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Regressive upper bounds

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REGRESSIVE UPPER BOUNDS

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1. Introduction.

The results presented in this paper were obtained when the following problem was considered: When does a denumerable collection of isols have a regressive upper bound? It was proven by J. C. E. Dekker and J. Myhill [6, Theorem 45 (b)] that a denumerable collection of isols always has an upper bound in the isols. However, it is easy to show that even finite collections of isols may have no regressive upper bound. There are two main aims of this paper. The first is to give a necessary and sufficient condition that a finite collection of isols have a regressive upper bound and the second is to discuss the existence of regressive upper bounds for three particular types of infinite collections of isols [1]).

2. Preliminaries.

We shall assume that the reader is familiar with the terminology and main results of the papers [1] through [6] and [9]; in particular with the concepts of regressive function, regressive set, regressive isol, infinite series of isols and the ordering relations

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$\leq^*$ (between functions and isols) and $\leq$ (between isols). When refering to an *upper bound* we shall always mean an upper bound which is an isol. The expression « regressive upper bound » will sometimes be abbreviated by writing r.u.b. *Denumerable* will mean either finite or denumerably infinite, and c will denote the cardinality of the continuum. We let

$E =$ the collection of all non-negative integers (*numbers*),

$\Lambda =$ the collection of all isols,

$\Lambda_R =$ the collection of all regressive isols.

It is known that $E \subset \Lambda_R \subset \Lambda$, where both $\Lambda_R - E$ and $\Lambda - \Lambda_R$ have cardinality c. Since every isol has only denumerably many predecessors it follows that if a collection of isols has an upper bound, then the collection is denumerable. It is for this reason that the problem concerning the existence of an upper bound is restricted to collections of isols which are denumerable.

3. Fundamental properties.

Throughout this section let Δ denote a denumerable collection of isols. We recall that an isol U will be an *upper bound* of Δ if $A \leq U$, for each $A \in \Delta$, i.e., if Δ is a subset of the collection of predecessors of the isol $U + 1$. The following two properties of isols and the relation $\leq$ were proven by Dekker in [4]. Let A, B and U be isols, then

$$(1) \qquad A \leq U,\ U \in \Lambda_R \Longrightarrow A \in \Lambda_R,$$

$$(2) \qquad A, B \leq U,\ U \in \Lambda_R \Longrightarrow A + B \in \Lambda_R.$$

It is readily seen that combining (1) and (2), yields

PROPOSITION 1. If Δ has a r.u.b., then

$$(\square) \qquad A, B \in \Lambda_R \Longrightarrow A, B,\ A + B \in \Lambda_R.$$

It was also proven in [4] that Λ_R is not closed under addition. If we let $A, B \in \Lambda_R$, such that $A + B \notin \Lambda_R$ then, in view of Proposition 1, we see that the collection (A, B) as well as any collection Δ with $A, B \in \Delta$ will have no regressive upper bound.

We have earlier noted that every isol has denumerably many predecessors. Further, it is well known that every isol has c successors in the isols. If an isol has a regressive successor then, by (1), the isol itself will also be regressive. Every finite isol will have c regressive successors, since there are c infinite regressive isols. On the other hand,

(3) every infinite regressive isol has exactly $\aleph_0$ regressive successors.

To verify (3), first note that if A is a regressive isol then each of the isols $A + 1, A + 2, \ldots$ is a regressive successor of A; hence every regressive isol has at least $\aleph_0$ regressive successors. Also, it was proven by T. G. McLaughlin [7, Lemma 1] that every infinite set has at most $\aleph_0$ regressive supersets. Since every non-zero isol contains $\aleph_0$ sets, it follows from this result that every infinite isol has at most $\aleph_0$ regressive successors. This proves (3). With respect to regressive upper bounds, it follows from (3), that if Δ has a r.u.b. and also contains at least one infinite isol then Δ will have exactly $\aleph_0$ regressive upper bounds.

We will now state another result that is related to regressive upper bounds, the proof of which may be obtained by a construction similar to that in the proof of [2, Theorem 1.3].

PROPOSITION 2. There exist infinite regressive isols A, U and V such that

$$A < U, A < V, U + V \notin \Lambda_R \text{ and } U \text{ and } V \text{ are not } \overset{*}{\vee} \text{ related.}$$

The $\overset{*}{\vee}$ relation was introduced in [2]; it is a binary relation defined between pairs of infinite regressive isols and it is weaker then each of the relations $\leq^*$ and $\leq$, in the sense that elements satisfying either of the latter already satisfy the former relation.

Assume that Δ has a r.u.b., and let Γ be the collection of all regressive upper bounds of Δ. Consider the two statements·

$$\text{(4)} \qquad \Gamma \text{ is closed under addition.}$$

$$\text{(5)} \qquad \Gamma \text{ is totally ordered by the relation } \leq^* \text{ or } \leq.$$

If $\Delta \subset E$ then both (4) and (5) are false, for in this case $\Lambda_R - E \subset \Gamma$. In addition, if we let $\Delta = (A)$, where A satisfies the hypothesis of Proposition 2, then it also follows that both (4) and (5) are false, even though in this event Γ is denumerable. We do not know if it is always the case that both statements (4) and (5) are false.

