that is, every universal sequence) is eventually constant, hence it is a Cauchy net whatever the space is.

Attempting to find a solution in this direction, it seems necessary to generalize the definition of universality for nets, hoping that among the new nets so obtained those which are defined on a suitably fixed directed set will suffice for a characterization of total boundedness.

Going on to the second consideration of no. 1.1., we will attempt to distinguish two elements in the Cauchy property, the first depending more than the second on the net while the second is linked more to the space.

1.3. We shall carry out the above outlined program by defining maximality for nets in uniform spaces in two ways, say (3) and (4)

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[3]) As it will be said in 2.1., we adhere to the terminology used in the Kelley's book on Topology. Moreover, we shall use without mention propositions which may be found in the same book.

[4]) This proposition does not appear in the Kelley's treatise. The author feels it is among the facts everybody knows, but he did not come across its proof in the literature he has knowledge of. However, a proof may be drawn directly from the proof of 4.2.2., bearing in mind that every net has a universal subnet.

since they are defined respectively in paragraphs 3. and 4., and by showing that in such spaces total boundedness is equivalent to the Cauchy property of all (4)-maximal nets and compactness is equivalent to convergence of all (3)-maximal nets. Moreover, we will show that in both cases it is sufficient to consider only nets defined on suitably chosen directed sets.

2. Terminology.

2.1. The terminology we use is the one adopted in the wellknown Kelley's book on general topology, however we shall consider all orderings to be reflexive.

If J is a directed set and X is any set, we say *J-net* to mean a mapping from J into X. If J is the set of positive integers with the usual ordering (set which we shall denote by N), we call *sequence* any N-net. The elements of the directed set J will be called *indices* and often instead of $\sigma(i)$, if $i \in J$, we shall write σ_i.

We recall that a J-net σ is, by definition, a *subnet* of the H-net μ if there exists a mapping $\pi : J \to H$ such that $\sigma = \mu \circ \pi$ and for every $h \in H$ there exists $i \in J$ in such a way that $p \in J$ and $i < p$ imply $h < \pi(p)$. We recall also that a net σ in a set X is defined to be *universal* whenever it is eventually either in Y or in $X - Y$, for every $Y \subset X$.

If $<$ is a reflexive and transitive relation among the objects of a class X and $Y \subset X$, we say that an element x is *order-maximal in Y* (with respect to $<$, if there is danger of misunderstanding) to mean that, for every $y \in Y, x < y$ implies $y < x$.

Finally, a uniform space will be denoted by a couple (T, K) where T is a set and K is a base for a uniformity in T. Unless otherwise stated, we suppose that the base K consists of symmetric elements. If $E \in K, x \in T, Y \subset T$ we shall write E_x to mean the set of all those $y \in T$ such that $(x, y) \in E$ and $E(Y)$ to mean the union of all the sets E_t for $t \in Y$. When no confusion is likely to arise, we shall write simply T instead of (T, K).

3. Maximality and compactness.

3.1. Given two nets σ, μ in the uniform space (T, K), we define σ *finer than* μ (and we shall write this $\mu < \sigma$) to mean that, for every $x \in T$, if σ is frequently in E_x for every $E \in K$, so does μ. We define next σ to be *equivalent to* μ whenever $\sigma < \mu$ and $\mu < \sigma$.

3.1.1. *Among all nets in T, $<$ is reflexive and transitive; moreover it depends on the uniformity of T and not on the actual base chosen for the uniformity itself.*

Proof: reflexivity and transitivity of $<$ are obvious. Let K, M be two bases for the space uniformity and let the net σ be finer than the net μ with respect to the base K; we want to prove that $\mu < \sigma$ also with respect to the base M. If σ is frequently in G_x for every $G \in M$ and some $x \in T$, given $E \in K$ choose $G \in M$ in such a way that $G \subset E$, then σ is frequently in $G_x \subset E_x$ and, by hypothesis, μ is frequently in E_x. Now, given any $H \in M$, choose $E \in K$ so that $E \subset H$, then μ is frequently in H_x being frequently in E_x, that is what we had to prove.

A net in the uniform space T is defined to be *maximal* if it is order-maximal in the class of all its subnets. From 3.1.1. it follows that also maximality depends on the space uniformity and not on the chosen base.

The following proposition lists some properties of the defined objects.

