Rendiconti
del
Seminario Matematico
della
Università di Padova

Martin Fan Cheng

On the approximation of continuous functions by polynomials on $(-\infty, \infty)$ and $(0, \infty)$ in terms of exponential weight factor

Rendiconti del Seminario Matematico della Università di Padova, tome 36, nº 2 (1966), p. 310-314

<http://www.numdam.org/item?id=RSMUP_1966__36_2_310_0>

ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY POLYNOMIALS ON $(-\infty, \infty)$ AND $(0, \infty)$ IN TERMS OF EXPONENTIAL WEIGHT FACTOR

di MARTIN FAN CHENG (*a Minneapolis*) *)

I. Introduction

On studying the completeness of the Hermite and Laguerre polynomials, one is led to the consideration of the following more general problem.

Suppose

$X = \{f \mid f$ a real-valued function continuous over the real line; $\lim_{x \to \pm\infty} |f(x)| e^{-\delta|x|^\alpha} = 0,\ \alpha > 0,\ \delta > 0\}$,

$Y = \{ p \mid p$ a polynomial in $x\}$.

Define, for $f \in X$

$$\|f\| = \sup_{-\infty < x < \infty} |f(x) e^{-\delta|x|^\alpha}| .$$

The problem is: can any f in X be approximated by polynomials in the above defined norm $\|\ \|$? In this paper, we show that the above mentioned f can be approximated by polynomials in this norm if $\alpha \geqslant 1$, but not for $0 < \alpha < 1$. The demarcation is shifted to $\alpha = 1/2$, if the norm is defined as

$$\|f\|_{1/2} = \sup_{x \geqslant 0} \{ |f(x)| e^{-\delta|x|^\alpha} \} .$$

*) Indirizzo dell'A.: School of Math., University of Minnesota, Minneapolis, Minnesota (U.S.A.).

The proof is based on reducing this problem to a corresponding problem in the theory of moments. From these results we fin that a theorem of Carlson [1] follows as a natural consequence. It is also clear that convergence in the norm $\| \ \|$ implies convergence in $L^p(\mu)$, $1 \leqslant p < \infty$, for any measure μ on $(-\infty, \infty)$ satisfying

$$\int_{-\infty}^{\infty} e^{\varepsilon |x|^\alpha} d\mu(x) < \infty, \qquad \varepsilon = p\delta .$$

II. The Main Theorem

The following is our main theorem where X, Y and the norm are defined as in the introduction.

THEOREM 1: *If $\alpha \geqslant 1$, then Y is dense in X. However this conclusion is false if $0 < \alpha < 1$.*

Before presenting the proof, we need the following well known theorem of Riesz [5, p. 115].

THEOREM 2 (Riesz): *Y is dense in X if and only if there does not exist a nontrivial finite signed measure μ such that*

$$\text{I)} \quad \int_{-\infty}^{\infty} x^n d\mu(x) = 0, \ n = 0, 1, 2, \ldots,$$

$$\text{II)} \quad \int_{-\infty}^{\infty} e^{\delta |x|^\alpha} d\, |\mu(x)| < \infty .$$

We also need

LEMMA 3: *If μ is a finite signed measure satisfying the conditions* I) *and* II) *of theorem* 2, *and if $\alpha \geqslant 1$, $\delta > 0$, then μ is trivial, i.e.*

$$\int_{-\infty}^{\infty} f d\mu = 0, \qquad \forall f \in X .$$

Proof.: Let μ^+ and μ^- be the positive and negative part of μ respectively. Then from I),

$$\text{III)} \quad \int_{-\infty}^{\infty} x^n d\mu^+ = \int_{-\infty}^{\infty} x^n d\mu^-, \; n = 0, 1, 2, \dots .$$

From II) we have

$$\text{IV)} \quad \int_{-\infty}^{\infty} e^{\delta|x|^\alpha} d\mu^+ < \infty \text{ and } \int_{-\infty}^{\infty} e^{\delta|x|^\alpha} d\mu^- < \infty .$$

By a theorem of Carleman [2], a sufficient condition that two positive measures μ^+ and μ^- satisfying III) are equal is that $\sum_1^\infty \mu_{2n}^{-1/2n}$ diverges, where

$$\mu_n = \int_{-\infty}^{\infty} x^n d\mu^+ = \int_{-\infty}^{\infty} x^n d\mu^-, \qquad n = 1, 2, \dots .$$

An easy estimation, based on IV), shows that $\mu_{2n} = 0[(2n)^{2n}]$ and therefore $\sum_1^\infty \mu_{2n}^{-1/2n}$ diverges. (The sufficiency of condition IV) is due to Hardy [3]. For more details, see [6, p. 19]).