4. **Finite collections.**

Assume also in this section that Δ is a denumerable collection of isols. We know that if Δ has a r.u.b. then

$$(\square) \qquad A, B \in \Delta \Longrightarrow A, B\ A + B \in \Lambda_R\,.$$

In § 6 it will be shown that every denumerable collection Δ of isols satisfying ($\square$) either will have no r.u.b. or else can be extended to a denumerable collection of isols which satisfies ($\square$) and having no r.u.b. . However, in the special case that Δ is a finite collection of isols, then ($\square$) is a sufficient as well as a necessary condition for Δ to have a r.u.b. ; and this is the main result of this section. We begin with the following proposition.

PROPOSITION 3. Let A, B and C be regressive isols such that

$$A + B,\ A + C,\ B + C \in \Lambda_R\,.$$

Then

$$A + B + C \in \Lambda_R\,.$$

PROOF. The result is immediate if either A, B or C is finite.

Assume now that each of the isols A, B, and C is infinite. Then there will be mutually separated (immune and regressive) sets α, β and δ belonging to A, B, and C respectively. In view of the hypothesis, each of the three sets

$$\alpha + \beta \in A + B, \quad \alpha + \delta \in A + C, \quad \beta + \delta \in B + C,$$

will also be regressive. Let

$$\pi = \alpha + \beta + \delta.$$

Then π is an immune set, because each of the sets $\alpha, \beta,$ and δ is immune. Also $\pi \in A + B + C$, and therefore to complete the proof it suffices to show that π is a regressive set. This will be our goal as we divide the remainder of the proof into three parts.

Part 1. Let a_n, b_n, c_n, u_n, v_n and w_n be (everywhere defined and one-to-one) regressive functions ranging over the sets $\alpha, \beta, \delta, \alpha + \beta, \alpha + \delta$ and $\beta + \delta$ respectively, and let Ξ denote the family consisting of these six functions. For each of the numbers $a^* \in \alpha$, $b^* \in \beta$ and $c^* \in \delta$, let

$$\Phi_{a^*} = (a_0, \dots, a_i, u_0, \dots, u_j, v_0, \dots, v_k), \quad \text{if} \quad a^* = a_i = u_j = v_k;$$

$$\Phi_{b^*} = (b_0, \dots, b_i, u_0, \dots, u_j, w_0, \dots, w_k), \quad \text{if} \quad b^* = b_i = u_j = w_k;$$

$$\Phi_{c^*} = (c_0, \dots, c_i, v_0, \dots, v_j, w_0, \dots, w_k), \quad \text{if} \quad c^* = c_i = v_j = w_k.$$

Suppose that a number $y \in \pi$ is given. Then y belongs to exactly one of the sets α, β, ond δ. Because the sets α, β and δ are mutually separated, we can determine the particular set to which y belongs. We can then also determine to which two of the three sets $\alpha + \beta, \alpha + \delta$ and $\beta + \delta$ that y also belongs. Taking into account that each of the functions in Ξ is regressive, it follows therefore that we can effectively find all of the numbers of the set Φ_y.

For each number $y \in \pi$, let

$$\omega_y^0 = \sum_{r \in \Phi_y} \Phi_r,$$

$$\omega_y^{k+1} = \sum_{r \in \omega_y} {}_k \, \Phi_r,$$

$$\omega_y = \sum_0^\infty \omega_y^k.$$

We note that for each function $t \in \Xi$ and number $k \in E$,

$$t_k \in \omega_y \Longrightarrow t_0, \ldots, t_k \in \omega_y .$$

Also by combining our earlier remarks with the definition of the set ω_y, we can conclude that from a given number $y \in \pi$, we can effectively generate all of the members of the set ω_y. Hence ω_y will be an r.e. subset of π. Since π is an immune set, it follows that ω_y will be a finite set for each $y \in \pi$.

Part 2. It turns out to be convenient for the discussion which follows to assume that for each $y \in \pi$, ω_y has a non-empty intersection with each of the sets α, β and δ. For this reason, we shall assume that the values of a_0, b_0 and c_0 are known to us and that these numbers are placed in each ω_y. It will be readily seen that this modification will cause no difficulty.

Some terminology. Let $y \in \pi$. Set

$$k, l, m = \begin{cases} \text{maximum numbers } k^*, l^* \text{ and } m^* \text{ respectively,} \\ \text{such that } a_{k^*}, b_{l^*}, c_{m^*} \in \omega_y . \end{cases}$$

We call the ordered triple of numbers (k, l, m) the *torre-number* of y, and denote it by $\widetilde{y}$. An ordered triple of numbers is a *torre-number* if it is the torre-number of some $y \in \pi$. If $t = (k, l, m)$ is a torre-number, the number $L(t) = k + l + m$ is called the *length* of t. It readily follows from the remarks in Part 1, that

given $y \in \pi$, we can effectively find $\widetilde{y}$,

$L(\widetilde{y})$ as well as $\widetilde{z}$ and $L(\widetilde{z})$ for any $z \in \omega_y$.

In addition, we have the following property,

LEMMA 1. If (k_0, l_0, m_0) and (k_1, l_1, m_1) are each torre-numbers, then

either, $\quad k_0 \leq k_1, l_0 \leq l_1$ and $m_0 \leq m_1$,

or, $\quad k_1 \leq k_0, l_1 \leq l_0$ and $m_1 \leq m_0$.

