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THE GENERATING FUNCTION FOR THE JACOBI POLYNOMIAL

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1. Put

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}.$$

Then it is familiar that

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta}, \tag{1}$$

where

$$R = (1 - 2\,x\,z + z^2)^{\frac{1}{2}}.$$

For proofs of (1) see [2, p. 127], [3, p. 140], [4, pp. 69-70].

If we put

$$u = \frac{1}{2}(x-1)\,z, \qquad v = \frac{1}{2}(x+1)\,z,$$

we have

$$R = ((1 - u - v)^2 - 4uv)^{\frac{1}{2}} \tag{2}$$

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and (1) becomes

$$(3) \qquad \sum_{j,\,k=0}^{\infty} \binom{\alpha+j+k}{j} \binom{\beta+j+k}{k} u^j v^k$$

$$= 2^{\alpha+\beta} R^{-1} (1 - u + v + R)^{-\alpha} (1 + u - v + R)^{-\beta} ,$$

where R is defined by (2).

The object of this note is to give a direct and elementary proof of (3).

2. Consider the sum

$$\frac{1}{(1-x)^{\alpha+1}(1-y)^{\beta+1}} \sum_{j,\,k=0}^{\infty} \frac{(\alpha+1)_{j+k}}{j!\,(\alpha+1)_k} \frac{(\beta+1)_{k+j}}{k!\,(\beta+1)_j} \frac{(-1)^{j+k} x^j y^k}{(1-x)^{j+k}(1-y)^{j+k}}$$

$$= \sum_{j,\,k=0}^{\infty} (-1)^{j+k} \frac{(\alpha+1)_{j+k}(\beta+1)_{k+j}}{j!\,(\alpha+1)_k\, k!\,(\beta+1)_j} x^j y^k$$

$$\cdot \sum_{r=0}^{\infty} \frac{(\alpha+j+k+1)_r}{r!} x^r \sum_{s=0}^{\infty} \frac{(\beta+j+k+1)_s}{s!} y^s =$$

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_m(\beta+1)_n}{m!\;n!} x^m y^n \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{(-m)_j\,(-n)_k}{j!\;k!} \frac{(\alpha+m+1)_k(\beta+n+1)_j}{(\alpha+1)_k(\beta+1)_j} .$$

The inner sum is equal to

$$\sum_{j=0}^{m} \frac{(-m)(\beta+n+1)_j}{j!\,(\beta+1)_j} \sum_{k=0}^{n} \frac{(-n)(\alpha+m+1)_k}{k!\,(\alpha+1)_k} = \frac{(-m)_m}{(\beta+1)_m} \frac{(-m)_n}{(\alpha+1)_n} ,$$

which vanishes unless $m = n$. It follows at once that

$$(4) \qquad \frac{1}{(1-x)^{\alpha+1}(1-y)^{\beta+1}} \sum_{j,\,k=0}^{\infty} \binom{\alpha+j+k}{j} \binom{\beta+j+k}{k}$$

$$\cdot \frac{(-1)^{j+k} x^j y^k}{(1-x)^{j+k}(1-y)^{j+k}} = \frac{1}{1-xy} .$$

If we put

$$(5)\qquad u = -\frac{x}{(1-x)(1-y)}, \qquad v = -\frac{y}{(1-x)(1-y)},$$

it is easily verified that

$$1-x = \frac{2}{1-u+v+R}, \qquad 1-y = \frac{2}{1+u-v+R}$$

and

$$\frac{1-xy}{(1-x)(1-y)} = R.$$

Thus (4) becomes

$$\sum_{j,k=0}^{\infty} \binom{\alpha+j+k}{j}\binom{\beta+j+k}{k} u^j v^k = R^{-1}(1-x)^{\alpha}(1-y)^{\beta}$$

$$= 2^{\alpha+\beta} R^{-1}(1-u+v+R)^{-\alpha}(1+u-v+R)^{-\beta}.$$

This evidently completes the proof of (3).
The above proof may be compared with [1, p. 82].

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