3.1.2. *In the uniform space (T, K) we have:*

(a) *every subnet of a net σ is finer than σ itself;*

(b) *the net σ is maximal if and only if it is equivalent to each of its own subnets;*

(c) *for the net σ to be maximal it is necessary and sufficient that if σ is frequently in E_x for every $E \in K$ and some $x \in T$, then σ converges to x (that is, σ converges to each of its adherence points);*

(d) *every universal net in T is maximal; hence, every net has a maximal subnet;*

(*e*) *every subnet of a maximal net is itself maximal*;

(*f*) *if for every $x \in T$ there exists $E \in K$ in such a way that the net σ is eventually outside of E_x, then the net σ is maximal*;

(*g*) *if the net σ is maximal and it is equivalent to a convergent net (in T), σ itself is convergent; hence, a maximal net converges if it has a convergent subnet.*

Proofs:

(*a*) obvious, since a net falls frequently in any set in which one of its subnet falls frequently;

(*b*) easy consequence of (*a*);

(*c*) let σ be a maximal H-net and let it be frequently in E_x for every $E \in K$ and some $x \in T$. If σ does not converge to x, there exists $F \in K$ such that σ is not eventually in F_x; this means that the set J of all those indices $i \in H$ for which $\sigma_i \notin F_x$ is cofinal in H, that is, the restriction of the mapping σ to J is a subnet of σ and it cannot be equivalent to σ itself since it does not fall frequently in F_x. Conversely, if μ is a J-subnet of σ and σ is frequently in E_x for every $E \in K$ and some $x \in T$, μ also falls frequently in the same E_x since μ converges to x being a subnet of a net which does converge to x;

(*d*) easy consequence of (*c*) since for any universal net to be frequently in a set implies to be eventually there; the second part is obvious since every net has a universal subnet;

(*e*) let σ be a subnet of the maximal net μ and let ϱ be a subnet of σ, then $\varrho < \mu < \sigma < \varrho$ since ϱ is also a subnet of μ; this shows the equivalence between ϱ and σ, hence the maximality of σ;

(*f*) easy consequence of (*c*), since for no $x \in T$ the net σ is frequently in E_x for every $E \in K$;

(*g*) let μ be a convergent net equivalent to σ; let μ converge to $x \in T$, then σ is frequently in E_x for every $E \in K$ since μ is; the conclusion follows then from (*c*).

We notice that maximality is not preserved under equivalence as the following example shows: let T be the space of reals with the usual metric topology, let σ and μ be the sequences respectively defined by $\sigma_n = 1/n$ for every $n \in N$, $\mu_n = 1/n$ for every even $n \in N$

and $\mu_n = n$ otherwise; it is easy to see then that σ and μ are equivalent, but while σ is maximal μ does not share this property.

In connection with 3.1.2. (g) we notice first that the proposition would not hold were it under the form: a maximal net equivalent to a Cauchy net is itself a Cauchy net (for example, in the open interval $(-1, 1)$ the sequences $\sigma = \{(-1)^n (1 - 1/n)\}_{n \in N}$ and $\mu = \{1 - 1/n\}_{n \in N}$ are equivalent and both maximal, but μ only enjoyes the Cauchy property), secondly we notice that the maximality hypothesis cannot be dispensed with (for example, in the space of reals the sequence σ such that $\sigma_n = n$ for every even $n \in N$ and $\sigma_n = 1$ otherwise is equivalent to the sequence μ defined in the preceding example, but it is not convergent, while μ converges to 1).

3.2. With regard to the Cauchy property, total boundedness and compactness the following propositions hold.

3.2.1. *Every Cauchy net in the uniform space* (T, K) *is maximal.*

Proof: it follows directly from 3.1.1. (c) since a Cauchy net converges to each of its adherence points.

3.2.2. *If every maximal sequence in the space* (T, K) *enjoyes the Cauchy property, the space* T *is totally bounded.*

Proof: were T not totally bounded, there would exist $E \in K$ such that for every finite set $Y \subset T$, $E(Y) \neq T$. Then, chosen any $x_1 \in T$, it would be possible to choose, for every positive integer n, a point $x_{n+1} \in T$ outside of $E(x_1, x_2, \dots, x_n)$. The sequence $\sigma = \{x_n\}_{n \in N}$ would be such that $(x_n, x_m) \in E$ implies $n = m$. This says that σ does not enjoy the Cauchy property. Next, chosen $F \in K$ in such a way that $F \circ F \subset E$, for no $y \in T$ the net σ could be frequently in F_y else there would exist inequal integers n, m for which $(x_n, x_m) \in F \circ F \subset E$. Then we could conclude from 3.1.2. (f) that σ is a maximal sequence.