PROOF. Assume otherwise and let (k_0, l_0, m_0) and (k_1, l_1, m_1) be the torre-numbers of y_0 and y_1 respectively. Without loss of generality, we may suppose that

$$(*) \qquad k_0 < k_1 \text{ and } l_1 < l_0 .$$

Consider the two numbers

$$a_{k_0+1} \text{ and } b_{l_1+1} .$$

Since $\alpha \cap \beta = \emptyset$, these are distinct numbers. Also each belongs to $\alpha + \beta$. Let

$$u_p = a_{k_0+1} \text{ and } u_q = b_{l_1+1}, \text{ where } p \neq q.$$

We consider separately two cases: *Case* 1. $q < p$. It follows from $(*)$ that $k_0 + 1 \leq k_1$, and therefore $u_p \in \omega_{y_1}$. Then combining the two facts

$$u_p \in \omega_{y_1} \text{ and } q < p,$$

implies that

$$u_q \in \omega_{y_1} .$$

This means that

$$b_{l_1+1} \in \omega_{y_1} ,$$

which contradicts the assumption that (k_1, l_1, m_1) is the torre-number of y_1.

Case 2. $q > p$. One can proceed here as in Case 1; we shall omit the details.

The contradictions obtained here establish the desired result of the lemma. This completes the proof.

We obtain directly from Lemma 1, the following two corollaries.

COROLLARY 1. For all $x, y \in \pi$,

$$L(x) \leq L(y) \Longrightarrow \omega_x \subset \omega_y .$$

COROLLARY 2. Any number $n \in E$ is the length of at most one torre-number.

Part 3. Combining our remarks in Parts 1 and 2 with Corollary 1, we can conclude that

> given $y \in \pi$, we can effectively determine all the numbers of the set $\{x \mid x \in \pi \text{ and } L(\tilde{x}) \leq L(\tilde{y})\}$ together with their respective torre-numbers.

There will be infinitely many torre-numbers ; and by Corollary 2, different torre-numbers will have different lengths. Let $t_0, t_1, t_2, \ldots$ be an enumeration of all torre-numbers, such that $L(t_n) < L(t_{n+1})$. Define, for each number $k \in E$,

$$\Delta_k = \{y \mid y \in \pi \text{ and } \tilde{y} = t_k\}.$$

Clearly,

$$\pi = \sum_{k=0}^{\infty} \Delta_k .$$

Let f_n be the unique one-to-one function having domain E and range π such that when reading from left to right the numbers in the enumeration,

$$f_0, f_1, f_2, \cdots,$$

every number of Δ_p appears before every number of Δ_{p+1}, and the numbers of Δ_p appear in their natural order according to size. If we combine the beginning remarks of this part together with the definition of the sets Δ_k, it follows that the function f_n has been defined in such a way as to be regressive i.e., the mapping,

$$f_{n+1} \rightarrow f_n ,$$

has a partial recursive extension. Hence π is a regressive set, and this completes the proof.

Corollary. Let $A_0, \ldots, A_k$ be regressive isols such that, for $0 \leq i, j \leq k$, $A_i + A_j \in \Lambda_R$. Then

$$A_0 + \ldots + A_k \in \Lambda_R .$$

Proof. If $k = 0$ or $k = 1$ the result is clear. For $k \geq 2$ use Proposition 2, associativity of isols under addition and finite induction.

We know that $A \leq A + B$ for any isols A and B. Combining this fact with Proposition 1 and the previous corollary yields,

THEOREM 1. Let Δ be a finite collection of isols. Then Δ has a r. u. b. if and only if

$$(\square) \qquad A, B \in \Delta \Longrightarrow A, B, A + B \in \Lambda_R .$$

5. Two collections.

We want to consider two particular types of denumerable collections of isols. Each of these will be defined in terms of recursive functions.

DEFINITIONS AND NOTATIONS. We recall from [1] that an (everywhere defined) function $f(x)$ is *increasing* if

$$x < y \Longrightarrow f(x) \leq f(y), \text{ for } x, y \in E;$$

and *eventually increasing* if for some number n, the function $g(x) = f(x + n)$ is increasing. Let $f(x)$ be an eventually increasing recursive function and let $D_f(x)$ denote its extension to Λ_R. One of the main results [1] is that D_f maps Λ_R into Λ_R. We let Φ denote the family of all increasing recursive functions. Let $h(i, x)$ be any recursive function of the two variables i and x, such that for each i, $h(i, x) \in \Phi$. Henceforth we assume that the function h is fixed; also we shall sometimes write $h_i(x)$ for $h(i, x)$. For $T \in \Lambda_R - E$, we define

$$\Delta_T = \{ D_{h_i}(T) \mid i \geq 0\}$$

$$\Gamma_T = \{D_f(T) \mid f \in \Phi\}.$$

Let T be an infinite regressive isol. We note that $\Delta_T \subset \Gamma_T$. Also it can be readily proven that Γ_T is a denumerable collection of regressive isols, infinitely many of which are infinite and which is closed under addition. This means that Γ_T as well as Δ_T will satisfy the condition ($\square$). We wish to establish the following two results.

THEOREM 2. Let $T \in \Lambda_R - E$, then Δ_T has a r. u. b..

THEOREM 3. There is an infinite regressive isol T such that Γ_T has a r.u.b..