Hence $\mu^+ = \mu^-$, i.e.

$$\int f d\mu^+ = \int f d\mu^-, \qquad \forall f \in X,$$

or

$$\int f d\mu = 0, \qquad \forall f \in X .$$

This means μ is trivial and the lemma is proved.

Now we are ready to prove the main theorem.

Proof of theorem 1: If $\alpha \geqslant 1$, $\delta > 0$, then by lemma 3, there does not exist a nontrivial measure satisfying I) and II) of theorem 2. Hence by the sufficiency of theorem 2, Y is dense in X.

However, for $0 < \alpha < 1$, there does exist a nontrivial finite measure $g(t)dt$ satisfying I) and II): namely let

$$g(x) = \mathrm{Re}\left[e^{-\left(\frac{x}{i}\right)^{\beta}}\right] = e^{-\left(\cos\frac{\beta\pi}{2}\right)|x|^{\beta}} \cos\left[\left(\sin\frac{\beta\pi}{2}\right)|x|^{\beta}\right]$$

where $\alpha < \beta < 1$. Hence, by the same theorem, Y is not dense in X if $0 < \alpha < 1$, and the proof is complete.

As an application, we give an independent proof of the follo-wuing theorem due to Carlson [1].

THEOREM 4: *If* $f(z)$ *is regular and of the form* $0(e^{k|z|})$ *for* $\mathrm{Im}\,(z) \geqslant 0$ *and*

$$f(z) = 0(e^{-a|z|^{\alpha}}), \quad a > 0, \quad \alpha \geqslant 1,$$

on the line $\mathrm{Im}\,(z) = 0$, *then* $f(z) = 0$ *identically.*

Proof.: Let $g(z) = e^{imz}f(z)$, where $m > k$, and

$$\bar{g}(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{i\beta x}dx .$$

Then by the Paley-Wiener Theorem [4], the derivatives

$$\bar{g}^{(n)}(0) = 0, \qquad n = 0, 1, 2, \ldots,$$

i.e.

$$\int_{-\infty}^{\infty} t^n g(t)dt = 0, \quad n = 0, 1, 2, \ldots .$$

Therefore $g(t)dt$ satisfies the conditions of lemma 3, provided the constant δ is chosen less than a. By lemma 3 $g(t)dt$ is a trivial measure, i.e. $g(x) = 0$ for every x. Thus by the unique continuation theorem $g(z) \equiv 0$; hence $f(z) \equiv 0$ and the theorem is proved.

In the following, we shall give the result corresponding to theorem 1 for the case where

$$\widetilde{X} = \{f \mid f \text{ a real-valued function continuous over } x \geqslant 0,$$

$$\lim_{x \to \infty} |f(x)| e^{-\delta x^{\alpha}} = 0, \alpha > 0, \delta > 0\},$$

$$\widetilde{Y} = \{p \mid p \text{ a polynomial in } x, x \geqslant 0\},$$

and

$$\| f \|_{1/2} = \sup_{x \geqslant 0} | f(x) e^{-\delta x^{\alpha}} | .$$

THEOREM 5: *If* $\alpha \geqslant 1/2$, *then* $\widetilde{Y}$ *is dense in* $\widetilde{X}$. *However this conclusion is false if* $0 < \alpha < 1/2$.

The proof follows from a modified form of Carleman's theorem (see [6, p. 20]), and is otherwise exactly similar to the proof of theorem 1.

BIBLIOGRAPHY

[1] CARLSON F.: *Sur Une Classe de Series de Taylor*, Thesis, Upsala, 1914.

[2] CARLEMAN T.: *Sur le Probleme des Moments*. Comptes Rendus, 1922, 174, 1680-1682.

[3] HARDY G. H.: *On Stieltjes' «Probleme des Moments»*. Messenger of Mathematics, 1917, 46, 175-182.

[4] PALEY R. E. A. C. and WIENER N.: *Fourier Transforms in the Complex Domain*. American Mathematical Society Colloquium Publication, 1934, 19, 545-554.

[5] RIESZ F. and NAGY B.: *Functional Analysis, English Translation, Frederick Ungar Publishing Co.*, 1955.

[6] SHOHAT J. A. and TAMARKIN J. D.: *The Problem of Moments*. American Mathematical Society, 1943.

University of Minnesota, Minneapolis, Minnesota (U.S.A.)

Manoscritto pervenuto in redazione il 15 ottobre 1965.