The converse implication in the preceding proposition is false as it may be seen in the following example: let T be the open interval $(-1, 1)$ with the usual metric topology, let be

$\sigma = \{(-1)^n (1 - 1/n)\}_{n \in N}$; then σ is a maximal sequence which does not enjoy the Cauchy property, nevertheless the space T is totally bounded.

Before going on to the main theorem 3.2.3., we introduce a definition. Given two directed sets J, H we denote by $J \times H$ the directed set of all ordered pairs (j, h) $(j \in J, h \in H)$ with the ordering given by $(j, h) < (r, s)$ if and only if $j < r$ and $h < s$. We notice that if σ is a J-net in a set X, σ has a subnet defined on $J \times H$ (define $\pi(j, h) = j$, then the definition of subnet applies). Analogously, every H-net has a $J \times H$-subnet.

3.2.3. *In the uniform space* (T, K) *the following propositions are equivalent*:

(*a*) *the space* T *is compact*;

(*b*) *every maximal net in* T *converges*;

(*c*) *every* $N \times K$*-net in* T *converges if it is maximal.*

Proof:

(*a*) $\Longrightarrow$ (*b*) the compactness hypothesis says that every net has a convergent subnet, proposition 3.1.2. (*g*) then allows the conclusion that every maximal net converges;

(*b*) $\Longrightarrow$ (*c*) obvious;

(*c*) $\Longrightarrow$ (*a*) it will suffice to show that the space T is totally bounded and complete. To show the total boundedness (according to 3.2.2.) we only have to prove that the Cauchy property is enjoyed by every maximal sequence. Now, let μ be a maximal sequence in T, it has then a $N \times K$-subnet, maximal again by 3.1.2. (*e*); this implies that μ has a convergent subnet hence, by 3.1.2. (*g*), μ itself converges. For what is proved in *SC* (see the introductory paragraph 1.1.), the completeness of T follows from the convergence of every Cauchy K-net in T. So, let μ be such a net, it has then a $N \times K$-subnet (which still enjoyes the Cauchy property, hence is maximal according to 3.2.1.); this in turn implies that μ has a convergent subnet and we may conclude that μ itself converges, being a Cauchy net.

4. Maximality and total boundedness.

4.1. In order to characterize total boundedness in uniform spaces we change the definition of a net finer than another as follows: given two nets σ, μ in the uniform space (T, K), we define σ *$*$-finer than* μ (we shall write this $\mu * < \sigma$) to mean that there exists $F \in K$ such that for every $E \in K$ and every $x \in T$, if $E \subset F$ and μ falls eventually in E_x, then also σ is eventually in E_x. As before, we define σ *$*$-equivalent to* μ to mean $\sigma * < \mu$ and $\mu * < \sigma$.

It is obvious that $* <$ is reflexive and transitive among all nets in T. We define a net σ to be *$*$-maximal* whenever it is uniformly order-maximal (with respect to $* <$, of course) in the class of its subnets: that is, there exists $F \in K$ such that for every subnet μ of σ, if $\sigma * < \mu$ and μ is eventually in E_x with $x \in T, E \in K, E \subset F$, then σ also is eventually in E_x.

Some properties of the defined objects are listed below.

4.1.1. *If σ and μ are nets in the uniform space (T, K), then the following propositions hold:*

(*a*) *every subnet of σ is $*$-finer than σ itself;*

(*b*) *the net σ is $*$-maximal if and only if it is uniformly $*$-equivalent to each of its own subnets; that is, there exists $F \in K$ (which may be taken as the set F in the definition of $*$-maximal net) such that if μ is a subnet of σ and $x \in T$, $E \in K$, $E \subset F$ then: μ is eventually in $E_x \Longleftrightarrow \sigma$ is eventually in E_x.*

(*c*) *$\sigma * < \mu \Longrightarrow$ (σ converges to $x \Longrightarrow \mu$ converges to x); hence a $*$-maximal net converges if it has a converging subnet;*

(*d*) *for the net σ to be $*$-maximal it is necessary and sufficient that there exists $F \in K$ such that for every $x \in T$ and every $E \in K$ with $E \subset F$, σ falls eventually in E_x if it falls there frequently;*

(*e*) *every universal net in T is $*$-maximal, hence every net has a $*$-maximal subnet;*

(*f*) *every subnet of a $*$-maximal net is itself $*$-maximal;*

(*g*) *every $*$-maximal net in T is maximal (in the sense of the definition given in 3.1.).*