PROOF OF THEOREM 2. Define the function $f(x)$ by,

$$f(x) = \sum_{0 \leq i,\, n \leq x} h(i, n).$$

Then $f(x)$ is a recursive function since $h(i, x)$ is a recursive function. In addition, it is readily seen that

(1) f is an (eventually) increasing function and for each number i,

(2) $$h_i(x) \leq f(x), \quad \text{for } x \geq i, \text{ and}$$

(3) the function $v_i(x) = f(x) \dot{-} h_i(x)$ is eventually increasing.

It follows from (1) that $D_f(T) \in \Lambda_R$. To complete the proof we now show that $D_f(T)$ is an upper bound of Δ_T. For this purpose let $A \in \Delta_T$ and assume that $A = D_h(T)$, where $h(x) = h(i, x)$. In view of (2), we obtain

(4) $$f(x) = v(x) + h(x), \quad \text{for } x \geq i,$$

where $v(x) = v_i(x)$. Identity (4) concerns only recursive functions and therefore yields, by a well known theorem of Nerode, that

(5) $$D_f(X) = D_v(X) + D_h(X), \quad \text{for} \quad X \in \Lambda_R - E.$$

Taking into account that T is an infinite regressive isol and that each of the functions f, v and h is eventually increasing and recursive, we can conclude from (5) that

$$D_f(T) = D_v(T) + D_h(T), \qquad \text{where } D_f(T), D_v(T), D_h(T) \in \Lambda_R.$$

Hence

$$D_h(T) \leq D_f(T).$$

We have thus shown that

$$A \in \Lambda_T \Longrightarrow A \leq D_f(T),$$

and this completes the proof.

With every regressive isol T and function a_n from E into E, Dekker [3] introduced and studied an infinite series $\Sigma_T a_n$. We recall the principal definition, in the special case that T is an infinite regressive isol.

NOTATIONS. Let j denote the familiar primitive recursive function defined by $j(x, y) = x + (x + y)(x + y + 1)/2$. For any number n and set α, let $\nu(n) = \{x \mid x < n\}$ and $j(n, \alpha) = \{j(n, y) \mid y \in \alpha\}$. We recall that the function j maps E^2 onto E in a one-to-one manner.

DEFINITION. Let a_n be any function from E into E, and T any infinite regressive isol. Then

$$\Sigma_T a_n = \text{Req} \sum_0^\infty j(t_n, \nu(a_n)),$$

where t_n is any regressive function ranging over any set in T.

By [3, Theorem 1], $\Sigma_T a_n$ is an isol, and depends on the infinite regressive isol T but not on the particular regressive function whose range is in T. In the special case that a_n is a recursive function, then $\Sigma_T a_n$ is a regressive isol [1, Theorem 1]. Infinite series of isols summed with respect to a recursive function and the extension to Λ_R of increasing recursive functions are closely related. Let $T \in \Lambda_R - E$; then a useful result which can be obtained from [1] is

$$\Gamma_T = \{D_f(T) \mid f \in \Phi\} = \{\Sigma_T a_n \mid a_n \text{ a recursive function}\}.$$

We shall use this representation of Γ_T in the course of proving Theorem 3. We also need the following lemma.

LEMMA 2. Let u_n be any function from E into E. Then there exist retraceable functions t_n such that, $u_n < t_n$ for $n \in E$.

PROOF. Let

$$t_n = 2^{1+u_0} \cdot 3^{1+u_1} \cdot \ldots \cdot p_n^{1+u_n} \,,$$

where p_n denotes the $n+1^{st}$ prime number. Clearly t_n is a retraceable function and satisfies the property, $u_n < t_n$ for $n \in E$.

PROOF OF THEOREM 3. Let $a_i(n)$ be an everywhere defined function of the two variables i and n such that, every recursive function of one variable, and no other function, appears in the sequence $\{a_i\}$. Let the function u_n be defined by

$$u_x = \sum_{0 \leq i,\, n \leq x} a_i(n).$$

Apply Lemma 2 and let t_n be a retraceable function such that

(1) $$u_n < t_n \,, \qquad \text{for } n \in E.$$

Let $\tau = \rho t_n$. Then τ is retraceable set. It was proven in [5] that retraceable sets are either recursive or immune. If τ is a recursive set then t_n will be a recursive function; but this is not possible in view of (1) and the definition of u_n. Hence τ is an immune regressive set. Let $T = \text{Req}\ \tau$. Then T is an infinite regressive isol. We claim that

$$\Gamma_T \text{ has a r. u. b.}.$$

We first note that combining the remark before Lemma 2 and the definition of the function $a_i(n)$ implies,

$$A \in \Gamma_T \Longrightarrow (\exists i)\,[A = \Sigma_T\, a_i(n)].$$

Define the set

(2) $$\sigma = \sum_0^{\infty} j\,(t_n \,,\, \nu(t_n)).$$

It follows from (1) that $t_n \geq 1$ for every $n \in E$. Hence,

(*) $$j(t_0\,, 0), \ldots, k(t_0\,, t_0 - 1)\,, j(t_1\,, 0), \ldots, j(t_1\,, t_1 - 1), j(t_2\,, 0), \ldots,$$

is a one-to-one enumeration of all the elements of the set σ. An easy consequence of the fact that t_n is a regressive function is that (*) represents a regressive enumeration of the elements of σ. Hence σ is a regressive set. Regressive sets are either. r. e. or immune [3, p. 90]. If σ were an r.e. set it would then follow that t_n is a recursive function, which we know is not the case. Thus σ is an immune regressive set. Set

$$U = \text{Req}\ \sigma. \tag{3}$$

Then $U \in \Lambda_R - E$. To complete the proof we will show that U is an upper bound of Γ_T. For this purpose let $A \in \Gamma_T$, and assume that

$$A = \Sigma_T\ a_i\,(n).$$

Let $a_n = a_i\,(n)$. Then

$$\sum_0^\infty j\,(t_n\,,\, \nu\,(a_n)) \in A. \tag{4}$$

It follows from (1) and the definition of the function u_n, that

$$a_0 + \ldots + a_i \leq t_i\,, \tag{5}$$

$$a_n \leq t_n\,, \text{ for } n \geq i. \tag{6}$$

Set

$$\delta = \sum_{n=i+1}^\infty j\,(t_n\,,\, \nu\,(a_n)), \quad M = \text{Req}\ \delta \tag{7}$$

$$\eta = \sum_{n=i+1}^\infty j\,(t_n\,,\, \nu\,(t_n)), \quad N = \text{Req}\ \eta. \tag{8}$$

It is easily seen that M and N are regressive isols and

$$A = a_0 + \ldots + a_i + M\,, \tag{9}$$

$$U = t_0 + \ldots + t_i + N. \tag{10}$$

We would like to prove that $A \leq U$. In view of (5), (9) and (10), it will be sufficient to show that $M \leq N$. For this purpose, we first note that (6) implies

$$\nu\,(a_n) \subset \nu\,(t_n), \qquad \text{for } n \geq i. \tag{11}$$

Hence

(12) $$\delta \subset \eta.$$

In addition since a_n is a recursive function and t_n is a regressive function, it readily follows from (7), (8) and (11) that

(13) $$\delta \mid \eta - \delta.$$

i. e., δ and $\eta - \delta$ are separated sets. Finally, combining (7), (8), (12) and (13) implies that $M \leq N$. This means that $A \leq U$. We have therefore shown

$$A \in \Gamma_T \Longrightarrow A \leq U,$$

and hence Γ_T has a r. u. b. This completes the proof.

6. **The collection Σ_T.**

In this section we wish to introduce and study a particular collection Σ_T of regressive isols associated with any infinite regressive isol T. The principal properties of Σ_T that will be proven are,

(a) $T \in \Sigma_T$,

(b) $E \subset \Sigma_T \subset \Lambda_R$,

(c) Σ_T is a denumerable collection,

(d) $A, B \in \Sigma_T \Longrightarrow A + B \in \Sigma_T$,

(e) $A \leq T \Longrightarrow A \in \Sigma_T$,

(f) $T \leq V$ and $V \in \Lambda_R = (\exists B)[B \in \Sigma_T, T \leq B$ and $V \leq^* B]$,

(g) Σ_T has no regressive upper bound.

Remark. Before we proceed to give the formal definition of the collection Σ_T, we want to state some observations about the collection Σ_T based on the above properties.

We first note that, by (a), (b) and (e), Σ_T will be a collection of regressive isols containing T as well as every (regressive) predecessor of T. Regarding regressive successors of T, some of these will belong to Σ_T; for example, according to (a), (b) and (d), Σ_T will contain $T+n$ and $n \cdot T$ for any number $n \geq 2$, and each of these regressive isols is a successor of T. However, it need not be true that every regressive successor of T belongs to Σ_T. To see this, recall that, by Proposition 2, there are infinite regressive isols having regressive successors whose sum is not regressive. If T is an isol of this type then, in view of (b) and (d) it follows that at least one regressive successor of T will not belong to Σ_T.

We also note that combining (b) and (d) gives,

$$A, B \in \Sigma_T \Longrightarrow A, B, A+B \in \Lambda_R .$$

Hence Σ_T satisfies the condition ($\square$). It is easily seen, that if a collection of isols has a regressive upper bound then it has a regressive upper bound which is infinite. Assume that Δ is a collection of isols having T as an (infinite) regressive upper bound. Then, in view of (e), it follows that $\Delta \subset \Sigma_T$. If we combine these previous remarks with (a), (b), (c) and (g), we can draw the following conclusion,

if a collection of isols has a r. u. b., then it is the subset of a denumerable collection of isols satisfying the condition ($\square$) and having no r. u. b..

We shall now introduce the preliminaries necessary for the definition of the collection Σ_T.

PRELIMINARIES. Throughout this part let a_n, b_n and c_n denote any functions from E into E.