Proofs :

(*a*) obvious, since a subnet of σ certainly falls eventually in a set where σ itself is eventually;

(*b*) easy consequence of (*a*);

(*c*) easy consequence of the definitions, the second part follows from (*b*);

(*d*) let σ be a $*$-maximal J-net falling frequently in the set E_x (with $x \in T$, $E \in K$, $E \subset F$, F being as in the definition of $*$-maximal net), then $H = \sigma^{-1}(E_x)$ is a cofinal subset of J, hence the restriction μ of the mapping σ to H is a subnet of σ which falls eventually in E_x; it follows from the $*$-equivalence between μ and σ that σ also is eventually in E_x. Conversely, let μ be a subnet of σ and let μ be eventually in E_x (with $x \in T$ and $E \subset F$), then σ is frequently in E_x (by the definition of subnet), hence, by hypothesis, σ is eventually in E_x, which together with (*a*) shows the $*$-equivalence between σ and μ;

(*e*) same proof as for 3.1.2. (*d*);

(*f*) easy consequence of the definitions;

(*g*) let σ be a $*$-maximal net in T and let $F \in K$ be as in the statement of (*d*); if for every $E \in K$ σ is frequently in E_x, then for every $E \in K$ contained in F, σ is eventually in E_x. This shows that σ converges to x and that σ is maximal (according to 3.1.2. (*c*)).

4.2. With regard to the Cauchy property and the total boundedness we have the following propositions.

4.2.1. *If σ is a $*$-maximal net in the uniform space (T, K), the following propositions are equivalent:*

(*a*) *σ is Cauchy net;*

(*b*) *there exists a subnet μ of σ which enjoyes the Cauchy property.*

Proof: (*a*) $\Longrightarrow$ (*b*) obvious;

(*b*) $\Longrightarrow$ (*a*) let be $\sigma : J \longrightarrow T$ and $\mu : H \longrightarrow T$, let $\pi : H \longrightarrow J$ be as in the definition of subnet. Since σ is $*$-maximal, proposition 4.1.1. (*d*) holds, so let $F \in K$ be as in its statement. Given $E \in K$, choose $G \in K$ in such a way that $G \circ G \subset E \cap F$. Then there exists $h \in H$ such that $p, q \in H$ and $h < p, h < q$ imply that $(\mu_p, \mu_q) \in G$.

This means that σ is frequently in G_t where $t = \sigma_{\pi(h)}$: for a given $j \in J$ let h_j be as in the definition of subnet, then, if $u \in H$ follows both h and h_j, $\pi(u)$ follows j and we have $(\sigma_{\pi(u)}, \sigma_{\pi(h)}) \in G$. By hypothesis then σ is eventually in G_t, say from the index j_0 on; so, if r, s are in J and both follow j_0, $(\sigma_r, \sigma_{\pi(h)})$ and $(\sigma_s, \sigma_{\pi(h)})$ are both in G, hence $(\sigma_r, \sigma_s) \in E$ which shows σ to have the Cauchy property.

4.2.2. *In the uniform space* (T, K) *the following propositions are equivalent:*

(*a*) *the space* T *is totally bounded;*

(*b*) *every* $*$ *- maximal net in* T *enjoyes the Cauchy property;*

(*c*) *every* $*$ *- maximal sequence in* T *enjoyes the Cauchy property.*

Proof:

$(a) \Longrightarrow (b)$ let σ be a $*$ - maximal J-net in T and let $F \in K$ be as in the statement of proposition 4.1.1. (*d*). Given $E \in K$, choose $G \in K$ in such a way that $G \circ G \subset E \cap F$. Let Y be a finite subset of T such that $T = G(Y)$: then there exists $y \in Y$ such that the net σ falls frequently, hence eventually, in G_y. This means that if r and s are in J and both follow a suitable index $j \in J$, then $(\sigma_r, \sigma_s) \in G \circ G \subset E$, and this shows σ to be a Cauchy net;

$(b) \Longrightarrow (c)$ obvious;

$(c) \Longrightarrow (a)$ [5] if the space T is not totally bounded, there exists $E \in K$ such that no finite subset Y of T has the property $T = E(Y)$. This allows us to choose, for every positive integer n, an element $y_n \in T$ in such a way that $y_{n+1} \notin E(y_1, \dots, y_n)$, that is in such a way that for every pair of positive integers n, m, if $n \neq m$ then $(y_n, y_m) \notin E$. Obviously, the sequence $\sigma = \{y_n\}_{n \in N}$ does not enjoy the Cauchy property, nevertheless it is $*$ - maximal since, if $F \in K$ is such that $F \circ F \subset E$, there are no $G \in K$ and $x \in T$ such that $G \subset F$ and σ is frequently in G_x (were, on the contrary, σ frequently in G_x, there would exist inequal integers n, m such that $(y_n, x) \in G$ and $(y_m, x) \in G$, hence (y_n, y_m) would be in $G \circ G \subset E$).