We write $a_n \leq^* b_n$, if there is a partial recursive function $p(x)$, such that

$$\text{(1)} \qquad \rho a_n \subset \delta p \text{ and } (\forall n)[p(a_n) = b_n].$$

We write $a_n \simeq b_n$, if there is one-to-one partial recursive function $p(x)$ such that (1) holds. It is readily verified that

$$\text{(2)} \qquad (a_n \leq^* b_n \text{ and } b_n \leq^* c_n) \Longrightarrow a_n \leq^* c_n ,$$

$$\text{(3)} \qquad (a_n \leq^* b_n \text{ and } a_n \leq^* c_n) \Longrightarrow a_n \leq^* (b_n + c_n).$$

Also, it was proven in [4] that, if a_n and b_n are each one-to-one functions, then

(4) $$(a_n \leq^* b_n \text{ and } b_n \leq^* a_n) <==> a_n \simeq b_n .$$

Let t_n be any (one to-one) regressive function. Using the definition of the relation $\leq^*$, it is readily shown that there are exactly $\aleph_0$ functions a_n such that $t_n \leq^* a_n$. Also, since t_n is a regressive function, it follows that $t_n \leq^* n$. From this fact, we can conclude that $t_n \leq^* a_n$, for every recursive function a_n.

Let T be any infinite regressive isol. We write $T \leq^* a_n$, if there is a regressive function t_n ranging over a set in T such that $t_n \leq^* a_n$. It is well know that if s_n and t_n are two regressive functions ranging over sets that belong to the same isol, then $s_n \simeq t_n$. In view of (2) and (4), this means that if $T \leq^* a_n$, then $t_n \leq^* a_n$ for every regressive function t_n ranging over a set in T. We also have the following properties,

(5) $$(T \leq^* a_n \text{ and } T \leq^* b_n) ==> T \leq^* (a_n + b_n),$$

(6) $$a_n \text{ a recursive function } ==> T \leq^* a_n ,$$

(7) $$(T + 1 \leq^* a_n) ==> T \leq^* a_n .$$

We obtain (5) from (3), and (6) follows from the fact that if t_n is any regressive function and a_n any recursive function, then $t_n \leq^* a_n$. To verify (7), let t'_n be a regressive function ranging over a set in $T + 1$. Let $t_n = t'_{n+1}$. Then t_n is a regressive function ranging over a set in T. We note that $t_n \leq^* t'_n$, since t'_n is a regressive aunction. Hence by (2),

$$t'_n \leq^* a_n ==> t_n \leq^* a_n ,$$

fnd this relation implies (7).

For each infinite regressive isol T, we let

$$F(T) = \{a_n \mid T \leq^* a_n\}.$$

It follows from our previous remarks that $F(T)$ is a denumerably infinite family of functions and every recursive function belongs to it. Also, by (5), we have

$$(8) \qquad a_n, b_n \in F(T) \Longrightarrow (a_n + b_n) \in F(T).$$

The principal definition. Let T be any infinite regressive isol. Then

$$\Sigma_T = \{\Sigma_{T+1} a_n \mid a_n \in F(T)\}.$$

REMARKS. We shall assume from now on that T is an infinite regressive isol and held fixed. We let $F = F(T)$, and let t'_n be a particular regressive function ranging over a set in $T+1$, and let $t_n = t'_{n+1}$. Then t_n is also a regressive function and it ranges over a set in T. We set $\tau' = \varrho t'$ and $\tau = \varrho t$.

In order to establish the properties (a) to (g) for the collection Σ_T we shall need several propositions and lemmas. The lemmas that follow can be verified in an easy manner and for this reason we shall omit their proofs.

NOTATION. Let S be an infinite regressive isol. Then

$$[a_0 + a_1 + a_2 + \dots]_S \text{ will denote } \Sigma_S a_n.$$

PROPOSITION 4. $T \in \Sigma_T$ and $F \subset \Sigma_T$.

PROOF. By (6), we know that $T \leq^* a_n$, for every recursive function a_n. Hence

$$\Sigma_{T+1} a_n \in \Sigma_T, \text{ for every recursive function } a_n.$$

In particular, $T \in \Sigma_T$ because

$$T = [0 + 1 + 1 + 1 + \dots]_{T+1},$$

and $k \in \Sigma_T$, for every number k, since

$$k = [k + 0 + 0 + 0 + 0 + \dots]_{T+1}.$$

Therefore $T \in \Sigma_T$ and $E \subset \Sigma_T$.

PROPOSITION 5. $\Sigma_T \subset \Lambda_R$.

Proof. Let $A \in \Sigma_T$, and assume that

$$A = \Sigma_{T+1}\, a_n \,, \text{ where } T \leq^* a_n \,. \tag{9}$$

Then

$$\sum_0^\infty j\,(t'_n\,, \nu\,(a_n)) \in A. \tag{10}$$

Let α denote the set appearing in (10). To prove the proposition, we want to show that $A \in \Lambda_R$, and this is equivalent to showing that α is an isolated and regressive set. By our remarks in § 5, we know that $A \in \Lambda$ and therefore α will be an isolated set. It remains to prove that α is regressive. Since finite sets are regressive, we may assume that α is an infiinite (immune) set. In view of the definition of α, it then follows that the set

$$\lambda = \{n \mid a_n > 0\}$$

will be infinite. Let $f\,(n)$ be the strictly increasing function ranging over λ. Then

$$j\,(t'_{f(0)}, 0), \dots, j\,(t'_{f(0)}, a_{f(0)} - 1), j\,(t'_{f(1)}, 0), \dots, (t'_{f(1)}, a_{f(1)} - 1), j\,(t_{f(2)}, 0), \dots \tag{11}$$

is an enumeration of all the elements of α. Also, this enumeration is one-to-one, because each of the functions t'_n and $j\,(x, y)$ is one-to-one. We now proceed to show that the enumeration in (11) is regressive. To begin, note that, by (9), $t_n \leq^* a_n$ and therefore also

$$t'_{n+1} \leq^* a_n \,. \tag{12}$$

Let the number $t'_{f(n+1)}$ be given. We wish to show that we can effectively find each of the numbers

$$t'_{f(n)} \text{ and } a_{f(n)} \,.$$

Using the regressiveness of the function t'_n we can find the value of $f\,(n + 1)$, as well as the numbers

$$t'_0\,, t'_1\,, \dots\,, t'_{f(n+1)}\,, \tag{13}$$

together with their respective indices. In view of (12), we can then compute the numbers,

$$(14) \qquad a_0, a_1, \dots, a_{f(n+1)-1}.$$

Because $f(n)+1 \leq f(n+1)$, it follows that by determining the numbers in (14) that are positive, we can effectively find the value of $a_{f(n)}$ and also the number $f(n)$. Therefore the value of $a_{f(n)}$ can be found. Since we know the value of $f(n)$, we can locate the number $t'_{f(n)}$, among the elements in (13). Hence the value of $t'_{f(n)}$ can also be found. We can conclude from these remarks that each of the mappings,

$$t'_{f(n+1)} \longrightarrow t'_{f(n)},$$

$$t'_{f(n+1)} \longrightarrow a_{f(n)},$$

has a partial recursive extension. Combining this property with the fact that $j(x, y)$ is a one-to-one recursive function it is readily seen that (11) represents a regressive enumeration of the set α. Therefore α is a regressive set, and this completes the proof.

LEMMA 3. Let $T \leq^* a_n$ and $T \leq^* b_n$; then $T \leq^* (a_n + b_n)$ and

$$\Sigma_{T+1}(a_n + b_n) = \Sigma_{T+1}\, a_n + \Sigma_{T+1}\, b_n.$$

PROPOSITION 6. $A, B \in \Sigma_T \Longrightarrow A + B \in \Sigma_T$.

PROOF. Use (5), the definition of Σ_T, and Lemma 3.

PROPOSITION 7. Let $T \leq^* a_n$; then $\Sigma_T\, a_n \in \Sigma_T$.

PROOF. Let the function b_n be defined by, $b_0 = 0$ and $b_{n+1} = a_n$. Then $T + 1 \leq^* b_n$; and by (7) also, $T \leq^* b_n$. Clearly,

$$[a_0 + a_1 + a_2 + \dots]_T = [b_0 + b_1 + b_2 + \dots]_{T+1}.$$

Therefore,

$$\Sigma_T\, a_n = \Sigma_{T+1}\, b_n, \text{ with } T \leq^* b_n,$$

and hence $\Sigma_T\, a_n \in \Sigma_T$.

Proposition 8. Let $A \leq T$. Then $A = \Sigma_T a_n$, for some function a_n with $T \leq^* a_n$.

Proof. If A is finite, say $A = k$, then the result follows by letting a_n be the (recursive) function given by, $a_0 = k$ and $a_{n+1} = 0$.

Assume now that A is an infinite (regressive) isol. Then there will be a strictly increasing function $f(x)$ such that, if $\alpha = \varrho t_{f(n)}$, then

$$\alpha \in A \text{ and } \alpha \mid \tau - \alpha.$$

In this case, let the function a_n be defined by,

$$a_n = \begin{cases} 1, & \text{if } t_n \in \alpha, \\ 0, & \text{if otherwise.} \end{cases}$$

Since $\alpha \mid \tau - \alpha$, it follows that $t_n \leq^* a_n$; and hence also

$$T \leq^* a_n. \tag{15}$$

In addition, it is readily seen that

$$\alpha \simeq \sum_0^\infty j(t_{f(n)}, 0) = \sum_0^\infty j(t_n, \nu(a_n)) \in \Sigma_T a_n. \tag{16}$$

Combining (15) and (16) with the fact that $\alpha \in A$, we obtain

$$A = \Sigma_T a_n \text{ and } T \leq^* a_n.$$

This completes the proof.

Corollary. $A \leq T \Longrightarrow A \in \Sigma_T$.

Proof. Use Propositions 7 and 8.

Lemma 4. Let a_n be any function such that, $a_n \geq 1$ for every number n. Then $T \leq \Sigma_T a_n$.

Remark. We wish to mention that while the infinite series referred to in Lemma 4 will be an isol it need not be a regressive isol.

LEMMA 5. Let δ be a regressive set such that,

$$\tau \subset \delta \text{ and } \tau \mid \delta - \tau.$$

Then there is a regressive function d_n that ranges over δ and which also preserves the order of the set τ as determined by the function t_n; in the sense that, if $d_m, d_n \in \tau$, then

$$(d_m = t_p, d_n = t_q\ m < n) \Longrightarrow p < q. \tag{17}$$

PROPOSITION 9. Let $V \in \Lambda_R$ and $T \leq V$. Then there is a regressive isol B such that,

$$B \in \Sigma_T, T \leq B \text{ and } V \leq^* B.$$

PROOF. Let δ be a regressive set belonging to V. Because $T \leq V$, it follows that there exists a regressive set $\tau^* \in T$, such that