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[5]) We remark that, in view of 4.1.1. (*g*), this proof gives an immediate proof of 3.2.2..

5. Concluding remarks.

5.1. Some differences between $*$ - maximality and maximality.

(*a*) A consequence of proposition 3.1.1. is that maximality does not depend on the actual base K of the space uniformity but on the uniformity itself. This is not the case for $*$ - maximality as it may be seen by considering in the space of the reals the sequence σ such that $\sigma_n = 1/n$ for every even $n \in N$ and $\sigma_n = 0$ otherwise; then, if we take as K the family of symmetric open strips about the diagonal, σ is not $*$ - maximal, whereas taking the strips closed (of course with non-zero width) σ becomes $*$ - maximal.

(*b*) Proposition 4.1.1. (*g*) says that every $*$ - maximal net is maximal: the converse implication does not hold, since $\sigma = \{(-1)^n/n\}_{n \in N}$ is a maximal sequence in the space of reals but it is not a $*$ - maximal one.

(*c*) Proposition 3.2.1. says that every Cauchy net is maximal: this property does not hold for $*$ - maximality. The sequence σ defined in (*a*) is a Cauchy sequence which is not $*$ - maximal in the uniformity defined by the open strips about the diagonal.

(*d*) The example given after the proof of 3.2.2. shows that the proposition 4.2.1. fails when maximality is substituted to $*$ - maximality.

5.2. What has been said in no. 5.1. (*a*) and the following proposition show in a sense both the weakness and the usefulness of $*$ - maximality. It is rather disturbing that $*$ - maximality does depend on the actual base chosen for the space uniformity, but this allows us to choose a base most suitable for a neat characterization of total boundedness (and herein lies its strength).

Precisely we prove that:

The uniform space T is totally bounded if and only if, with respect to its whole uniformity K [6]*, every $*$ - maximal sequence is*

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[6]) Notice that in this case K does not consist of symmetric subsets of $T \times T$.

eventually fixed in the meaning that the sequence converges and falls eventually in the set of its limit points [7]).

Proof: If in the space (T, K) every $*$ - maximal sequence is eventually fixed (in the meaning made precise above), proposition 4.2.2. insures total boundedness.

Conversely, let σ be a $*$ - maximal sequence in the totally bounded space (T, K). Let σ be not eventually fixed (in the above meaning) and let K be the whole uniformity of T. Then (by 4.2.2.) σ is a Cauchy sequence and, as we are going to prove, it has an injective subsequence μ. To this end, call two positive integers n, m equivalent if and only if $\sigma_n = \sigma_m$; this gives a partition of N in equivalence classes $\{N_i\}_{i \in J}$ and the set J cannot be finite (so that we may take $J = N$) otherwise the sequence σ would fall eventually in the set of its limit points (for, whenever N_i is not finite, which certainly happens if J is finite, $t = \sigma(N_i)$ is an adherence (hence also a limit) point of σ, then, no neighborhood of t may exclude $\sigma(N_j)$ if N_j is not finite and this says that σ is eventually in the set $\bar{t}$ of its limit points). Then, chosing a number n_i in every N_i, we may define $\mu_i = \sigma(n_i)$: the sequence μ so defined [8]) is an injective subsequence of σ, still enjoying the Cauchy property.

Now, given any set $F \in K$, choose $p \in N$ such that μ falls eventually in $F_{\mu(p)}$ and $\mu(p)$ is not a limit point of μ. Then there exist $G \in K$ and an infinite set $B \subset N$ such that $G \subset F$ and $n \in B$ implies $\mu_n \notin G_{\mu(p)}$. Take $L \subset B$ such that neither L nor $B - L$ are finite and no integer less than p belongs to L Consider next the set $H = G \cup \{(\mu_p, \mu_n)\}_{n \in L}$: certainly $H \in K$ and $H \subset F$; moreover the sequence μ is frequently but not eventually in $H_{\mu(p)}$ since no μ_n with index $n \in B - L$ belongs to $H_{\mu(p)}$. This shows that the sequence μ is not $*$ - maximal and this is impossible since μ is a subsequence of a $*$ - maximal sequence (see no. 4.1.1. (f)). This contradiction completes the proof.