$$\tau^* \subset \delta \text{ and } \tau^* \mid \delta - \tau^*. \tag{18}$$

Both τ and τ^* belong to T and therefore $\tau \simeq \tau^*$. For reasons of notation, we wish to assume that $\tau = \tau^*$. Because $\tau \simeq \tau^*$, it will be seen that this modification does not cause any difficulty in the proof. With this identification, (18) can be rewritten as,

$$\tau \subset \delta \text{ and } \tau \mid \delta - \tau. \tag{19}$$

Apply Lemma 5, and let d_n be a regressive function ranging over δ and satisfying (17), i.e.,

$$(d_m = t_p, d_n = t_q, m < n) \Longrightarrow p < q.$$

It will be seen that there is no loss in generality if we assume that $d_0 = t_0$; and once again for reasons of notation, we shall assume from now on that this is true. Let $f(n)$ be the strictly increasing function such that,

$$t_n = d_{f(n)}. \tag{20}$$

We note that combining (16), with the regressiveness of the functions t_n dnd d_n, yields

(21) $$t_n \leq^* f(n).$$

Define the function $e(n)$ by,

(22) $$\begin{cases} e(o) = f(o) = 0 \\ e(n+1) = f(n+1) - f(n). \end{cases}$$

Then $e(o) = 0$, and for every number n, $e(n+1) \geq 1$. Taking into account that t_n is a regressive function it follows from (20), that

(23) $$t_n \leq^* e_n,$$

and hence also that

(24) $$T \leq^* e_n.$$

Set

$$B = \Sigma_{T+1}\, e_n.$$

Then, in view of (24), $B \in \Sigma_T$. In addition, since $e(o) = 0$, we have that

$$\Sigma_{T+1}\, e_n = \Sigma_T\, e_{n+1},$$

and therefore also,

(25) $$B = \Sigma_T\, e_{n+1}.$$

Combining Lemma 4 with the fact that, $e(n+1) \geq 1$ for every number n, it follows from (25) that

(26) $$T \leq B.$$

We have shown so far that both $B \in \Sigma_T$ and $T \leq B$. To complete the proof we now verify that,

(27) $$V \leq^* B.$$

Let

$$\beta = \sum_0^\infty j(t_n, \nu(e_{n+1})).$$

In view of (25), $\beta \in B$. We know that both $T \leq B$ and T is an infinite isol. Because $B \in \Sigma_T$, it follows therefore that B is an infinite regressive isol and β is an infinite regressive set. By employing an argument similar to the one in the proof of Proposition 5, one can readily show that

$$(28) \qquad j(t_0, 0), \dots, j(t_0, e_1 - 1), j(t_1, 0), \dots, j(t_1, e_2 - 1), j(t_2, 0), \dots,$$

represents a regressive enumeration of the set β.

Let b_n be the regressive function ranging over β that is determined by this enumeration. Taking into account the definitions of the functions $f(n)$ and $e(n)$, we note that the function b_n has the following property;

$$(29) \qquad \begin{cases} \text{if } f(k) \leq n < f(k+1) \text{ and } f(k) + r = n, \\ \text{then } b_n = j(t_k, r). \end{cases}$$

In order to establish (27), it suffices to prove that

$$(30) \qquad d_n \leq^* b_n,$$

and this will be our approach here. For this purpose let the number d_n be given. We wish to find the value of b_n. Since d_n is a regressive function we can find the number n. Let k be the largest number such that $f(k) \leq n$; and recall that $t_n = d_{f(k)}$. Because $\tau \mid \delta - \tau$ and d_n is a regressive function, we can determine both of the numhers $f(k)$ and t_k; and hence also the number r such that,

$$f(k) + r = n.$$

We can then compute the number,

$$j(t_k, r).$$

In view of (29), this means that we can effectively find the value of b_n. We can conclude from these remarks that the mapping,

$$d_n \longrightarrow b_n,$$

has a partial recursive extension, and therefore $d_n \leq^* b_n$. This proves (30), and completes the proof of the proposition.

PROPOSITION 10. Σ_T has no regressive upper bound.

PROOF. Assume otherwise and let U be a regressive upper bound of Σ_T.

We first note that $U \notin \Sigma_T$. For if $U \in \Sigma_T$, then also $U+1 \in \Sigma_T$, which would mean that $U+1 \leq U$, since U is an upper bound of Σ_T. But it is well known that the relation $U+1 \leq U$ is false for isols.

Because $T \in \Sigma_T$ and U is a regressive upper bound of Σ_T, it follows that $T \leq U$. Hence by Proposition 9, there will be a regressive isol B such that $B \in \Sigma_T$, and

$$U \leq^* B. \tag{31}$$

Since $B \in \Sigma_T$, it follows that $B \leq U$ and hence also that

$$B \leq^* U. \tag{32}$$

Combining (31) and (32) implies that

$$B = U.$$

This would mean that $U \in \Sigma_T$, which we already know is not the case. We can conclude therefore, that Σ_T has no regressive upper bound.

REMARKS. This completes the results necessary in order to verify the properties (a) to (g) for the collection Σ_T. We note that properties (a) and (b) follows from Propositions 4 and 5. Property (c) follows directly from the definition of Σ_T and the fact that F is a denumerable collection of functions. Property (d) is Proposition 6, and property (e) is the corollary to Proposition 8. Lastly, properties (f) and (g), are Propositions 9 and 10, respectively.

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