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[7]) In Hausdorff spaces our meaning of « eventually fixed net » coincides with the usual one of « eventually constant net ».

[8]) With a suitable renumbering of the i's, of course.

5.3. Let us consider the following propositions:

C *The space (T, K) is compact;*
TB *The space (T, K) is totally bounded;*
I. *Every maximal net in (T, K) is a Cauchy net;*
II. *Every maximal net in (T, K) converges;*
III. *Every maximal sequence in (T, K) is a Cauchy sequence;*
IV. *Every $N \times K$-net in (T, K) converges if it is maximal.*

Let us denote by $I^*., II^*., III^*., IV^*.$ the propositions corresponding respectively to I., II., III., IV. when - maximality is substituted for maximality.

Then, we may sum up what has been said in the preceding paragraphs as follows:

$$\begin{array}{ccccccccc} C & \Longleftrightarrow & II. & \Longleftrightarrow & IV. & \Longrightarrow & I. & & \\ & & \Downarrow & & & & \Downarrow & & \\ & & II^*. & & & & III. & & \\ & & \Downarrow & & & & \Downarrow & & \\ & & IV^*. & \Longrightarrow & I^*. & \Longleftrightarrow & III^*. & \Longleftrightarrow & TB. \end{array}$$

We notice next that the following implications do *not* hold:

(*a*) TB $\Longrightarrow$ III. (counterexample shown after the proof of proposition 3.2.2.);

(*b*) I. $\Longrightarrow$ IV*. (counterexample: take as space T the half open interval $(0, 1]$ with K consisting of all the open strips of width $1/n$ about the diagonal; in this situation I. holds since a maximal net in T is maximal also in the closed interval $[0, 1]$ therefore it converges and it is a Cauchy net, nevertheless IV*. does not hold since the net $\mu : N \times N \longrightarrow T$ defined by $\mu(n, m) = 1/n$ is $*$-maximal but does not converge);

(*c*) IV*. $\Longrightarrow$ I. (counterexample: take as space T the open interval $(-1, 1)$ and let K be the whole uniformity induced by the usual metric of the reals. Now, if σ is a $*$-maximal $N \times N$-net in T, it has a subsequence μ (take, for instance, $\mu_n = \sigma(n, n)$ for every $n \in N$) which is again $*$-maximal so, by the results of no. 5.2, the sequence μ converges and so does σ (by 4.1.1. (*c*)). This means that in T proposition IV*. holds. However proposition I. fails to hold since the sequence $\sigma = \{(-1)^n (1 - 1/n)\}_{n \in N}$ is maximal in T but it is not a Cauchy sequence).

d) III. $\Longrightarrow$ I. (counterexample: let X be the ordered set of all ordinals less than the first uncountable ordinal number. Take the set Y of all couples $+\alpha$ or $-\alpha$ for every $\alpha \in X$ and introduce in Y the usual ordering, that is, for every $\alpha, \beta \in X$,

$$-\alpha < +\beta,$$

$$\alpha < \beta \Longleftrightarrow +\alpha < +\beta \Longleftrightarrow -\beta < -\alpha.$$

In the order topology the set Y becomes a sequentially compact topological space, so that every sequence in Y has adherence points, hence converges if it is maximal. Let us consider now the net $\mu : X \longrightarrow Y$ defined as follows: for every $\alpha \in X$, $\mu(\alpha) = +\alpha$ if α has no preceding element (in X); if, on the contrary, α has a preceding element β, $\mu(\alpha)$ equals $+\alpha$ or $-\alpha$ according to $\mu(\beta)$ being $-\beta$ or $+\beta$. The net μ, of course, does not enjoy the Cauchy property, nevertheless it is a maximal net having no adherence points).

5.4. $*$ - maximality may enter in a characterization of compactness in the following way:

The uniform space (T, K) *is compact if and only if all* $N \times K$*-nets in* T *converge if they are either* $*$ *- maximal or Cauchy nets.*

5.5. Finally we remark that had the definition of σ $*$ - finer than μ read: for every set $Y \subset T$ if μ is eventually in Y then so is σ, $*$ - maximality would have reduced to universality, since universal nets are those nets which are eventually wherever any of their subnets falls eventually.

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